An Optimization-based Framework for Controlling Discretization Error through Anisotropic $h$-Adaptation

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Outline

1. Introduction
2. Optimization Framework: Anisotropic $h$-Adaptation
3. Results: $L^2$ Error Control
4. Results: Functional Output Error Control
5. Conclusion
Objective: *Realize the full potential of high order methods through automated mesh adaptation and error control*

Our choice of technologies:
- Discontinuous Galerkin (DG) method
  - $hp$-flexible on unstructured meshes
  - Convergence limited by regularity
    - shocks, geometric singularities, turbulence models
- Simplex mesh adaptation
  - Complex geometries
  - Arbitrarily oriented anisotropy
- Adjoint-based error estimator
  - Error estimate / localization for output of interest (e.g. lift, drag)
Problem: Find $J(u)$ with $u \in V$ s.t.

$$R(u, v) = 0 \quad \forall v \in V$$

Dual-weighted residual (DWR) error representation [Becker,2001]

$$\mathcal{E} \equiv J_{h,p}(u_{h,p}) - J(u) = R_{h,p}(u_{h,p}, \psi), \quad \psi = \text{adjoint}$$

Local error indicator

$$\eta_K \equiv R_{h,p}(u_{h,p}, \psi_{h,p+1}|_K)$$

Error indicator assigns 1 value to each element

- Only sufficient for 1 parameter adaptation

<table>
<thead>
<tr>
<th>type</th>
<th>$n_{\text{param}}$</th>
<th>parameters</th>
<th>$\eta_K$ sufficient</th>
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<tbody>
<tr>
<td>isotropic $h$</td>
<td>1</td>
<td>size</td>
<td>✓</td>
</tr>
<tr>
<td>isotropic $hp$</td>
<td>2</td>
<td>size, poly. order</td>
<td>✗</td>
</tr>
<tr>
<td>anisotropic $h$</td>
<td>3 (6)</td>
<td>size, shape</td>
<td>✗</td>
</tr>
<tr>
<td>anisotropic $hp$</td>
<td>4 (7)</td>
<td>size, shape, poly. order</td>
<td>✗</td>
</tr>
</tbody>
</table>
Need additional info to make anisotropy decision

Previous work on output-based anisotropic $h$-adaptation

<table>
<thead>
<tr>
<th>Method</th>
<th>$p &gt; 1$</th>
<th>$\psi$ ani. simplex</th>
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</thead>
<tbody>
<tr>
<td>Mach Hessian [Venditti, 2003]</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Mach “Hessian” [Fidkowski, 2007]</td>
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<td>x</td>
</tr>
<tr>
<td>Mach jump [Hartmann, 2008]</td>
<td>✓</td>
<td>x</td>
</tr>
<tr>
<td>Edge comp. ref [Park, 2008][Sun, 2009]</td>
<td>✓*</td>
<td>✓*</td>
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<tr>
<td>Comp. ref. [Georgoulis, 2009][Ceze, 2010]</td>
<td>✓</td>
<td>x</td>
</tr>
<tr>
<td>Flux reconstruction + C.V. [Loseille, 2010]</td>
<td>x</td>
<td>✓</td>
</tr>
<tr>
<td>Anisotropic error estimate [Richter, 2010]</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

**Contribution:** *Anisotropic $h$-adaptation framework for simplex meshes that implicitly synthesizes information about*

- primal and adjoint solution behaviors
- interpolation order
- solution regularity
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Problem Definition / Example

- **Objective:** Develop “optimal” anisotropic mesh
- Example problem: $L^2$-projection error control of boundary layer

$$
\mathcal{E}^D(T_h) = \| u - u_{h,p} \|^2_{L^2(\Omega)} \quad \text{where} \quad u = \exp\left(-\frac{x_1}{\delta}\right)
$$

$$
\mathcal{C}^D(T_h) = \text{DOF}(T_h)
$$

\[ u = \exp\left(-\frac{x_1}{\delta}\right) \quad (\delta = 0.01) \]

optimal isotropic mesh
\[ \mathcal{E}^\frac{1}{2} = 1.6 \times 10^{-4} \]

optimal anisotropic mesh
\[ \mathcal{E}^\frac{1}{2} = 5.1 \times 10^{-8} \]

*each mesh has $\approx 200$ elements ($p = 3$)
Continuous Relaxation

- (Intractable) discrete optimization problem

\[ T_{\text{h, opt}} = \arg \min_{T_{\text{h}}} E^D(T_{\text{h}}) \quad \text{s.t.} \quad C^D(T_{\text{h}}) = \text{DOF}_{\text{target}} \]

- Approximability of \( T_{\text{h}} \) encoded in *metric tensor field*

\[ \mathcal{M}(x) \in \text{Sym}_d^+ \quad (d \times d \text{ SPD matrix}) \]

- Continuous relaxation [cf. Loseille, 2009]

\[ \mathcal{M}_{\text{opt}} = \arg \min_{\mathcal{M}} E(\mathcal{M}) \quad \text{s.t.} \quad C(\mathcal{M}) = \text{DOF}_{\text{target}} \]
Problem Statement: Seek optimal metric field

\[ \mathcal{M}_{\text{opt}} = \arg \min_{\mathcal{M}} \mathcal{E}(\mathcal{M}) \quad \text{s.t.} \quad \mathcal{C}(\mathcal{M}) = \text{DOF}_{\text{target}} \]

Assume locality of error and cost functionals

\[ \mathcal{E}(\mathcal{M}) = \int_{\Omega} e(x, \mathcal{M}(x)) \, dx \quad \text{and} \quad \mathcal{C}(\mathcal{M}) = \int_{\Omega} c(\mathcal{M}(x)) \, dx \]

- Local error function: \( e(\cdot, \cdot) : \mathbb{R}^d \times \text{Sym}_d^+ \rightarrow \mathbb{R}^+ \)
- Local cost function: \( c(\mathcal{M}) = C_p \sqrt{\det(\mathcal{M})} \)

First order optimality

\[ \frac{\partial e}{\partial \mathcal{M}} (x, \mathcal{M}(x)) - \lambda \frac{\partial c}{\partial \mathcal{M}} (x, \mathcal{M}(x)) = 0 \quad \forall x \in \Omega \]

Need to estimate behavior of \( e(\mathcal{M}) \)
Local Sampling

- Split edges of $K_0$ to form $K_i$’s, $i = 0, \ldots, m$
- Obtain $\eta_{K_i}$ on split config.
  - Freeze neighbor states
  - Solve local problem
- Obtain metric $\mathcal{M}_{K_i}$ implied by split config.
- Associate $\eta_{K_i} \approx e(\mathcal{M}_{K_i})|K_i|$

Original edge: 1 split edge 2 split edge 3 split uniform split

- $\mathcal{M}_0$, $\eta_0$
- $\mathcal{M}_1$, $\eta_1$
- $\mathcal{M}_2$, $\eta_2$
- $\mathcal{M}_3$, $\eta_3$
- $\mathcal{M}_u$, $\eta_u$
Manipulation of SPD Matrix Space

- Original problem on $\text{Sym}_d^+$
  \[
  \frac{\partial e}{\partial M}(x, M) - \lambda \frac{\partial c}{\partial M}(x, M) = 0 \quad M \in \text{Sym}_d^+
  \]

- SPD matrices define cone in Euclidean space
- Hard to manipulate SPD matrices directly

Consider substitutions

- $M(\Sigma) = M_0^{1/2} \exp(\Sigma) M_0^{1/2} \Rightarrow M(\Sigma) \in \text{Sym}_d^+ \quad \forall \Sigma \in \text{Sym}_d$
- $e(f) = e_0 \exp(f) \Rightarrow e(f) \in \mathbb{R}^+ \quad \forall f \in \mathbb{R}$
- $c(g) = c_0 \exp(g) \Rightarrow c(g) \in \mathbb{R}^+ \quad \forall g \in \mathbb{R}$

New problem on $\text{Sym}_d$

\[
e_0 \frac{\partial f}{\partial \Sigma}(x, \Sigma) - \lambda c_0 \frac{\partial g}{\partial \Sigma}(x, \Sigma) = 0 \quad \Sigma \in \text{Sym}_d
\]

- Manipulation of symmetric matrices
- Approximate $f(\Sigma)$ and move toward optimality
Local Error Model / Parameter Identification

- Construct linear approximation of $f(\Sigma)$

$$f(\Sigma) = \text{tr}(R\Sigma), \quad R \in Sym_d = \text{rate tensor}$$

- $R$ is generalization of convergence rate to tensor space
  - $\text{tr}(R)$: sensitivity to uniform scaling
  - $R - \frac{\text{tr}(R)}{d} I$: sensitivity to shape change

- Identify $R$ from local samples $\{M_{K_i}, \eta_{K_i}\}_{i=1}^m$

$$
\begin{bmatrix}
- & \Sigma_{K_1} & - \\
\vdots & \vdots & \vdots \\
- & \Sigma_{K_m} & -
\end{bmatrix}
\begin{bmatrix}
R_{11} \\
2R_{12} \\
R_{22}
\end{bmatrix}
= 
\begin{bmatrix}
f(\Sigma_{K_1}) \\
\vdots \\
f(\Sigma_{K_m})
\end{bmatrix}
$$

- $\Sigma(K_i) = \log(M_0^{-1/2} M_{K_i} M_0^{-1/2})$ and $f(\Sigma_{K_i}) = \log(\eta_{K_i}/\eta_{K_0})$
Global isotropic error balance: \( \exists \lambda \in \mathbb{R} \text{ s.t.} \)
\[
\eta_K \text{tr}(R_K) - \lambda \frac{d}{2} \text{dof}(K) = 0 \quad \forall K \in T_h
\]

Local anisotropic optimality:
\( R_K \propto I \quad \forall K \in T_h \)

E.g. \( R \neq I \): more reduction by increasing \( M_{11} \) than \( M_{22} \)
\( \Rightarrow \) decrease \( h_1 \) and increase \( h_2 \)
Algorithm

1. Obtain primal solution $u_{h,p}$ on $\mathcal{T}_{h,p}^{(i)}$
2. Obtain adjoint solution $\psi_{h,p+1}$ on $\mathcal{T}_{h,p+1}^{(i)}$ (for DWR)
3. Solve local problems and obtain $\{M_{K_i}, \eta_{K_i}\}_{i=0}^m$, $\forall K \in \mathcal{T}_{h}^{(i)}$
4. Construct local error function $e(M) \approx \tilde{e}(M; \{M_{K_i}, \eta_{K_i}\}_{i=0}^m)$
5. Update metric field $M^{(i+1)}$
6. Generate new mesh $\mathcal{T}^{(i+1)}$ from $M^{(i+1)}$ and go to 1

Adapt 0 : $E_2 = 9.4 \times 10^{-3}$  
Adapt 2 : $E_2 = 1.8 \times 10^{-5}$  
Adapt 7 : $E_2 = 7.9 \times 10^{-8}$

*each mesh has $\approx 200$ elements ($p = 3$)
Assessment of Boundary Layer $L^2$ Error Control

- Function: $u = \exp(-x_1/\delta)$ ($\delta = 0.01$)
- Analytical optimal $h_\perp$ vs. $x_1$

$$h_{\perp,\text{opt}} = C\exp\left(k_{\perp,\text{opt}}x\right), \quad k_{\perp,\text{opt}} = \frac{1}{\delta(p + 3/2)}$$

- Weaker grading for higher $p$

\[\begin{array}{c}
p = 1 \quad (k_{\perp,\text{opt}} = 40.0) \\
\text{dof}=500 \quad \text{dof}=1000 \quad \text{dof}=2000 \quad \text{dof}=4000
\end{array}\]

\[\begin{array}{c}
p = 3 \quad (k_{\perp,\text{opt}} = 22.2) \\
\text{dof}=1000 \quad \text{dof}=2000 \quad \text{dof}=4000 \quad \text{dof}=8000
\end{array}\]
Assessment of Singular Corner $L^2$ Error Control

- Function: $u = r^\alpha \sin \left[\alpha \left(\theta + \frac{\pi}{2}\right)\right]$ ($\alpha = 2/3$)
- Analytical optimal $h$ vs. $r$
  \[ h_{opt}(r) = C r^{k_{opt}}, \quad k_{opt} = 1 - \frac{\alpha + 1}{p + 2} \]
- Stronger grading for higher $p$

\[ p = 1 \ (k_{opt} = 0.44) \]
\[ p = 3 \ (k_{opt} = 0.67) \]
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Objective: Application to transonic turbulent flow

Functional: Drag

Comparison: Mach-based anisotropy ($\nabla^p M$)
**Anisotropic resolution of features**

**Primal:** BLs, wake, shock

**Adjoint:** BLs, stagnation streamline, shock-induced perturbations

Unified ($p = 2$, dof = 60k)  
Mach anisotropy ($p = 2$, dof = 60k)
Objective: Application to sonic-boom prediction

Functional: Pressure $50c$ below airfoil, $J = \int_l (p - p_\infty)^2 ds$
NACA0006 Shock Propagation

Euler: $M_{\infty} = 2.0; p = 2$

Unified ($p = 2$, dof = 40k)

Mach anisotropy ($p = 2$, dof = 40k)
Three-Element MDA Airfoil

RANS-SA: $M_\infty = 0.2$, $Re_c = 9 \times 10^6$, $\alpha = 8.10^\circ$; $p = 2$

**Objective:** Application to complex turbulent flow

**Functional:** Drag

![Mach](image1)

![Mass adjoint](image2)

![Drag error](image3)

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19 / 22
Three-Element MDA Airfoil

\[ M_\infty = 0.2, \quad Re_c = 9 \times 10^6, \quad \alpha = 8.10^{\circ}; \quad p = 2 \]

Unified \((p = 2, \text{dof} = 90k)\)

Mach anisotropy \((p = 2, \text{dof} = 90k)\)
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Conclusion:

- Presented optimization framework for anisotropic $h$-adaptation that *implicitly synthesizes* information about
  - primal and adjoint solution behaviors
  - interpolation order
  - solution regularity
- Verified optimality of algorithm for $L^2$ error control
- Compared performance of algorithm against primal derivative based anisotropy detection

Ongoing Work:

- Extension to $hp$-adaptation
- Improving robustness of error estimator for highly nonlinear problems