5 Linear Systems

5.1 State-Space Equations

Consider a system with m inputs, $u_1(t), \ldots, u_m(t)$, and p outputs $y_1(t), \ldots, y_p(t)$. If the system is linear and time invariant (LTI) (and proper as well as finite dimensional), then $\mathbf{y} \in \Re^p$ can be related to $\mathbf{u} \in \Re^m$ by the state-space equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{1}$$

where $\mathbf{x} \in \Re^n$ is the state vector. Note that

$$\mathbf{A} \in \Re^{n\times n}, \quad \mathbf{B} \in \Re^{n\times m}, \quad \mathbf{C} \in \Re^{p\times n}, \quad \mathbf{D} \in \Re^{p\times m}$$

are constant matrices. If $m = p = 1$, it is a single-input/single-output (SISO) system. Otherwise it is a multi-input/multi-output (MIMO) system.

The solution of (1) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau \tag{2}$$

where

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k t^k / k!$$

Therefore,

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau + \mathbf{D}\mathbf{u}(t) \tag{3}$$

Let

$$\hat{\mathbf{x}}(s) = \mathcal{L}\{\mathbf{x}(t)\} = \int_0^\infty e^{-st}\mathbf{x}(t)\, dt$$

denote the Laplace transform (L.T.). Taking L.T.'s in (1) gives

$$s\hat{\mathbf{x}}(s) - \mathbf{x}_0 = \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s)$$

and hence

$$\hat{\mathbf{x}}(s) = (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) \tag{4}$$

Comparing (2) with (4), we see that

$$\begin{aligned}\mathcal{L}\{e^{\mathbf{A}t}\mathbf{x}_0\} &= (s\mathbf{1}-\mathbf{A})^{-1}\mathbf{x}_0 \\ \mathcal{L}\{e^{\mathbf{A}t} * \mathbf{B}\mathbf{u}(t)\} &= (s\mathbf{1}-\mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s)\end{aligned}$$

where $(*)$ denotes temporal convolution. We conclude that

$$\mathcal{L}\{e^{\mathbf{A}t}\} = (s\mathbf{1}-\mathbf{A})^{-1} \tag{5}$$

Setting $\mathbf{x}_0 = \mathbf{0}$ gives

$$\begin{aligned}\hat{\mathbf{y}}(s) &= \mathbf{C}\hat{\mathbf{x}}(s) + \mathbf{D}\hat{\mathbf{u}}(s) \\ &= \underbrace{\left[\mathbf{C}(s\mathbf{1}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]}\hat{\mathbf{u}}(s) \\ &= \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)\end{aligned} \tag{6}$$

$\hat{\mathbf{G}}(s)$ is called the transfer matrix (or transfer function in the SISO case). Writing

$$\mathbf{C} = \begin{bmatrix}\mathbf{c}_1 \\ \vdots \\ \mathbf{c}_p\end{bmatrix}, \quad \mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_m], \quad \mathbf{D} = \text{matrix}\{D_{ij}\}$$

it follows that

$$\hat{\mathbf{G}}(s) = \text{matrix}\{\hat{G}_{ij}(s)\}, \quad \hat{G}_{ij}(s) = \mathbf{c}_i(s\mathbf{1}-\mathbf{A})^{-1}\mathbf{b}_j + D_{ij} \tag{7}$$

We also define

$$\left[\begin{array}{c|c}\mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right] = \mathbf{C}(s\mathbf{1}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

Physical Significance of $\hat{\mathbf{G}}(s)$

Let us explicitly exhibit the magnitude and phase of $\hat{G}_{ij}(j\omega)$:

$$\hat{G}_{ij}(j\omega) = |\hat{G}_{ij}(j\omega)|e^{j\phi_{ij}(\omega)}, \quad \phi_{ij}(\omega) = \arg\{\hat{G}_{ij}(j\omega)\}$$

Let

$$u_j(t) = \sin\omega t, \quad u_i = 0, i \neq j$$

Then as $t \to \infty$,

$$y_i(t) = |\hat{G}_{ij}(j\omega)|\sin(\omega t + \phi_{ij})$$

Hence, $\hat{G}_{ij}(j\omega)$ contains the frequency response between the jth input and the ith output.

Also the inverse Laplace transform of $\hat{\mathbf{G}}(s)$ is given by

$$\mathbf{G}(t) = \mathcal{L}^{-1}\{\hat{\mathbf{G}}(s)\} = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

which is the impulse response, *i.e.*, $y_i(t) = G_{ij}(t)$ when $u_j(t) = \delta(t), u_i = 0, i \neq j$. Using (6), we have

$$\begin{aligned} \mathbf{y}(t) &= \mathcal{L}^{-1}\{\hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)\} \\ &= \mathbf{G}(t) * \mathbf{u}(t) \\ &= \int_0^t [\mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B} + \mathbf{D}\delta(t-\tau)]\mathbf{u}(\tau)\, d\tau \\ &= \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

which agrees with (3) when $\mathbf{x}_0 = \mathbf{0}$.

Stability

Assume that the eigenvalues of $\mathbf{A}$, λ_i, are distinct with corresponding eigenvectors $\mathbf{e}_i$. Define

$$\mathbf{\Lambda} = \mathrm{diag}\{\lambda_i\}, \quad \mathbf{E} = \mathrm{row}\{\mathbf{e}_i\}$$

Recall that

$$\begin{aligned} \mathbf{A} &= \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1} \\ e^{\mathbf{A}t} &= \mathbf{E}e^{\mathbf{\Lambda}t}\mathbf{E}^{-1} \\ e^{\mathbf{\Lambda}t} &= \mathrm{diag}\{e^{\lambda_i t}\} \end{aligned} \tag{8}$$

The system in (1) with $\mathbf{u}(t) = \mathbf{0}$ is aymptotically stable if the eigenvalues of $\mathbf{A}$ have negative real parts. In this case, Eqs. (2) and (8) imply that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for any $\mathbf{x}_0$.

Also, note that

$$\begin{aligned} (s\mathbf{1} - \mathbf{A})^{-1} &= (s\mathbf{E}\mathbf{E}^{-1} - \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1})^{-1} \\ &= [\mathbf{E}(s\mathbf{1} - \mathbf{\Lambda})\mathbf{E}^{-1}]^{-1} \\ &= \mathbf{E}(s\mathbf{1} - \mathbf{\Lambda})^{-1}\mathbf{E}^{-1} \end{aligned} \tag{9}$$

where

$$(s\mathbf{1} - \mathbf{\Lambda})^{-1} = \text{diag}\left\{\frac{1}{s - \lambda_i}\right\}$$

Hence, combining (7) and (9) gives

$$\hat{G}_{ij}(s) = \mathbf{c}_i\mathbf{E}(s\mathbf{1} - \mathbf{\Lambda})^{-1}\mathbf{E}^{-1}\mathbf{b}_j + D_{ij}$$

It is clear then that the eigenvalues of $\mathbf{A}$ are the poles of each $\hat{G}_{ij}(s)$.

Controllability

The system in Eq. (1) is controllable if for any initial state $\mathbf{x}(0) = \mathbf{x}_0$, $t_1 > 0$, and final state $\mathbf{x}_1$, there exists a control $\mathbf{u}$ so that the solution of (1) satisfies $\mathbf{x}(t_1) = \mathbf{x}_1$. If (1) is controllable we say that $(\mathbf{A}, \mathbf{B})$ is controllable.

The following are equivalent:

(i) $(\mathbf{A}, \mathbf{B})$ is controllable;

(ii) rank $[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}] = n$;

(iii) The eigenvalues of $(\mathbf{A} + \mathbf{BF})$ can arbitrarily assigned (with complex eigenvalues in complex-conjugate pairs) through proper choice of $\mathbf{F}$.

Item (iii) provides an easy way to stabilize (control) a controllable system. Consider the state feedback

$$\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t), \quad \mathbf{F} \in \Re^{m \times n} \tag{10}$$

Then, (1) becomes

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BF})\mathbf{x}$$

which is asymptotically stable if the eigenvalues of $\mathbf{A} + \mathbf{BF}$ have negative real parts.

Observability

The system in (1) is observable if for any $t_1 > 0$, the initial state $\mathbf{x}_0$ can be determined from the time histories of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ on $[0, t_1]$. If (1) is observable we say that $(\mathbf{C}, \mathbf{A})$ is observable.

The following are equivalent:

(i) $(\mathbf{C}, \mathbf{A})$ is observable;

(ii) $\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n;$

(iii) $(\mathbf{A}^T, \mathbf{C}^T)$ is controllable;

(iv) The eigenvalues of $(\mathbf{A} + \mathbf{LC})$ can arbitrarily assigned (with complex eigenvalues in complex-conjugate pairs) through proper choice of $\mathbf{L}$.

5.2 Observers

In order to implement the state feedback in (10) we require measurements of the states $\mathbf{x}(t)$. What if we only have access to the output $\mathbf{y}(t)$? An observer is a model of the system which uses knowledge of $\mathbf{y}(t)$ and $\mathbf{u}(t)$ to generate an estimate of the state, $\hat{\mathbf{x}}(t)$, which has the property that

$$\lim_{t \to \infty} [\mathbf{x}(t) - \hat{\mathbf{x}}(t)] = \mathbf{0} \tag{11}$$

Consider (1) with $\mathbf{D} = \mathbf{O}$ (this is not essential but simplifies things somewhat). An observer is a model of (1) that takes the form

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{12}$$

where $\mathbf{L}$ is selected so that the eigenvalues of $(\mathbf{A} + \mathbf{LC})$ have negative real parts. Define the estimation error by $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. Subtracting Eq. (12) from Eq. (1) gives

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \dot{\mathbf{e}} = (\mathbf{A} + \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}}) = (\mathbf{A} + \mathbf{LC})\mathbf{e}$$

Therefore, if $\mathbf{L}$ is selected as above, $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$.

Observer-Based Controller

Assuming that the states $\mathbf{x}(t)$ are unavailable for feedback in (10), we can use the estimate from (12) in place of $\mathbf{x}$:

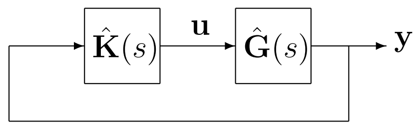
$$\mathbf{u} = \mathbf{F}\hat{\mathbf{x}} \tag{13}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} + \mathbf{BF} + \mathbf{LC})\hat{\mathbf{x}} - \mathbf{Ly} \tag{14}$$

This is a dynamical system of the form of Eq. (1) but the state vector is $\hat{\mathbf{x}}$, the "input" is $\mathbf{y}$ and the "output" is $\mathbf{u}$. Taking Laplace transforms we have

$$\begin{aligned} \hat{\mathbf{u}}(s) &= \hat{\mathbf{K}}(s)\hat{\mathbf{y}}(s), \\ \hat{\mathbf{K}}(s) &= \left[\begin{array}{c|c} \mathbf{A}+\mathbf{BF}+\mathbf{LC} & -\mathbf{L} \\ \hline \mathbf{F} & \mathbf{O} \end{array}\right] \end{aligned}$$

The closed-loop system can be represented by the above block diagram.



Combining Eqs. (1) and (14) gives

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{BF} \\ -\mathbf{LC} & \mathbf{A}+\mathbf{BF}+\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{15}$$

The stability of the closed-loop system is governed by the eigenvalues of the system matrix given here. Let us consider a transformation of the state vector:

$$\begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

This leads to

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}+\mathbf{BF} & -\mathbf{BF} \\ \mathbf{O} & \mathbf{A}+\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{16}$$

Given the zero partition, the eigenvalues of this matrix satisfy

$$\lambda\{\mathbf{A}+\mathbf{BF}\} \cup \lambda\{\mathbf{A}+\mathbf{LC}\}$$

Therefore, if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{C}, \mathbf{A})$ is observable, we can can choose $\mathbf{F}$ and $\mathbf{L}$ so that $(\mathbf{A}+\mathbf{BF})$ and $(\mathbf{A}+\mathbf{LC})$ (and hence the entire system) are stable.

We say that the observer-based controller has a separation structure.