3 Rigid Spacecraft Attitude Control

Consider a rigid spacecraft with body-fixed frame $\boldsymbol{\mathcal{F}}_b$ with origin O at the mass centre. Let $\boldsymbol{\omega}$ denote the angular velocity of $\boldsymbol{\mathcal{F}}_b$ with respect to an inertial frame $\boldsymbol{\mathcal{F}}_i$ with components expressed in $\boldsymbol{\mathcal{F}}_b$. The external torques about O are due to control torques $\mathbf{u}$ and disturbances $\mathbf{d}$. The motion equation describing the evolution of $\boldsymbol{\omega}$ is Euler's equation:

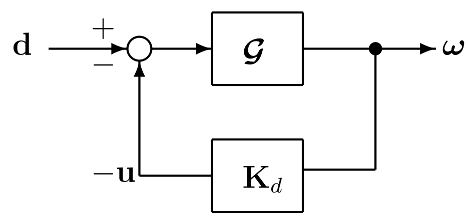
$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \mathbf{u} + \mathbf{d} \tag{1}$$

where $\mathbf{I} = \mathbf{I}^T > \mathbf{O}$ is the moment of inertia matrix.

We have seen previously that when $\mathbf{d} = \mathbf{0}$, the mapping from $\mathbf{u}$ to $\boldsymbol{\omega}$ is passive. It follows that the strictly passive feedback,

$$\mathbf{u} = -\mathbf{K}_d\boldsymbol{\omega}, \quad \mathbf{K}_d = \mathbf{K}_d^T > \mathbf{O} \tag{2}$$

yields L_2-stability for $\boldsymbol{\omega}$, *i.e.*, $\mathbf{d} \in L_2 \Rightarrow \boldsymbol{\omega} \in L_2$.



Furthermore, for $\mathbf{d} = \mathbf{0}$, $\boldsymbol{\omega} = \mathbf{0}$ is a globally asymptotically stable equilibrium point. This follows from considering the Lyapunov function $V = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega}$ and showing that $\dot{V} = -\boldsymbol{\omega}^T\mathbf{K}_d\boldsymbol{\omega}$.

3.1 Attitude Description

The above provides the ingredients for controlling the spacecraft rate motion. We now consider the attitude. Recall that

$$\int_0^t \boldsymbol{\omega}(\tau)\, d\tau \tag{3}$$

is, in general, not meaningful. The attitude can be described with the rotation matrix

$$\mathbf{C} = \mathbf{C}_{bi} = \underrightarrow{\boldsymbol{\mathcal{F}}}_b \cdot \underrightarrow{\boldsymbol{\mathcal{F}}}_i^T$$

We know that $\mathbf{C}$ is an orthogonal matrix, *i.e.*, $\mathbf{C}^{-1} = \mathbf{C}^T$ and evolves according to

$$\dot{\mathbf{C}} = -\boldsymbol{\omega}^\times \mathbf{C} \tag{4}$$

The nine entries in $\mathbf{C}$ can be parametrized by a minimum of three parameters. One possibility is Euler angles:

$$\mathbf{C} = \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3) \tag{5}$$

where

$$\begin{aligned}
\mathbf{C}_3(\theta_3) &= \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{C}_2(\theta_2) &= \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \\
\mathbf{C}_1(\theta_1) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}
\end{aligned}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$, $i = 1, 2, 3$. Rewriting (4) as $\boldsymbol{\omega}^\times = -\dot{\mathbf{C}}\mathbf{C}^T$ and substituting (5) leads to (eventually)

$$\boldsymbol{\omega} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{6}$$

where $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \theta_3]^T$ and

$$\mathbf{S}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 & -s_2 \\ 0 & c_1 & s_1 c_2 \\ 0 & -s_1 & c_1 c_2 \end{bmatrix}, \quad \mathbf{S}^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & s_1 \tan\theta_2 & c_1 \tan\theta_2 \\ 0 & c_1 & -s_1 \\ 0 & s_1 \sec\theta_2 & c_1 \sec\theta_2 \end{bmatrix} \tag{7}$$

Note that $\mathbf{S}^{-1}$ is undefined at $\theta_2 = \pm\pi/2$, the singularity of this parametrization. When using it, it will be assumed that the singularity is not reached.

Lastly we note that for small angles $\boldsymbol{\theta}$ and rates $\boldsymbol{\omega}$, (6) degenerates to

$$\boldsymbol{\omega} \doteq \dot{\boldsymbol{\theta}} \tag{8}$$

and the result of the integration in (3) is $\boldsymbol{\theta}(t) - \boldsymbol{\theta}(0)$.

3.2 Attitude Control

It is now assumed that the frames $\boldsymbol{\mathcal{F}}_b$ and $\boldsymbol{\mathcal{F}}_i$ have been selected so that the desired attitude corresponds to $\boldsymbol{\mathcal{F}}_b = \boldsymbol{\mathcal{F}}_i$, hence $\mathbf{C} = \mathbf{1}$, hence $\boldsymbol{\theta} = \mathbf{0}$. In order to stabilize this equilibrium, we must feed back $\boldsymbol{\theta}$ in some form. Building on our rate feedback result, it is tempting to suggest

$$\mathbf{u}(t) = -\mathbf{K}_d\boldsymbol{\omega}(t) - \mathbf{K}_p\boldsymbol{\theta}(t), \quad \mathbf{K}_p = \mathbf{K}_p^T > \mathbf{O} \tag{9}$$

This certainly makes senses in the limit of small angles, since the map from $\mathbf{u}$ to $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$ is passive and the above represents a strictly passive feedback with respect to $\dot{\boldsymbol{\theta}}$. Note that a proportional-integral (PI) feedback is the sum of a strictly passive feedack (P) and a passive one (I); hence it is strictly passive.

For large angles $\boldsymbol{\theta}$ let us use the passivity from $\mathbf{u}$ to $\boldsymbol{\omega}$:

$$\begin{aligned} \int_0^T \boldsymbol{\omega}^T \mathbf{u}\, dt &\geq 0 \\ \Rightarrow \int_0^T (\mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}})^T \mathbf{u}\, dt &\geq 0 \\ \Rightarrow \int_0^T \dot{\boldsymbol{\theta}}^T \mathbf{S}^T(\boldsymbol{\theta})\mathbf{u}\, dt &\geq 0 \\ \Rightarrow \int_0^T \dot{\boldsymbol{\theta}}^T \hat{\mathbf{u}}\, dt &\geq 0, \quad \hat{\mathbf{u}} \triangleq \mathbf{S}^T(\boldsymbol{\theta})\mathbf{u} \end{aligned}$$

Hence the map from $\hat{\mathbf{u}}$ to $\dot{\boldsymbol{\theta}}$ is passive. Note that $\mathbf{u}$ can recovered from $\hat{\mathbf{u}}$ according to

$$\mathbf{u}(t) = \mathbf{S}^{-T}(\boldsymbol{\theta})\hat{\mathbf{u}}(t) \tag{10}$$

Given this new passivity result, let us employ a PI law for $\hat{\mathbf{u}}$:

$$\begin{aligned} \hat{\mathbf{u}} &= -\mathbf{K}_d\dot{\boldsymbol{\theta}}(t) - \mathbf{K}_p\boldsymbol{\theta}(t) \qquad (11) \\ \Rightarrow \mathbf{u} &= -\mathbf{S}^{-T}(\boldsymbol{\theta})(\mathbf{K}_d\dot{\boldsymbol{\theta}} + \mathbf{K}_p\boldsymbol{\theta}) \end{aligned}$$

where $\mathbf{K}_p$ and $\mathbf{K}_d$ are symmetric and positive definite.

Theorem. Assuming $\theta_2 \neq \pm\pi/2$, the system represented by (1) ($\mathbf{d} = \mathbf{0}$), (6), and (11) has an asymptotically stable equilibrium $\boldsymbol{\omega} = \boldsymbol{\theta} = \mathbf{0}$.

Proof. We adopt as a Lyapunov function

$$V = \tfrac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\theta}^T\mathbf{K}_p\boldsymbol{\theta} > 0$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \boldsymbol{\omega}^T \mathbf{I}\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\theta}}^T \mathbf{K}_p \boldsymbol{\theta} \\ &= \boldsymbol{\omega}^T(-\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} + \mathbf{u}) + \dot{\boldsymbol{\theta}}^T \mathbf{K}_p \boldsymbol{\theta} \\ &= \dot{\boldsymbol{\theta}}^T (\mathbf{S}^T(\boldsymbol{\theta})\mathbf{u} + \mathbf{K}_p \boldsymbol{\theta}) \\ &= -\dot{\boldsymbol{\theta}}^T \mathbf{K}_d \dot{\boldsymbol{\theta}} \leq 0\end{aligned}$$

The result follows from LaSalle's theorem. □

3.3 Spacecraft Magnetic Control

Consider Euler's equation with the external torque stemming from on-board dipole moments immersed in a magnetic field:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} = \mathbf{m}^\times \mathbf{B} \tag{12}$$

where $\mathbf{B} = [B_1 \ B_2 \ B_3]^T$ are the components of the magnetic field expressed in the body-fixed frame and $\mathbf{m}(t) = [m_1 \ m_2 \ m_3]^T$ is the magnetic dipole moment of the spacecraft expressed in the body-fixed frame.

Consider the kinetic energy

$$H(t) = \tfrac{1}{2}\boldsymbol{\omega}^T \mathbf{I}\boldsymbol{\omega} \tag{13}$$

Then

$$\begin{aligned}\dot{H} &= \boldsymbol{\omega}^T \mathbf{I}\dot{\boldsymbol{\omega}} \\ &= \boldsymbol{\omega}^T[-\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} - \mathbf{B}^\times \mathbf{m}]\end{aligned} \tag{14}$$

Integrating both sides from $t = 0$ to $t = T$ and taking $H(0) = 0$ yields

$$\int_0^T \mathbf{m}^T \mathbf{B}^\times \boldsymbol{\omega}\, dt = H(T) \geq 0$$

Hence, the dynamical system with input $\mathbf{m}$ and output $\mathbf{B}^\times \boldsymbol{\omega} = -\boldsymbol{\omega}^\times \mathbf{B}$ is passive.

On the basis of the passivity theorem, we can use the strictly passive (negative) feedback:

$$\mathbf{m} = \mathbf{K}\boldsymbol{\omega}^\times \mathbf{B}, \ \ \mathbf{K} = \mathbf{K}^T > \mathbf{O} \tag{15}$$

Writing the magnetic field vector as

$$\underset{\rightarrow}{\mathbf{B}} = \underset{\rightarrow}{\mathcal{F}}_i^T \mathbf{B}_i = \underset{\rightarrow}{\mathcal{F}}_b^T \mathbf{B}$$

we have

$$\dot{\underset{\rightarrow}{\mathbf{B}}} = \overset{\circ}{\underset{\rightarrow}{\mathbf{B}}} + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathbf{B}}$$

The magnetic field as seen in the body-fixed frame appears to change because of two effects: (i) the field is changing as a function of spacecraft position which is itself changing and (ii) the local field as seen in the spacecraft changes because the spacecraft is rotating. If we assume the first effect is much slower than the second one and neglect it, we have

$$\dot{\underset{\rightarrow}{\mathbf{B}}} = \underset{\rightarrow}{\mathbf{0}} \;\Rightarrow\; \overset{\circ}{\underset{\rightarrow}{\mathbf{B}}} = -\underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathbf{B}} \tag{16}$$

or in terms of components, $\dot{\mathbf{B}} = -\boldsymbol{\omega}^\times \mathbf{B}$. Using this expression in Eq. (15), we have

$$\mathbf{m}(t) = -\mathbf{K}\dot{\mathbf{B}}(t) \tag{17}$$

which is termed *B-dot* control.

A Lyapunov stability proof is also possible for the combination of Eqs. (12) and (15). Using $H(t)$ in Eq. (13) as a Lyapunov function, we have using (14)

$$\begin{aligned} \dot{H} &= -\boldsymbol{\omega}^T \mathbf{B}^\times \mathbf{m} \\ &= -\boldsymbol{\omega}^T \mathbf{B}^\times \mathbf{K} \boldsymbol{\omega}^\times \mathbf{B} \\ &= \boldsymbol{\omega}^T \mathbf{B}^\times \mathbf{K} \mathbf{B}^\times \boldsymbol{\omega} \\ &= -\boldsymbol{\omega}^T \mathbf{B}^{\times T} \mathbf{K} \mathbf{B}^\times \boldsymbol{\omega} \end{aligned}$$

Since $\mathbf{B}^{\times T}\mathbf{K}\mathbf{B}^\times$ is symmetric and positive-semidefinite, we have established Lyapunov stability of the angular velocity. We can use LaSalle's theorem to determine the invariant set. Setting $\dot{H} = \mathbf{0}$ implies that

$$\mathbf{B}^\times \boldsymbol{\omega} = \mathbf{0}$$

Hence the angular velocity does not necessarily converge to zero but can spin about an axis parallel to the **B**-field. In practice, this problem can be overcome since the direction of the **B**-field changes with time owing to the change in spacecraft position with time.

3.4 Tracking Control

The previous controllers are suitable for setpoint ($\boldsymbol{\theta} = \mathbf{0}$) regulation. Let us now assume that we are given trajectories $\{\boldsymbol{\theta}_d(t), \dot{\boldsymbol{\theta}}_d(t), \ddot{\boldsymbol{\theta}}_d(t)\}$ which we wish the spacecraft to track. The corresponding tracking error is denoted by

$$\tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_d(t) \tag{18}$$

which we desire to keep small and ideally zero. In preparation for deriving a tracking controller, let us remind ourselves that $\tilde{\boldsymbol{\theta}} \in L_2$ does not necessarily imply that $\tilde{\boldsymbol{\theta}}(t) \to \mathbf{0}$ as $t \to \infty$. However, if $\tilde{\boldsymbol{\theta}} \in L_2$ and $\dot{\tilde{\boldsymbol{\theta}}} \in L_2$, then it is so. In an effort to create this situation, we define the filtered error by

$$\boldsymbol{\rho}(t) = \dot{\tilde{\boldsymbol{\theta}}}(t) + \lambda\tilde{\boldsymbol{\theta}}, \quad \lambda > 0$$

Taking Laplace transforms of each side, we have

$$\hat{\boldsymbol{\rho}}(s) = \hat{\tilde{\boldsymbol{\theta}}}(s)(s + \lambda)$$

Therefore,

$$\hat{\tilde{\boldsymbol{\theta}}}(s) = \frac{1}{s+\lambda}\hat{\boldsymbol{\rho}}(s), \quad s\hat{\tilde{\boldsymbol{\theta}}}(s) = \frac{s}{s+\lambda}\hat{\boldsymbol{\rho}}(s),$$

Since $1/(s+\lambda) \in \mathcal{H}_\infty$ and $s/(s+\lambda) \in \mathcal{H}_\infty$ we have that $\boldsymbol{\rho} \in L_2$ implies that $\tilde{\boldsymbol{\theta}}, \dot{\tilde{\boldsymbol{\theta}}} \in L_2$ and hence $\tilde{\boldsymbol{\theta}}(t) \to \mathbf{0}$.

The dynamics are assumed to be of the form

$$\begin{aligned} \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} &= \mathbf{u}, \\ \boldsymbol{\omega} &= \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \end{aligned} \tag{19}$$

In addition to the tracking errors in (18), we define the following quantities

$$\begin{aligned} \dot{\boldsymbol{\theta}}_r &= \dot{\boldsymbol{\theta}}_d - \lambda\tilde{\boldsymbol{\theta}} \\ \boldsymbol{\rho} &= \dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_r = \dot{\tilde{\boldsymbol{\theta}}} + \lambda\tilde{\boldsymbol{\theta}} \\ \boldsymbol{\omega}_r &= \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}_r \\ \dot{\boldsymbol{\omega}}_r &= \mathbf{S}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_r + \dot{\mathbf{S}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}_r \\ \tilde{\boldsymbol{\omega}} &= \boldsymbol{\omega} - \boldsymbol{\omega}_r = \mathbf{S}(\boldsymbol{\theta})\boldsymbol{\rho} \end{aligned} \tag{20}$$

Let $\gamma = \min_i \lambda_i\{\mathbf{S}^T(\boldsymbol{\theta})\mathbf{S}(\boldsymbol{\theta})\}$. Then,

$$\gamma \int_0^T \boldsymbol{\rho}^T\boldsymbol{\rho}\, dt \leq \int_0^T \boldsymbol{\rho}^T\mathbf{S}^T\mathbf{S}\boldsymbol{\rho}\, dt = \int_0^T \tilde{\boldsymbol{\omega}}^T\tilde{\boldsymbol{\omega}}\, dt$$

Hence, letting $T \to \infty$ shows that

$$\widetilde{\boldsymbol{\omega}} \in L_2 \;\Rightarrow\; \boldsymbol{\rho} \in L_2 \;\Rightarrow\; \lim_{t\to\infty} \widetilde{\boldsymbol{\theta}}(t) = \mathbf{0}$$

Our basic approach to the tracking problem is

(i) choose a feedforward so that the error dynamics (with output $\widetilde{\boldsymbol{\omega}}$) are passive;

(ii) close a strictly passive feedback to render the error dynamics L_2-stable so that $\widetilde{\boldsymbol{\omega}} \in L_2$.

The feedforward torque is defined by

$$\mathbf{u}_d = \mathbf{I}\dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}_r^{\times}\mathbf{I}\boldsymbol{\omega} \tag{21}$$

and the feedback part of the control torque is defined by

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_d \tag{22}$$

so that the total torque has two parts, $\mathbf{u} = \mathbf{u}_d + \tilde{\mathbf{u}}$.

Subtracting (21) from (19) yields

$$\mathbf{I}\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^{\times}\mathbf{I}\boldsymbol{\omega} = \tilde{\mathbf{u}} \tag{23}$$

<u>Claim</u>: The mapping $\widetilde{\boldsymbol{\omega}} = \boldsymbol{\mathcal{G}}\tilde{\mathbf{u}}$ is passive ($\widetilde{\boldsymbol{\omega}}(0) = \mathbf{0}$).

<u>Proof</u>: Let $V(t) = \frac{1}{2}\widetilde{\boldsymbol{\omega}}^T\mathbf{I}\widetilde{\boldsymbol{\omega}} \geq 0$. Therefore

$$\begin{aligned} \dot{V} &= \widetilde{\boldsymbol{\omega}}^T\mathbf{I}\dot{\widetilde{\boldsymbol{\omega}}} \\ &= \widetilde{\boldsymbol{\omega}}^T[\tilde{\mathbf{u}} - \widetilde{\boldsymbol{\omega}}^{\times}\mathbf{I}\boldsymbol{\omega}] \\ &= \widetilde{\boldsymbol{\omega}}^T\tilde{\mathbf{u}} \end{aligned}$$

Integrating both sides from $t = 0$ to $t = T$ gives

$$\int_0^T \widetilde{\boldsymbol{\omega}}^T\tilde{\mathbf{u}}\,dt = V(T) - V(0) = V(T) \geq 0$$

which yields the desired result. □

We say that the chosen feedforward preserves passivity in the error dynamics. Based on the passivity theorem, we can render $\widetilde{\boldsymbol{\omega}} \in L_2$ by choosing a strictly passive feedback:

$$\tilde{\mathbf{u}} = -\mathbf{K}\widetilde{\boldsymbol{\omega}},\; \mathbf{K} = \mathbf{K}^T > \mathbf{O} \tag{24}$$

Hence

$$\begin{aligned}\mathbf{u}-\mathbf{u}_d &= -\mathbf{K}\mathbf{S}(\boldsymbol{\theta})\boldsymbol{\rho}\\ \Rightarrow \mathbf{u} &= \mathbf{u}_d-\mathbf{K}\mathbf{S}(\boldsymbol{\theta})\boldsymbol{\rho}\\ \Rightarrow \mathbf{u} &= \mathbf{I}\dot{\boldsymbol{\omega}}_r+\boldsymbol{\omega}_r^{\times}\mathbf{I}\boldsymbol{\omega}-\mathbf{K}\mathbf{S}(\boldsymbol{\theta})[\dot{\widetilde{\boldsymbol{\theta}}}+\lambda\widetilde{\boldsymbol{\theta}}]\end{aligned}$$

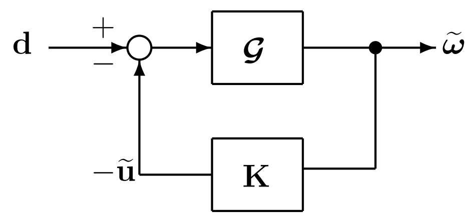
If the dynamics (19) are changed to

$$\mathbf{I}\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega}=\mathbf{u}+\mathbf{d}$$

where $\mathbf{d}$ is a disturbance torque, then the error dynamics become

$$\mathbf{I}\dot{\widetilde{\boldsymbol{\omega}}}+\widetilde{\boldsymbol{\omega}}^{\times}\mathbf{I}\boldsymbol{\omega}=\tilde{\mathbf{u}}+\mathbf{d}$$

and we can draw the following block diagram:



Applying the passivity theorem, $\mathbf{d}\in L_2 \;\Rightarrow\; \widetilde{\boldsymbol{\omega}}\in L_2$. Therefore, $\boldsymbol{\rho}\in L_2$ which implies that $\widetilde{\boldsymbol{\theta}}(t)\to\mathbf{0}$ as $t\to\infty$.

A Lyapunov Approach ($\mathbf{d}=\mathbf{0}$, $\widetilde{\boldsymbol{\omega}}(0)\neq\mathbf{0}$)

Adopting $V(t)=\frac{1}{2}\widetilde{\boldsymbol{\omega}}^T\mathbf{I}\widetilde{\boldsymbol{\omega}}$ as a Lyapunov function we have

$$\dot{V}=\widetilde{\boldsymbol{\omega}}^T\tilde{\mathbf{u}}=-\widetilde{\boldsymbol{\omega}}^T\mathbf{K}\widetilde{\boldsymbol{\omega}} \tag{25}$$

Hence $\widetilde{\boldsymbol{\omega}}=\mathbf{0}$ is a globally asymptotically stable equilibrium point. Therefore, $\widetilde{\boldsymbol{\omega}}\to\mathbf{0}$. Also integrating (25) with respect to time gives

$$V(t)-V(0)=-\int_0^t\widetilde{\boldsymbol{\omega}}^T\mathbf{K}\widetilde{\boldsymbol{\omega}}\,dt\leq 0$$

If $\epsilon=\min_i\lambda_i(\mathbf{K})$, we can write

$$\epsilon\int_0^t\widetilde{\boldsymbol{\omega}}^T\widetilde{\boldsymbol{\omega}}\,dt\leq\int_0^t\widetilde{\boldsymbol{\omega}}^T\mathbf{K}\widetilde{\boldsymbol{\omega}}\,dt=V(0)-V(t)\leq V(0),\ \forall t\geq 0 \tag{26}$$

where we have noted that $V\geq 0$ and $\dot{V}\leq 0$. Letting $t\to\infty$, we have $\widetilde{\boldsymbol{\omega}}\in L_2$ and hence, as above, $\widetilde{\boldsymbol{\theta}}\to\mathbf{0}$.

3.5 Adaptive Attitude Control

The above control scheme requires accurate knowledge of $\mathbf{I}$ in the feedforward. What if $\mathbf{I}$ is poorly known? Is it possible to identify $\mathbf{I}$ "on the fly" and use the estimated $\mathbf{I}$?

Note that the feedforward in Eq. (21) is linear in $\mathbf{I}$. Define

$$\boldsymbol{\alpha} = [I_{11}\ I_{22}\ I_{33}\ I_{12}\ I_{13}\ I_{23}]^T$$

Then (21) can be written in the form

$$\mathbf{u}_d = \mathbf{W}(\dot{\boldsymbol{\omega}}_r, \boldsymbol{\omega}_r, \boldsymbol{\omega})\boldsymbol{\alpha} \tag{27}$$

where $\mathbf{W}$ is called the regressor matrix.

<u>**Example**</u>. Let us assume that the frame used to calculate $\mathbf{I}$ is a principal axis frame (in general, a bad assumption if $\mathbf{I}$ is unknown!) and I_{11}, I_{22}, I_{33} are unknown. Hence,

$$\begin{aligned}
\mathbf{u}_d &= \mathbf{I}\dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}_r^{\times}\mathbf{I}\boldsymbol{\omega} \\
&= \begin{bmatrix} I_{11}\dot{\omega}_{r1} \\ I_{22}\dot{\omega}_{r2} \\ I_{33}\dot{\omega}_{r3} \end{bmatrix} + \begin{bmatrix} 0 & -\omega_{r3} & \omega_{r2} \\ \omega_{r3} & 0 & -\omega_{r1} \\ -\omega_{r2} & \omega_{r1} & 0 \end{bmatrix} \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix} \\
&= \begin{bmatrix} \dot{\omega}_{r1} & -\omega_2\omega_{r3} & \omega_{r2}\omega_3 \\ \omega_{r3}\omega_1 & \dot{\omega}_{r2} & -\omega_{r1}\omega_3 \\ -\omega_{r2}\omega_1 & \omega_{r1}\omega_2 & \dot{\omega}_{r3} \end{bmatrix} \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix}
\end{aligned}$$

which is in the form of (27).

Let $\widehat{\boldsymbol{\alpha}}(t)$ be an estimate for $\boldsymbol{\alpha}$ (constant) and define the parameter error by

$$\widetilde{\boldsymbol{\alpha}}(t) = \widehat{\boldsymbol{\alpha}}(t) - \boldsymbol{\alpha} \tag{28}$$

Since $\boldsymbol{\alpha}$ is unknown, let us replace it with its estimate in the feedforward. Therefore, we take as a control signal for the torques

$$\mathbf{u}(t) = \mathbf{W}(\dot{\boldsymbol{\omega}}_r, \boldsymbol{\omega}_r, \boldsymbol{\omega})\widehat{\boldsymbol{\alpha}}(t) + \bar{\mathbf{u}}(t) \tag{29}$$

where $\bar{\mathbf{u}}$ is the feedback part to be determined. Using the definition of $\tilde{\mathbf{u}}$ and that of $\mathbf{u}_d$ in (27), we have

$$\begin{aligned}
\tilde{\mathbf{u}} &= \mathbf{u} - \mathbf{u}_d \\
&= \mathbf{W}\widehat{\boldsymbol{\alpha}} + \bar{\mathbf{u}} - \mathbf{W}\boldsymbol{\alpha} \\
&= \mathbf{W}\widetilde{\boldsymbol{\alpha}} + \bar{\mathbf{u}}
\end{aligned} \tag{30}$$

Recall that $\widetilde{\boldsymbol{\omega}} = \boldsymbol{\mathcal{G}}\tilde{\mathbf{u}}$ is passive. Based on the passivity theorem, we want to use a feedback of the form $\tilde{\mathbf{u}} = -\boldsymbol{\mathcal{H}}\widetilde{\boldsymbol{\omega}}$ with $\boldsymbol{\mathcal{H}}$ strictly passive, i.e.,

$$\begin{aligned} -\int_0^T \tilde{\mathbf{u}}^T\widetilde{\boldsymbol{\omega}}\,dt = \int_0^T \widetilde{\boldsymbol{\omega}}^T\boldsymbol{\mathcal{H}}\widetilde{\boldsymbol{\omega}}\,dt \;&\geq\; \epsilon\int_0^T \widetilde{\boldsymbol{\omega}}^T\widetilde{\boldsymbol{\omega}}\,dt, \quad \epsilon > 0 \\ \Rightarrow -\left[\int_0^T \bar{\mathbf{u}}^T\widetilde{\boldsymbol{\omega}}\,dt + \int_0^T \widetilde{\boldsymbol{\alpha}}^T\mathbf{W}^T\widetilde{\boldsymbol{\omega}}\,dt\right] \;&\geq\; \epsilon\int_0^T \widetilde{\boldsymbol{\omega}}^T\widetilde{\boldsymbol{\omega}}\,dt \end{aligned}$$

This latter inequality can be satisfied if the map from $-\widetilde{\boldsymbol{\omega}}$ to $\bar{\mathbf{u}}$ is strictly passive and the map from $-\mathbf{W}^T\widetilde{\boldsymbol{\omega}}$ to $\widetilde{\boldsymbol{\alpha}}$ is passive. Possible choices are

$$\bar{\mathbf{u}} = -\mathbf{K}\widetilde{\boldsymbol{\omega}}, \qquad \mathbf{K} = \mathbf{K}^T > \mathbf{O} \tag{31}$$

$$\dot{\widetilde{\boldsymbol{\alpha}}} = -\boldsymbol{\Gamma}\mathbf{W}^T\widetilde{\boldsymbol{\omega}}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{O} \tag{32}$$

Since

$$\dot{\widetilde{\boldsymbol{\alpha}}} = \dot{\widehat{\boldsymbol{\alpha}}} - \dot{\boldsymbol{\alpha}} = \dot{\widehat{\boldsymbol{\alpha}}}$$

the adaptation law can be rewritten as

$$\widehat{\boldsymbol{\alpha}}(t) = \widehat{\boldsymbol{\alpha}}(0) - \boldsymbol{\Gamma}\int_0^t \mathbf{W}^T\widetilde{\boldsymbol{\omega}}\,dt \tag{33}$$

where we have integrated (32) with respect to time. Hence, we have used an integrator for the passive map from $-\mathbf{W}^T\widetilde{\boldsymbol{\omega}}$ to $\widetilde{\boldsymbol{\alpha}}$. This is the simplest passive operation which avoids knowledge of the $\boldsymbol{\alpha}$ (the true parameters) in the adaptation law.

Based on the passivity theorem, $\widetilde{\boldsymbol{\omega}} \in L_2$ and hence, as before, $\widetilde{\boldsymbol{\theta}} \to \mathbf{0}$. A Lyapunov-style proof of this can be produced by taking

$$V(t) = \tfrac{1}{2}\widetilde{\boldsymbol{\omega}}^T\mathbf{I}\widetilde{\boldsymbol{\omega}} + \tfrac{1}{2}\widetilde{\boldsymbol{\alpha}}^T\boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\alpha}}$$

Hence, using (30), (31), (32), we have

$$\begin{aligned} \dot{V} &= \widetilde{\boldsymbol{\omega}}^T\mathbf{I}\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\alpha}}^T\boldsymbol{\Gamma}^{-1}\dot{\widetilde{\boldsymbol{\alpha}}} \\ &= \widetilde{\boldsymbol{\omega}}^T\tilde{\mathbf{u}} + \widetilde{\boldsymbol{\alpha}}^T\boldsymbol{\Gamma}^{-1}\dot{\widetilde{\boldsymbol{\alpha}}} \\ &= \widetilde{\boldsymbol{\omega}}^T[\mathbf{W}\widetilde{\boldsymbol{\alpha}} - \mathbf{K}\widetilde{\boldsymbol{\omega}}] - \widetilde{\boldsymbol{\alpha}}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}\mathbf{W}^T\widetilde{\boldsymbol{\omega}} \\ &= -\widetilde{\boldsymbol{\omega}}^T\mathbf{K}\widetilde{\boldsymbol{\omega}} \end{aligned}$$

We cannot apply LaSalle's theorem since the system is nonautonomous. In general, we cannot conclude that $\widetilde{\boldsymbol{\alpha}} \to \mathbf{0}$. However,

$$\int_0^t \dot{V}(\tau)\,d\tau = -\int_0^t \widetilde{\boldsymbol{\omega}}^T\mathbf{K}\widetilde{\boldsymbol{\omega}}\,dt$$

Therefore, as in the nonadaptive case, if $\epsilon = \min_i \lambda_i(\mathbf{K})$, we can write

$$\epsilon \int_0^t \widetilde{\boldsymbol{\omega}}^T \widetilde{\boldsymbol{\omega}}\, dt \leq \int_0^t \widetilde{\boldsymbol{\omega}}^T \mathbf{K} \widetilde{\boldsymbol{\omega}}\, dt = V(0) - V(t) \leq V(0), \;\forall t \geq 0 \tag{34}$$

Therefore, letting $t \to \infty$, $\widetilde{\boldsymbol{\omega}} \in L_2$ and $\breve{\boldsymbol{\theta}} \to \mathbf{0}$.