A REDUCED BASIS METHOD FOR COERCIVE EQUATIONS WITH AN EXACT SOLUTION CERTIFICATE AND SPATIO-PARAMETER ADAPTIVITY: ENERGY-NORM AND OUTPUT ERROR BOUNDS

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Abstract. We develop a reduced basis method for linear coercive parametrized partial differential equations (PDEs) with two objectives: providing an \textit{a posteriori} error bound with respect to the exact weak solution of the PDE as opposed to the typical finite element “truth” solution; providing reliable and efficient construction of a reduced basis model through automatic adaptivity in both physical and parameter spaces. Our error bounds build on two key ingredients. The first is a minimum-residual mixed formulation which provides an approximate solution as well as an upper bound of the dual norm of the residual with respect to the infinite-dimensional function space. The second is an extension of the successive constraint method (SCM) to evaluate a lower bound of the stability constant with respect to the infinite-dimensional function space; the approach builds on a computable lower bound of the minimum eigenvalue associated with the stability constant. Both the minimum-residual mixed formulation and the extended SCM admit offline-online computational decomposition. The offline stage incorporates spatial mesh adaptation and greedy parameter sampling for both the solution approximation and the stability eigenproblem to yield a reliable online system in an efficient manner. The online stage provides an approximate solution and an \textit{a posteriori} error bound with respect to the exact solution for any parameter value in complexity independent of the size of the finite element spaces. We demonstrate the effectiveness of the approach for a thermal block problem, which exhibits parameter-dependent spatial singularities.

Key words. reduced basis method, \textit{a posteriori} error bounds, minimum-residual mixed method, coercivity constant, offline-online decomposition, spatio-parameter adaptivity

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1. Introduction. The goal of the certified reduced basis method is to provide rapid and reliable solution of parametrized partial differential equations (PDEs) in real-time and many-query applications [15]. The rapidness is provided by an offline-online computational decomposition; the reliability is provided by \textit{a posteriori} error bounds. The classical certified reduced basis method provides \textit{a posteriori} error bounds with respect to the finite element “truth” solution computed in a finite-dimensional “truth” space, which is assumed to be sufficiently rich such that the difference between the “truth” solution and the exact weak solution of the PDE is negligible. In practice, this assumption is not rigorously verified and may be violated for problems that exhibit complex parameter-dependent spatial features. An under-refined mesh results in a reduced basis error bound that is not reliable with respect to the exact solution. Conversely, an over-refined mesh results in unnecessarily expensive finite element snapshot solves in the offline stage. A typical “truth” mesh suffers from both of these problems as the mesh is under-refined in some regions and over-refined in other regions. The goal of this work is to eliminate the issue of “truth” within the classical reduced basis framework and to develop a reduced basis method that provides error bounds with respect to the exact weak solution in the energy norm as well as for linear functional outputs.

The error bounds we present build on two key ingredients: an upper bound of the dual norm of the residual and a lower bound of the stability constant. Our method provides, in the online stage, uniform (as opposed to asymptotic) bounds for the dual norm of the residual and the stability constant, both of which are with respect to the infinite-dimensional function space as opposed to the typical finite element “truth” space. We adhere to the standard reduced basis goals as regard the offline-online computational efficiency and online certification: i) the online solution is computed in complexity independent of the underlying finite element discretization; ii) the online error bound is provided for any parameter value and not just those in the training set. Providing an error bound with respect to the exact solution is a recent idea in the reduced basis community; we here note related work by Ali \textit{et al.} [2], Ohlberger and Schindler [10], and ourselves [17, 18].

Our approximation of the solution field is based on a version of the minimum-residual mixed reduced basis method [18], which belongs to a family of least-squares methods (see, e.g., [3, 11]). The mixed formulation simultaneously computes both the (primal) solution and a dual solution, the latter of which provides a bound for the dual norm of the (primal) residual. The use of the dual solution to provide an error bound

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is similar to the complementary variational principle, which has been extensively applied to finite element methods by, for example, Ladevèze and Leguillion [9], Ainsworth and Oden [1], and Sauer-Budge et al. [16] and more recently to a reduced basis method [17]. However, our approach [18] differs from the classical complementary variational principle in that it does not require a space that exactly satisfies dual-feasibility conditions, construction of which in an offline-online efficient manner can be done in some limited cases [17] but is formidable in general. This “relaxation” is accomplished by considering a dual norm with respect to a norm that is different from the energy norm. However, as a consequence of not using the energy norm, the stability constant is not unity and we must compute a lower bound of the stability constant to bound the error; in this work we extend our minimum-residual mixed formulation [18] to incorporate a carefully chosen, parameter-dependent norm which facilitates the computation of the stability constant.

Providing a lower bound of the stability constant requires the evaluation of a lower bound of the minimum eigenvalue associated with an eigenproblem. It is well-known that an upper bound of the minimum eigenvalue can be readily computed by any Galerkin method, but a lower bound is much more complicated to compute; for brevity, we will not attempt to provide a comprehensive overview of eigenvalue bounds and instead refer to a review paper by Plum [12]. Our formulation is based on the method by Weinstein (see [4], Chapter 6), which provides a lower bound using the eigenproblem residual or “defect” under the assumption that the minimum eigenvalue associated with the finite element approximation is closer to the minimum eigenvalue of the exact problem than to the second smallest eigenvalue. In the offline stage, we provide a bound for the eigenproblem residual using a mixed formulation similar to that for the solution residual. We then appeal to the successive constraint method (SCM) of Huynh et al. [7] — revised to incorporate lower bounds of stability constants with respect to the infinite-dimensional function space — to provide an offline-online computational decomposition of the stability constant. Our formulation also incorporates a transformation of the original stability constant such that the effectivity of the stability constant is desensitized from the relatively poor effectivity of the SCM lower bound.

The contributions of this work are fivefold. First, we develop a minimum-residual mixed formulation whose dual variable provides a built-in bound of a dual norm of the residual with respect to a parametrized norm. Second, we extend the SCM to evaluate a lower bound of the stability constant with respect to an infinite-dimensional function space; here, we appeal to Weinstein’s method to express a lower bound of the minimum eigenvalue in terms of the eigenproblem residual and then develop a mixed formulation to bound the dual norm of the residual. Third, we develop a transformation of the stability constant such that the bound gap for the stability constant is desensitized from the SCM bound gap. Fourth, we develop spatio-parameter adaptation strategies based on isotropic-$h$ mesh adaptation and greedy parameter sampling for both the stability eigenproblem and the solution approximation. Fifth, we demonstrate the effectiveness of the spatio-parameter adaptive strategy for the classical thermal block problem [15], which is defined in a high-dimensional parameter space and exhibits parameter-dependent spatial singularities.

We note one crucial assumption, and hence a limitation, associated with error bounds developed in this work. As noted, one of the ingredients of our error bounds is a lower bound of the stability constant, which is equivalent to a lower bound of the minimum eigenvalue associated with a parametrized eigenproblem. We appeal to Weinstein’s method to construct a lower bound. As a consequence, in the offline approximation of the minimum eigenvalue, we assume that the minimum eigenvalue associated with the Galerkin finite element approximation of the eigenproblem is closer to the minimum eigenvalue than the second smallest eigenvalue of the exact eigenproblem. We unfortunately do not have any means to verify if this assumption is satisfied. However, in practice for many elliptic equations, the eigenfunction associated with the minimum eigenvalue is the easiest to approximate, and hence this assumption is often satisfied even on a very coarse mesh.

This paper is organized as follows. Section 2 introduces the problem considered throughout this paper, defines function spaces, and presents a well-known proposition that identifies key ingredients of our error bounds. Section 3 introduces a minimum-residual mixed formulation and devise an associated offline-online computational strategy. Section 4 presents an extension of the SCM to infinite-dimensional function spaces, including the computation of a lower bound of the stability constant with respect to the exact space. Section 5 presents our error bounds. Section 6 presents two spatio-parameter adaptive strategies, one associated with the stability constant and the other associated with the solution field. Section 7 assesses the effectiveness of the proposed strategy using the classical thermal block problem. Finally, we summarize the work in
Section 8.

2. Preliminaries.

2.1. Problem statement. We first introduce a $d$-dimensional bounded spatial domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $\partial \Omega$; the boundary is decomposed into a Dirichlet boundary $\Gamma_D$, which is assumed non-empty, and a Neumann boundary $\Gamma_N$ such that $\overline{\Omega} = \Gamma_D \cup \Gamma_N$. We also introduce a $P$-dimensional bounded parameter domain $\mathcal{D} \subset \mathbb{R}^P$. We then introduce a Sobolev space

$$
V \equiv \{ v \in H^1(\Omega) \mid v|_{\partial \Omega} = 0 \},
$$

where $H^1(\Omega)$ is the standard $H^1$ Sobolev space (see, e.g., [13]). We now consider the following weak statement: given $\mu \in \mathcal{D}$, find $u(\mu) \in V$ such that

$$
(1) \quad a(u(\mu), v; \mu) = \ell(v; \mu) \quad \forall v \in V,
$$

where

$$
a(w, v; \mu) = \int_{\Omega} \nabla v \cdot K(\mu) \nabla w \, dx \quad \forall w, v \in V,
$$

$$
\ell(v; \mu) = \int_{\Omega} vf(\mu) \, dx + \int_{\Gamma_N} vg(\mu) \, ds \quad \forall v \in V.
$$

Here, $K : \mathcal{D} \to [L^\infty(\Omega)]^{d \times d}$ is a diffusivity field, $f : \mathcal{D} \to L^2(\Omega)$ is a source function, and $g : \mathcal{D} \to L^2(\Gamma_N)$ is a Neumann boundary data; note that each function is parametrized.

We make a few assumptions about the forms that define the problem. First, we assume that $K(\mu)$ is uniformly positive in the sense that minimum eigenvalue of $K(\mu)(x)$ is bounded from the below: $\lambda_{\min}(K(\mu)(x)) \geq K_{\min} > 0$, $\forall x \in \Omega, \forall \mu \in \mathcal{D}$. Second, we assume that $K(\mu), f(\mu)$, and $g(\mu)$ admit decompositions that are affine in functions of the parameter:

$$
K(\mu) = \sum_{q=1}^{Q_K} \Theta_{q}^K(\mu)K_q, \quad f(\mu) = \sum_{q=1}^{Q_f} \Theta_{q}^f(\mu)f_q, \quad g(\mu) = \sum_{q=1}^{Q_g} \Theta_{q}^g(\mu)g_q,
$$

where $K_q \in (L^\infty(\Omega))^{d \times d}$, $f_q \in L^2(\Omega)$, and $g_q \in L^2(\Gamma_N)$ are parameter-independent fields, and $\Theta_{q}^K \in L^\infty(\mathcal{D})$, $\Theta_{q}^f \in L^\infty(\mathcal{D})$, and $\Theta_{q}^g \in L^\infty(\mathcal{D})$ are parameter-dependent functions. Third, we assume that $K^{-1}(\mu)$ admits a decomposition that is affine in functions of parameter:

$$
K^{-1}(\mu) = \sum_{q=1}^{Q_{K^{-1}}} \Theta_{q}^{K^{-1}}(\mu)K_q^{\text{inv}},
$$

where $K_q^{\text{inv}} \in (L^\infty(\Omega))^{d \times d}$ is a parameter-independent field, and $\Theta_{q}^{K^{-1}} \in L^\infty(\mathcal{D})$ is a parameter-dependent function; note that $K_q^{\text{inv}}$ is in general not related to $K_q$. Finally, we assume that fields $K_q$, $f_q$, $g_q$, and $K_q^{\text{inv}}$ are piecewise polynomials such that we can evaluate integrals involving the fields exactly using quadrature rules.

In this work, we develop a reduced basis method that approximates the solution to (1) and provides an associated error bound. Specifically, we provide a sequence of approximations $u_N(\mu)$, $N = 1, 2, \ldots$, to $u(\mu)$ and associated error bounds $\Delta_N(\mu)$, $N = 1, 2, \ldots$, such that

$$
\| u(\mu) - u_N(\mu) \|_\mu \leq \Delta_N(\mu), \quad N = 1, 2, \ldots,
$$

where $\| \cdot \|_\mu$ is the energy norm given by $\| w \|_\mu \equiv \sqrt{a(w, w; \mu)}$. In addition, given a bounded linear functional $\ell(\cdot; \mu)$ that admits affine parameter decomposition and the associated true output $s(\mu) \equiv \ell(u(\mu); \mu)$, we provide a sequence of approximate outputs $s_N(\mu)$, $N = 1, 2, \ldots$, and associated error bounds $\Delta_N^s(\mu)$, $N = 1, 2, \ldots$, such that

$$
| s(\mu) - s_N(\mu) | \leq \Delta_N^s(\mu), \quad N = 1, 2, \ldots;
$$

we note that the output bound is of interest in engineering practice.
Remark 1. Compared to the standard reduced basis method [15] that does not require the third assumption, the proposed method limits the class of problems that can be addressed. However, this assumption allows us to choose a norm for the residual and stability constant such that our error bound is effective. In addition, many problems of practical interest admit an affine decomposition of $K^{-1}(\mu)$, including those problems involving geometry transformations.

Remark 2. In this work we consider scalar equations to simplify the presentation. However, our approximation and error bound procedures readily extend to vector-valued equations.

2.2. Inner products and norms. We introduce two different inner products that simplify the presentation of the proposed method. First is a parametrized inner product

$$(w,v)_{\mathcal{V}(\mu;\delta)} \equiv \int_{\Omega} \nabla v \cdot K(\mu) \nabla w \, dx + \delta \int_{\Omega} v u \, dx + \delta \int_{\Gamma_N} v u \, ds \quad \forall w,v \in \mathcal{V}$$

and the associated induced norm $\|w\|_{\mathcal{V}(\mu;\delta)} \equiv \sqrt{(w,w)_{\mathcal{V}(\mu;\delta)}}$ for a parameter $\mu \in \mathcal{D}$ and a weight $\delta \in \mathbb{R}_{\geq 0}$. Note that this parametrized norm is related to the energy norm by $\|w\|^2_{\mathcal{V}(\mu;\delta)} = \|w\|^2_{\mathcal{V}(\mu)} + \delta \|w\|^2_{L^2(\Omega)} + \delta \|w\|^2_{L^2(\Gamma_N)}$. In addition, by the Poincaré-Friedrichs inequality and the trace theorem, the norm $\|\cdot\|_{\mathcal{V}(\mu;\delta)}$ is equivalent to the energy norm $\|\cdot\|_{\mathcal{V}(\mu)}$, which in turn is equivalent to $\|\cdot\|_{H^1(\Omega)}$ for all $\mu \in \mathcal{D}$.

Second is a parametrized inner product

$$(p,q)_{\mathcal{K}(\mu)} \equiv \int_{\Omega} q \cdot K(\mu) p \, dx \quad \forall p,q \in (L^2(\Omega))^d$$

and the associated induced norm $\|p\|_{\mathcal{K}(\mu)} \equiv \sqrt{(p,p)_{\mathcal{K}(\mu)}}$. Note that the energy norm may be expressed as $\|w\|_\mu = \|\nabla w\|_{\mathcal{K}(\mu)}$. Because the field $K(\mu)$ is bounded and positive, the norm $\|\cdot\|_{\mathcal{K}(\mu)}$ is equivalent to $\|\cdot\|_{L^2(\Omega)}$ for all $\mu \in \mathcal{D}$.

2.3. Energy-norm error bound. We now introduce the two ingredients that constitute our error bound: the dual-norm of the residual and the stability constant. We first introduce the residual functional: for $\mu \in \mathcal{D}$ and $\hat{u} \in \mathcal{V}$,

$$r(v;\hat{u};\mu) \equiv \ell(v;\mu) - a(\hat{u},v;\mu) \quad \forall v \in \mathcal{V};$$

the dual norm of the residual is given by

$$(2) \quad \|r(\cdot;\hat{u};\mu)\|_{\mathcal{V}(\mu;\delta)} \equiv \sup_{v \in \mathcal{V}} \frac{r(v;\hat{u};\mu)}{\|v\|_{\mathcal{V}(\mu;\delta)}}.$$

We then introduce a stability constant

$$(3) \quad \alpha(\mu;\delta) \equiv \inf_{v \in \mathcal{V}} \frac{\|v\|^2_\mu}{\|v\|^2_{\mathcal{V}(\mu;\delta)}}.$$

The following well-known proposition provides an energy-norm error bound. (See, e.g., [13]).

Proposition 3. For any $\mu \in \mathcal{D}$ and $\hat{u} \in \mathcal{V}$, the energy norm of the error is bounded by

$$\|u(\mu) - \hat{u}\|_\mu \leq \frac{1}{(\alpha(\mu;\delta))^{1/2}} \|r(\cdot;\hat{u};\mu)\|_{\mathcal{V}(\mu;\delta)}.$$

Proof. For $e \equiv u(\mu) - \hat{u}$, we obtain

$$\|e\|_\mu = \frac{\alpha(e,e;\mu)}{\|\hat{u}\|_\mu} = \frac{r(e;\hat{u};\mu)}{\|\hat{u}\|_\mu} \leq \sup_{v \in \mathcal{V}} \frac{r(v;\hat{u};\mu)}{\|v\|_{\mathcal{V}(\mu;\delta)}} \leq \frac{1}{(\alpha(\mu;\delta))^{1/2}} \|r(\cdot;\hat{u};\mu)\|_{\mathcal{V}(\mu;\delta)}.$$
here, the first equality follows from the definition of the energy norm \( \| \cdot \|_H \), the second equality follows from the error-residual relationship, and the last equality follows from the definitions of the stability constant and the dual-norm of the residual.

The proposition shows that the evaluation of an upper bound of the energy-norm of the error requires an upper bound of the dual norm of the residual and a lower bound of the stability constant.

### 2.4. Error bound for linear functional outputs

We briefly note that our energy-norm error bound may be extended to provide an error bound for a linear functional output, \( s(\mu) = \ell^o(v(\mu); \mu) \). Our approach follows that for the standard reduced basis method [15] and appeals to the following well-known proposition regarding the output error. (See, e.g., Giles and S{"u}li [5].)

**Proposition 4.** We define the adjoint problem associated with a parametrized linear functional \( \ell^o(\cdot; \cdot) : V \times D \to \mathbb{R} \): given \( \mu \in D \), find \( \psi(\mu) \in V \) such that

\[
a(v, \psi(\mu); \mu) = \ell^o(v; \mu) \quad \forall v \in V.
\]

Then, for a primal approximation \( \tilde{u} \in V \) and an adjoint approximation \( \tilde{\psi} \in V \), we introduce an adjoint-corrected output prediction

\[
\tilde{s}(\mu) \equiv \ell^o(\tilde{u}) + r(\tilde{\psi}; \tilde{u}; \mu).
\]

The error in the output prediction is bounded by

\[
|s(\mu) - \tilde{s}(\mu)| \leq \frac{1}{\alpha(\mu; \delta)} \| r(\cdot; \tilde{u}; \mu) \|_{V(\mu; \delta)} \| r_{\text{adj}}(\cdot; \tilde{\psi}; \mu) \|_{V(\mu; \delta)},
\]

where \( \alpha(\mu; \delta) \) is the stability constant (3) and the adjoint residual \( r_{\text{adj}}(\cdot; \cdot; \cdot) \) is defined by

\[
r_{\text{adj}}(v; \tilde{\psi}; \mu) \equiv \ell^o(v; \mu) - a(v, \tilde{\psi}; \mu) \quad \forall v \in V.
\]

**Proof.** See, for instance, Giles and S{"u}li [5].

The proposition shows that the evaluation of a bound for a functional output requires a lower bound of the stability constant, an upper bound of the dual norm of the (primal) residual, and an upper bound of the dual norm of the adjoint residual. As the same technique can be used to bound the dual norm of the primal and adjoint residuals, bounding the output error requires the same set of techniques as bounding the energy norm of the error.

### 3. An offline-online computable upper bound of the dual norm of the residual

#### 3.1. Residual bound form

We now develop a computable upper bound of the dual-norm of the residual (2). The proposed residual bound procedure is a generalization of the procedure introduced in [18] to the parametrized norm \( \| \cdot \|_{V(\mu; \delta)} \).

We first introduce a vector-valued Hilbert space

\[
Q \equiv H^1(\text{div}; \Omega) \equiv \{ q \in (L^2(\Omega))^d \mid \nabla \cdot q \in L^2(\Omega) \},
\]

endowed with an inner product \( (w, v)_Q \equiv (w, v)_{H^1(\text{div}; \Omega)} \equiv \int_\Omega (\nabla \cdot w)(\nabla \cdot v)dx + \int_\Omega w \cdot vdx \) and the induced norm \( \| \cdot \|_Q \equiv \sqrt{\langle w, w \rangle_Q} \). The following proposition provides a bound of the dual norm of the residual.

**Proposition 5.** For any \( \mu \in D, \delta \in \mathbb{R}_{\geq 0} \), and \( \tilde{u} \in V \), the dual norm of the residual is bounded by

\[
\| r(\cdot; \tilde{u}; \mu) \|_{V(\mu; \delta)} \leq (F(\tilde{u}, q; \mu; \delta))^{1/2} \quad \forall q \in Q,
\]

where the bound form is given by

\[
(5) \quad F(\tilde{u}, q; \mu; \delta) = \| \nabla \tilde{u} - K^{-1}(\mu)p \|_{L^2(\Omega)}^2 + \delta^{-1} \| f(\mu) + \nabla \cdot p \|_{L^2(\Omega)}^2 + \delta^{-1} \| g(\mu) - p \cdot n \|_{L^2(\Gamma_N)}^2.
\]
Proof. We note that for any \( \tilde{u} \in \mathcal{V}, v \in \mathcal{V}, \) and \( q \in \mathcal{Q}, \)

\[
r(v; \tilde{u}; \mu) = \ell(v; \mu) - a(\tilde{u}, v; \mu)
= \int_{\Omega} vf(\mu)dx + \int_{\Gamma_N} vg(\mu)ds - \int_{\Omega} \nabla v \cdot K(\mu) \nabla \tilde{u} dx
= \int_{\Omega} vf(\mu)dx + \int_{\Gamma_N} vg(\mu)ds - \int_{\Omega} \nabla v \cdot (K(\mu) \nabla \tilde{u} + q) dx + \int_{\Gamma_N} v g(\mu) - q \cdot nds
\leq \|v\|_{L^2(\Omega)} \|f(\mu) + \nabla \cdot q\|_{L^2(\Omega)} + \|\nabla v\|_{K(\mu)} \|\nabla \tilde{u} - K^{-1}(\mu) q\|_{K(\mu)} + \|v\|_{L^2(\Gamma_N)} \|g(\mu) - q \cdot n\|_{L^2(\Gamma_N)}
\leq (\delta \|v\|_{L^2(\Omega)}^2 + \|\nabla\|_{K(\mu)}^2 + \delta \|v\|_{L^2(\Gamma_N)}^2)^{1/2}
= \|v\|_{\mathcal{V}(\mu; \delta)}^2 \leq (F(\tilde{u}, q; \mu; \delta))^{1/2},
\]

here the third equality follows from the Green’s theorem. It thus follows that

\[
\|r(v; \tilde{u}; \mu)\|_{\mathcal{V}(\mu; \delta)} = \sup_{v \in \mathcal{V}} \frac{r(v; \tilde{u}; \mu)}{\|v\|_{\mathcal{V}(\mu; \delta)}} \leq (F(\tilde{u}, q; \mu; \delta))^{1/2},
\]

which is the desired inequality. \(\square\)

We make a few observations as regard the behavior of the primal bound form (5) with the weight \( \delta \). In the case of vanishing parameter \( \delta = 0 \), the bound form (5) evaluates to

\[
F(\tilde{u}, q; \mu; \delta = 0) = \begin{cases} \|\nabla \tilde{u} - K^{-1}(\mu) q\|_{K(\mu)}^2, & \forall \tilde{u} \in \mathcal{V}, \forall q \in \mathcal{Q}^*(\mu), \\
\infty & \text{otherwise}, \end{cases}
\]

where the constrained space \( \mathcal{Q}^*(\mu) \) is given by

\[
\mathcal{Q}^*(\mu) = \{q \in \mathcal{Q} \mid f(\mu) + \nabla \cdot q = 0 \text{ in } L^2(\Omega) : g(\mu) - q \cdot n = 0 \text{ in } L^2(\Gamma_N)\}.
\]

We note that the condition (6) is precisely the dual-feasibility condition for the complementary variational principle. The exact satisfaction of the dual-feasibility condition is possible in the finite-element context, as pursued by, for instance, Ladevèze and Leguillion [9], Ainsworth and Oden [1], and Sauer-Budge et al. [16]. On the other hand, the exact satisfaction of the dual-feasibility condition in the reduced-basis context, as pursued in [17], requires a special construction of reduced-basis spaces with limited practicality. Hence, in the present context, we choose a small but non-zero \( \delta \) to relax the dual-feasibility requirement; a practical rule for selecting a \( \delta \) is discussed in Section 6.3.

3.2. Affine decomposition of the bound form. In order to facilitate the development of finite-element and reduced-basis approximations, we introduce a decomposition of the form \( F(\cdot, ; \cdot; \mu; \delta) \) defined in (5) into quadratic, linear, and constant terms. The form (5) may be expressed as

\[
F(w, p; \mu; \delta) = G((w, p), (w, p); \mu; \delta) - 2L((w, p); \mu; \delta) + H(\mu; \delta),
\]

where the bilinear form, linear form, and constant term are given by

\[
G((w, p), (v, q); \mu; \delta) = G_0((w, p), (v, q); \mu) + \delta^{-1} G_\Omega((w, p), (v, q); \mu) + \delta^{-1} G_{\Gamma_N}((w, p), (v, q); \mu),
\]

\[
L((v, q); \mu; \delta) \equiv \delta^{-1} L_\Omega((v, q); \mu) + \delta^{-1} L_{\Gamma_N}((v, q); \mu),
\]

\[
H(\mu; \delta) \equiv \delta^{-1} H_\Omega(\mu) + \delta^{-1} H_{\Gamma_N}(\mu),
\]
for
\[
G_0((w,p),(v,q);\mu) = \int_{\Omega} \left[ q \cdot K^{-1}(\mu)p - q \cdot \nabla w - \nabla v \cdot p + \nabla v \cdot K(\mu)\nabla w \right] dx,
\]
\[
G_\Omega((w,p),(v,q);\mu) = \int_{\Omega} (\nabla \cdot q)(\nabla \cdot p) dx,
\]
\[
G_{\Gamma_N}((w,p),(v,q);\mu) = \int_{\Gamma_N} (q \cdot n)(p \cdot n) ds,
\]
\[
L_\Omega((v,q);\mu) = \int_{\Omega} (-\nabla \cdot q)f(\mu) dx,
\]
\[
L_{\Gamma_N}((v,q);\mu) = \int_{\Gamma_N} (q \cdot n)g(\mu) ds,
\]
\[
H_\Omega(\mu) = \int_{\Omega} f(\mu)f(\mu) dx,
\]
\[
H_{\Gamma_N}(\mu) = \int_{\Gamma_N} g(\mu)g(\mu) ds.
\]
We note that the parametrized forms associated with the bound form \(F(\cdot, \cdot; \mu; \delta)\) inherit the affine parameter decomposition of the fields \(K(\mu), K^{-1}(\mu), f(\mu), \) and \(g(\mu)\). As we will see shortly, and as in the standard reduced basis methods [15], this affine parameter decomposition plays a key role in offline-online computational decomposition.

3.3. Minimum-bound mixed finite element method. We first introduce a sequence of conforming, non-degenerate triangulations \(T_h\) of \(\Omega\). We then introduce conforming finite-element approximation spaces for \(\mathcal{V}\) and \(\mathcal{Q}\):
\[
\mathcal{V}^{N_{\mathcal{V}}} = \{v \in \mathcal{V} \mid v|_\kappa \in \mathbb{P}^p, \ \forall \kappa \in T_h\},
\]
\[
\mathcal{Q}^{N_{\mathcal{Q}}} = \{q \in \mathcal{Q} \mid q|_\kappa \in \mathbb{RT}^{p-1} = (\mathbb{P}^{p-1})^d + x\mathbb{P}^{p-1}, \ \forall \kappa \in T_h\};
\]

note that \(\mathcal{V}^{N_{\mathcal{V}}}\) consists of the standard \(H^1\) conforming elements and \(\mathcal{Q}^{N_{\mathcal{Q}}}\) consists of Raviart-Thomas elements [14] of degree \(p - 1\). The superscripts \(N_{\mathcal{V}}\) and \(N_{\mathcal{Q}}\) denote the number of degrees of freedom associated with the spaces \(\mathcal{V}^{N_{\mathcal{V}}}\) and \(\mathcal{Q}^{N_{\mathcal{Q}}}\), respectively. We in addition set \(N = N_{\mathcal{V}} + N_{\mathcal{Q}}\), which serves as a measure of the complexity of the finite-element approximation.

We now consider the minimum-bound solution: given \(\mu \in \mathcal{D}\) and \(\delta \in \mathbb{R}_{>0}\), find \((u^N(\mu), p^N(\mu)) \in \mathcal{V}^{N_{\mathcal{V}}} \times \mathcal{Q}^{N_{\mathcal{Q}}}\) such that
\[
(u^N(\mu), p^N(\mu)) = \arg \inf_{u \in \mathcal{V}^{N_{\mathcal{V}}}} \inf_{q \in \mathcal{Q}^{N_{\mathcal{Q}}}} \mathcal{F}(w, q; \mu; \delta).
\]

We identify the associated Euler-Lagrange equation: given \(\mu \in \mathcal{D}\) and \(\delta \in \mathbb{R}_{>0}\), find \((u^N(\mu), p^N(\mu)) \in \mathcal{V}^{N_{\mathcal{V}}} \times \mathcal{Q}^{N_{\mathcal{Q}}}\) such that
\[
G((u^N(\mu), p^N(\mu)), (v, q); \mu; \delta) = L((v, q); \mu; \delta) \quad \forall v \in \mathcal{V}^{N_{\mathcal{V}}}, \ \forall q \in \mathcal{Q}^{N_{\mathcal{Q}}},
\]
where \(G\) is the bilinear form (8) and \(L\) is the linear form (9). The problem is well posed thanks to the coercivity of the bilinear form \(G(\cdot, \cdot; \mu; \delta)\) with respect to \(\mathcal{V} \times \mathcal{Q}\). Once we obtain the finite element approximation, we readily bound the dual norm of the residual using \(\|r(\cdot; u^N(\mu); \mu)\|_{\mathcal{V}^{N_{\mathcal{V}}}}; \delta) \leq (F(u^N(\mu), p^N(\mu); \mu; \delta))^{1/2}\).

3.4. Minimum-bound mixed reduced basis method. We consider our minimum-output-bound reduced basis approximation. Towards this end, we introduce an \(N\)-dimensional primal approximation space spanned by a basis \(\{\xi_n\}_{n=1}^N\),
\[
\mathcal{V}_N = \text{span}\{\xi_n\}_{n=1}^N \subset \mathcal{V}^{N_{\mathcal{V}}},
\]
and an $N$-dimensional dual approximation space spanned by a basis \( \{ \eta_n \}_{n=1}^N \),
\[
Q_N = \text{span} \{ \eta_n \}_{n=1}^N \subset \mathbb{Q}^N.
\]

We then consider the minimum-bound solution: given \( \mu \in \mathcal{D} \) and \( \delta \in \mathbb{R}_{>0} \), find \( (u_N(\mu), p_N(\mu)) \in \mathcal{V} \times \mathbb{Q}^N \) such that
\[
(u_N(\mu), p_N(\mu)) = \arg \inf_{w \in \mathcal{V}, \eta \in \mathbb{Q}} F(w, \eta; \mu),
\]
identified as Theorem 3.5. Offline-online computational decomposition. We now present an offline-online computational procedure for the reduced basis method. We here appeal to the decomposition of the primal bound (7) into bilinear forms, linear forms, and constant terms. Specifically, for $w \equiv \sum_{n=1}^N \xi_n w_n \in \mathcal{V}_N$, $p \equiv \sum_{n'}^N \eta_{n'} p_{n'} \in \mathbb{Q}_N$, $v \equiv \sum_{m=1}^N \xi_m v_m \in \mathcal{V}_N$, and $q \equiv \sum_{m'=1}^N \eta_{m'} q_{m'} \in \mathbb{Q}_N$ in the reduced basis spaces, the terms that constitute (7) evaluate to
\[
G_0((w, p), (v, q); \mu) = \left( q_{m'} \right)^T \left( \Theta_a^K \right) \left( \int \nabla \cdot \eta_{m'} \nabla \cdot \eta_{n'} \, dx \right) - \left( \int \nabla \cdot \xi_{m'} \nabla \cdot \eta_{n'} \, dx \right) \left( \Theta_a^K \right) \left( \int \nabla \xi_{m} \cdot \nabla \xi_{n} \, dx \right) \left( p_{n'} \right) \left( w_n \right),
\]
\[
G_1((w, p), (v, q); \mu) = \left( q_{m'} \right)^T \left( \int \nabla \cdot \eta_{m'} \nabla \cdot \eta_{n'} \, dx \right) \left( p_{n'} \right) \left( w_n \right),
\]
\[
G_{\Gamma_N}((w, p), (v, q); \mu) = \left( q_{m'} \right)^T \left( \int \nabla \cdot \eta_{m'} \nabla \cdot \eta_{n'} \, dx \right) \left( p_{n'} \right) \left( v_m \right),
\]
\[
L_{\Omega}((v, q); \mu) = \left( q_{m'} \right)^T \left( \int \nabla \cdot \eta_{m'} \nabla \cdot \eta_{n'} \, dx \right) \left( p_{n'} \right) \left( v_m \right),
\]
\[
L_{\Gamma_N}((v, q); \mu) = \left( q_{m'} \right)^T \left( \int \eta_{m'} \cdot \eta_{m'} \, ds \right) \left( p_{n'} \right) \left( v_m \right),
\]
\[
H_{\Omega}(\mu) = \Theta_a^f(\mu) \left( \int f_a f_b \, dx \right),
\]
\[
H_{\Gamma_N}(\mu) = \Theta_a^b(\mu) \left( \int g_a g_b \, dx \right),
\]

Here, the summation on the repeated indices of \( \alpha \) and \( \beta \) are implied. In the offline stage, we compute the terms appealing in brackets, [ ]. In the online stage, we first assemble the matrices and vectors with appropriate weights \( \Theta(\mu) \) evaluated for the parameter value of interest. We then solve the primal reduced basis system (14) of the size $2N$ to obtain the coefficients $u_N(\mu) \in \mathbb{R}^N$ and $p_N(\mu) \in \mathbb{R}^N$ of the primal solution. We finally appeal to the decomposition (7) to evaluate the bound of the dual-norm of the primal residual, $F(u_N(\mu), p_N(\mu); \mu, \delta)$. 

In Remark 6, our formulation readily accommodates primal and dual reduced basis spaces of different dimensions. However, in practice, we use the spaces of the equal dimension because of the way these bases are generated in the offline stage.
4. An offline-online computable lower bound of the stability constant.

4.1. Transformation of the stability constant. We now consider a offline-online computable lower bound of the stability constant \( (3) \), \( \alpha(\mu; \delta) \equiv \inf_{v \in \mathcal{V}} \|v\|_{\mu}^2/\|v\|_{\mathcal{V}(\mu; \delta)}^2 \). By way of preliminaries, we introduce a norm

\[
\|v\|_{\mathcal{W}}^2 \equiv \|v\|_{H^1(\Omega)}^2 + \|v\|_{L^2(\Gamma_N)}^2
\]

for notational convenience. We now present a proposition that relates the stability constant \( \alpha(\mu; \delta) \) to another quantity \( \tau(\mu) \), which is amenable to offline-online computational decomposition.

\textbf{Proposition 7.} The stability constant \( \alpha(\mu; \delta) \equiv \inf_{v \in \mathcal{V}} \|v\|_{\mu}^2/\|v\|_{\mathcal{V}(\mu; \delta)}^2 \) is bounded from the below by

\[
\alpha(\mu; \delta) \geq \left( 1 + \frac{\delta}{\tau_{\text{LB}}(\mu)} \right)^{-1}
\]

where \( \tau_{\text{LB}}(\mu) \) satisfies

\[
\tau_{\text{LB}}(\mu) \leq \tau(\mu) \equiv \inf_{v \in \mathcal{V}} \|v\|_{\mu}^2/\|v\|_{\mathcal{W}}^2.
\]

\textbf{Proof.} We note that

\[
\frac{1}{\alpha(\mu; \delta)} \equiv \left( \inf_{v \in \mathcal{V}} \frac{\|v\|_{\mu}^2}{\|v\|_{\mathcal{V}(\mu; \delta)}^2} \right)^{-1} = \sup_{v \in \mathcal{V}} \frac{\|v\|_{\mu}^2}{\|v\|_{\mathcal{V}(\mu; \delta)}^2} = \sup_{v \in \mathcal{V}} \frac{\|v\|_{\mu}^2 + \delta \|v\|_{L^2(\Omega)}^2 + \delta \|v\|_{L^2(\Gamma_N)}^2}{\|v\|_{\mu}^2} \leq 1 + \delta \sup_{v \in \mathcal{V}} \frac{\|v\|_{\mathcal{W}}^2}{\|v\|_{\mu}^2} = 1 + \frac{\delta}{\tau(\mu)};
\]

here, the first equality follows from the definition of the stability constant, the third equality follows from the definition of \( \| \cdot \|_{\mu} \) and \( \| \cdot \|_{\mathcal{V}(\mu; \delta)} \), the inequality follows from \( \|v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Gamma_N)}^2 \leq \|v\|_{\mathcal{W}}^2 \), and the last equality follows from the definition of \( \tau(\mu) \). We obtain the desired result by noting that \( \tau_{\text{LB}}(\mu) \leq \tau(\mu) \). \( \square \)

We make four observations. First, the proposition shows that if we can compute a lower bound of \( \tau(\mu) \) in an offline-online efficient manner, then we can rapidly evaluate a lower bound of \( \alpha(\mu; \delta) \). Second, we observe that the lower bound of \( \alpha(\mu; \delta) \) is close to unity if \( \delta \) is chosen small with respect to \( \tau_{\text{LB}}(\mu) \); more specifically, the transformation allows us to desensitize the effectivity of a lower bound of \( \alpha(\mu; \delta) \) from the effectivity of a lower bound of \( \tau(\mu) \) by choosing \( \delta \ll \tau_{\text{LB}}(\mu) \). Third, we observe that for \( \delta = 0 \) the stability constant is unity; we recall that the minimum-bound solution for \( \delta = 0 \) must satisfy the dual-feasibility condition \( (6) \) exactly, and for such a solution, by complementary variational principle, the dual norm of the residual is the same as the energy-norm of the error, which implies the stability constant is unity. Fourth, we observe that \( \tau(\mu) \equiv \|v\|_{\mu}^2/\|v\|_{\mathcal{W}}^2 \) admits an affine parameter decomposition, as \( \| \cdot \|_{\mu} \equiv a(\cdot, \cdot; \mu) \) admits an affine decomposition and the norm \( \| \cdot \|_{\mathcal{W}}^2 \) is independent of the parameter; this property of \( \tau(\mu) \) makes it suitable for offline-online computation by the successive constraint method.

Using an argument based on the Rayleigh quotient, we can readily show the stability constant \( \tau(\mu) \) is the minimum eigenvalue of the following eigenproblem: given \( \mu \in \mathcal{D} \), find \( (z_i(\mu), \lambda_i(\mu)) \in \mathcal{V} \times \mathbb{R} \) such that \( \|z_i(\mu)\|_{\mathcal{W}} = 1 \) and

\[
a(z_i(\mu), v; \mu) = \lambda_i(\mu)(z_i(\mu), v)_{\mathcal{W}} \quad \forall v \in \mathcal{V}.
\]

We will henceforth refer to this eigenproblem as the stability eigenproblem. Without loss of generality, we order the eigenvalues in ascending order; hence \( \tau(\mu) = \lambda_{\text{min}}(\mu) = \lambda_1(\mu) \), where the subscript 1 denotes the first eigenvalue. Hence, to provide a lower bound of \( \tau(\mu) \), we need to provide a lower bound of \( \lambda_1(\mu) \).
4.2. Successive constraint method (SCM). Our approach to compute a lower bound of $\tau(\mu)$ in an offline-online efficient manner is based on the successive constraint method (SCM) of Huynh et al. [7]. The original SCM was introduced to compute a lower bound of the stability constant with respect to the finite-dimensional finite element space; here we extend the method to provide a lower bound of the stability constant with respect to the infinite-dimensional space $V$. We here present only a brief overview of the SCM and refer to [7] for a more detailed presentation; we will however highlight the key differences between the original SCM and our extension.

The SCM, as the name suggests, recasts the minimization problem associated with $\tau(\mu)$ as a linear constrained optimization problem. Towards this end, we first introduces a space

$$
V = \left\{ v \in \mathbb{R}^{Q_K} \mid \exists y \in V \text{ s.t. } y_q = \frac{\int_{\Omega} \nabla v_y \cdot K_q \nabla v_{y} dx}{\|u_{y}\|_{L^2(V)}}, q = 1, \ldots, Q_K \right\}
$$

and a functional

$$
J(\mu; y) \equiv \sum_{q=1}^{Q_K} \Theta_q^K(\mu)y_q.
$$

We then note that the stability constant can be expressed as

$$
\tau(\mu) = \inf_{y \in V} J(\mu; y).
$$

To evaluate a lower bound of the stability constant, we now introduce a space $V_{LB} \supset V$ given by

$$(17) \quad V_{LB}(\mu; \Xi_{con}) \equiv \left\{ y \in B_{Q_K} \mid \sum_{q=1}^{Q_K} \Theta_q^K(\mu') \geq \tau(\mu'), \forall \mu' \in \Xi_{con} \right\},$$

here $B_{Q_K}$ is a bounding box defined by

$$
B_{Q_K} = \prod_{q=1}^{Q_K} [\gamma^-_q, \gamma^+_q]
$$

for

$$(18) \quad \gamma^-_q \equiv \inf_{v \in V} \int_{\Omega} \nabla v \cdot K_q \nabla v dx, \quad q = 1, \ldots, Q_K,$$

$$(19) \quad \gamma^+_q \equiv \sup_{v \in V} \int_{\Omega} \nabla v \cdot K_q \nabla v dx, \quad q = 1, \ldots, Q_K,$$

and $\Xi_{con} \subset \mathcal{D}$ is a set of “SCM constrained points” that are chosen in a careful manner (e.g., using a greedy algorithm). We then define a lower bound of the stability constant as

$$
\tau_{LB}(\mu) = \inf_{y \in V_{LB}(\mu; \Xi_{con})} J(\mu; y);
$$

here, $\tau_{LB}(\mu) \leq \tau(\mu)$ because $V_{LB} \supset V$.

We note that $\tau_{LB}(\mu)$ admits offline-online computational decomposition: in the offline stage, we compute the set $\tau(\mu')$, $\mu' \in \Xi_{con}$, and the set $\gamma^\pm_q$, $q = 1, \ldots, Q_K$, which together define the space $V_{LB}(\mu; \Xi_{con})$; in the online stage, we solve the linear programming problem $\tau_{LB}(\mu) = \inf_{y \in V_{LB}(\mu; \Xi_{con})} J(\mu; y)$ to find a lower bound for a given $\mu$.

The two key differences between the original SCM [7] and our extension for $\tau(\mu)$ with respect to the infinite-dimensional space $V$ are the following. First, the constants $\gamma^\pm_q$, $q = 1, \ldots, Q_K$, defined in (18) and (19) require infimization and suprmization over $V$, as opposed to a finite element space $V^{N}$ for the original SCM. Second, the constraints $\tau(\mu')$ defined in (15) require infimization over $V$, as opposed to $V^{N}$. Because these suprmization and infimization problems are defined over an infinite-dimensional space $V$, they cannot be computed directly. We now present our strategies to provide computable bounds for these quantities.
4.3. Offline evaluation of an upper bound of $\gamma^+_q$. To evaluate $\gamma^+_q$, we appeal to

$$
\gamma^+_q \equiv \sup_{v \in \mathcal{V}} \frac{\int_\Omega \nabla v \cdot K_q \nabla v dx}{\|v\|_{H^1(\Omega)}} \leq \sup_{v \in \mathcal{V}} \frac{\int_\Omega \nabla v \cdot K_q \nabla v dx}{\|v\|_{H^1(\Omega)}} \leq \|\lambda_{\max}(K_q(x))\|_{L^\infty(\Omega)} \equiv \tilde{\gamma}^+_q.
$$

Because the fields $K_q \in [L^\infty(\Omega)]^{d \times d}$ are known, we can directly compute the upper bounds. Similarly, to evaluate $\gamma^-_q$, we appeal to

$$
\gamma^-_q \equiv \inf_{v \in \mathcal{V}} \frac{\int_\Omega \nabla v \cdot K_q \nabla v dx}{\|v\|_{H^1(\Omega)}} \geq -\inf_{v \in \mathcal{V}} \frac{\int_\Omega \nabla v \cdot K_q \nabla v dx}{\|v\|_{H^1(\Omega)}} \geq -\|\lambda_{\max}(K_q(x))\|_{L^\infty(\Omega)} \equiv \tilde{\gamma}^-_q.
$$

We note that $K_q$ sometimes possess a special structure such that a tighter bound for $\gamma^-_q$ can be obtained; for instance, if $K_q$ is positive, then $\gamma^-_q$ is bounded from the below by 0.

4.4. An abstract lower bound of the minimum eigenvalue. In order to express a lower bound of the minimum eigenvalue using Weinstein’s method, we define a residual associated with the eigenproblem (16): for $\mu \in \mathcal{D}$, $\tilde{z} \in \mathcal{V}$ with $\|\tilde{z}\|_\mathcal{W} = 1$, and $\tilde{\lambda} \in \mathbb{R}$,

$$
r_{\text{eig}}(v; \tilde{z}, \tilde{\lambda}; \mu) \equiv a(\tilde{z}, v; \mu) - \tilde{\lambda}(\tilde{z}, v)_\mathcal{W} \quad \forall v \in \mathcal{V}.
$$

The associated dual norm of the eigenproblem residual with respect to $\|\cdot\|_\mathcal{W}$ is

$$
\|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)\|_{\mathcal{W}'} \equiv \sup_{v \in \mathcal{V}} \frac{r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)}{\|v\|_{\mathcal{W}}}
$$

The following proposition by Weinstein relates the dual norm of the residual — which is called the defect in the original work — to the distance to the closest eigenvalue (see [4], Chapter 6).

**Proposition 8.** Let $(\tilde{z}, \tilde{\lambda}) \in \mathcal{V} \times \mathbb{R}$ be an approximate eigenpair where $\|\tilde{z}\|_{\mathcal{W}} = 1$. Then, the distance between $\tilde{\lambda}$ and the closest eigenvalue is bounded from the above by

$$
\min_{j=1,2,...} |\lambda_j(\mu) - \tilde{\lambda}| \leq \|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)\|_{\mathcal{W}'}.
$$

**Proof.** For notational convenience, throughout this proof, we suppress the dependence of the bilinear form $a(\cdot, \cdot; \mu)$, the residual form $r_{\text{eig}}(\cdot, \cdot; \cdot; \mu)$, and the eigenvalues $\{\lambda_j(\mu)\}$ on the parameter $\mu \in \mathcal{D}$. We denote the coefficients associated with the representation of an arbitrary vector $v \in \mathcal{V}$ in the eigenbasis $\{z_i\}_{i=1}^\infty$ by $\hat{v}$. Similarly, we denotes the coefficients associated with the representation of the approximate eigenvector $\tilde{z}$ in the eigenbasis by $\hat{\tilde{z}}$ such that $\tilde{z} = \sum_{n=1}^\infty \hat{\tilde{z}}_n z_n$. The inequality then follows from

$$
\|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda})\|_{\mathcal{W}'} = \sup_{v \in \mathcal{V}} \frac{a(\tilde{z}, v) - \tilde{\lambda}(\tilde{z}, v)_\mathcal{W}}{\|v\|_{\mathcal{W}}} = \sup_{\hat{v}} \frac{\sum_{n=1}^\infty \hat{v}_m \hat{\tilde{z}}_n [a(z_n, z_m) - \tilde{\lambda}(z_n, z_m)_\mathcal{W}]}{(\sum_{n=1}^\infty \hat{v}_n^2)^{1/2}}
$$

$$
= \sup_{\hat{v}} \frac{\sum_{n=1}^\infty \hat{\tilde{z}}_n (\lambda_n - \tilde{\lambda})}{(\sum_{n=1}^\infty \hat{v}_n^2)^{1/2}} = (\sum_{n=1}^\infty \hat{\tilde{z}}_n^2)^{1/2}
$$

$$
\geq \min_{n=1,2,...} |\lambda_n - \tilde{\lambda}| (\sum_{n=1}^\infty \hat{\tilde{z}}_n^2)^{1/2} = \min_{n=1,2,...} |\lambda_n - \tilde{\lambda}| \|\tilde{z}\|_{\mathcal{W}} = \min_{n=1,2,...} |\lambda_n - \tilde{\lambda}|.
$$

A few explanations are in order. The third equality follows from the definition of the eigenvalues and the orthogonality of the eigenbasis with respect to $a(\cdot, \cdot)$ and $(\cdot, \cdot)_\mathcal{W}$: $a(z_n, z_m) = \lambda_n$ if $n = m$ and $a(z_n, z_m) = 0$ if $n \neq m$; $(z_n, z_m) = 1$ if $n = m$ and $(z_n, z_m) = 0$ if $n \neq m$. The fourth equality follows from the fact that the supremizer is $\hat{v}_n = \hat{\tilde{z}}_n (\lambda_n - \tilde{\lambda})$, $n = 1, 2, \ldots$. The last equality follows from $\|\tilde{z}\|_{\mathcal{W}} = 1$. \hfill □

We appeal to the proposition to provide a lower bound of the minimum eigenvalue under one crucial assumption:
Corollary 9. Let \((\tilde{z}, \tilde{\lambda}) \in \mathcal{V} \times \mathbb{R}\) be an approximate eigenpair where \(\|\tilde{z}\|_\mathcal{W} = 1\). If \(|\lambda_1(\mu) - \tilde{\lambda}| \leq |\lambda_2(\mu) - \tilde{\lambda}|\) then
\[
\lambda_1(\mu) \geq \tilde{\lambda} - \|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)\|_\mathcal{W}.
\]

We make two remarks. First, we unfortunately have no means to rigorously verify whether the condition \(|\lambda_1(\mu) - \tilde{\lambda}| \leq |\lambda_2(\mu) - \tilde{\lambda}|\) is satisfied; this, as noted in the Introduction, is a crucial limitation as regard the rigorousness of error bounds provided by our method. However, we have found that in practice the first two eigenvalues of (16) are sufficiently well separated for a typical elliptic problem that even a crude finite element approximation of the eigenproblem is sufficient to meet the condition. Hence, if we could compute an upper bound of the dual norm of the eigenproblem residual, then we can provide a lower bound of the \(\lambda_1(\mu) = \tau(\mu)\).

Second, the bound based on Weinstein’s method is not very effective. Specifically, there exists another bound technique by Kato [8] which provides a lower bound whose gap scales as the square of the residual instead of linearly with the residual [12]. However, Kato’s method introduces an additional assumption regarding the second eigenvalue of the eigenproblem. Here, we sacrifice the effectivity for the robustness of making one fewer assumptions and build our algorithm on Weinstein’s method. We also note that the thanks to the desensitization provided by the transformation of \(\alpha(\mu; \delta)\) to \(\tau(\mu)\), the loss of effectivity in a lower bound of \(\tau(\mu)\) does not significantly affect the effectivity of the resulting lower bound of \(\alpha(\mu; \delta)\).

4.5. Eigenproblem residual bound form. In order to construct a computable lower bound of \(\lambda_1(\mu)\), we now develop a computable upper bound of the dual norm of the eigenproblem residual (22). Our approach is similar to the bounding technique developed in Section 3.1 to bound the dual norm of the (solution) residual. The following proposition provides a bound of the dual norm of the eigenproblem residual.

Proposition 10. For any \(\mu \in \mathcal{D}\), \(\tilde{z} \in \mathcal{V}\) such that \(\|\tilde{z}\|_\mathcal{W} = 1\), and \(\tilde{\lambda} \in \mathbb{R}\), the dual norm of the eigenproblem residual is bounded by
\[
\|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)\|_\mathcal{W} \leq (F_{\text{eig}}(\tilde{z}, \tilde{\lambda}, q; \mu))^{1/2} \quad \forall w \in \mathcal{Q},
\]
where the bound form is given by
\[
(23) \quad F_{\text{eig}}(\tilde{z}, \tilde{\lambda}, q; \mu) = \tilde{\lambda} (\|\nabla \cdot q + \tilde{z}\|_{L^2(\Omega)}^2 + \|q - \tilde{\lambda}^{-1} K(\mu) \nabla \tilde{z} + \nabla \tilde{z}\|_{L^2(\Omega)}^2 + \|q \cdot n - \tilde{z}\|_{L^2(\Gamma_N)}^2).
\]

Proof. We note that, for all \(\tilde{z} \in \mathcal{V}\), \(v \in \mathcal{V}\), \(\tilde{\lambda} \in \mathbb{R}\), and \(q \in \mathcal{Q}\),
\[
\begin{align*}
\text{r}_{\text{eig}}(v; \tilde{z}, \tilde{\lambda}; \mu) &= \int_\Omega \nabla v \cdot K(\mu) \nabla \tilde{z} dx - \tilde{\lambda} \int_\Omega (\nabla v \cdot \nabla \tilde{z} + v \tilde{z}) dx - \tilde{\lambda} \int_{\Gamma_N} v \tilde{z} ds \\
&= \int_\Omega \nabla v \cdot K(\mu) \nabla \tilde{z} dx - \tilde{\lambda} \int_\Omega (\nabla v \cdot \nabla \tilde{z} + v \tilde{z}) dx - \tilde{\lambda} \int_{\Gamma_N} v \tilde{z} ds \\
&= \tilde{\lambda} \int_\Omega v (- \tilde{z} - \nabla \cdot q) dx + \tilde{\lambda} \int_\Omega \nabla v \cdot (\tilde{\lambda}^{-1} K(\mu) \nabla \tilde{z} - \nabla \tilde{z} - q) dx - \tilde{\lambda} \int_{\Gamma_N} v (\tilde{z} - q \cdot n) ds \\
&\leq (\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Gamma_N)}^2)^{1/2} \\
&\quad \tilde{\lambda} (\|\nabla \cdot q + \tilde{z}\|_{L^2(\Omega)}^2 + \|q - \tilde{\lambda}^{-1} K(\mu) \nabla \tilde{z} + \nabla \tilde{z}\|_{L^2(\Omega)}^2 + \|q \cdot n - \tilde{z}\|_{L^2(\Gamma_N)}^2)^{1/2} \\
&= \|v\|_{\mathcal{W}(F_{\text{eig}}(\tilde{z}, \tilde{\lambda}, q; \mu))}^{1/2},
\end{align*}
\]
here, the second equality follows from the Green’s theorem. It thus follows
\[
\|r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)\|_\mathcal{W} \equiv \sup_{v \in \mathcal{V}} \frac{r_{\text{eig}}(\cdot; \tilde{z}, \tilde{\lambda}; \mu)}{\|v\|_{\mathcal{W}}} \leq (F_{\text{eig}}(\tilde{z}, \tilde{\lambda}, q; \mu))^{1/2},
\]
which is the desired inequality.
4.6. Finite element approximation of $\tau_{LB}(\mu)$. We now present a practical algorithm to compute $\tau_{LB}(\mu)$ for select values of $\mu$ in the offline stage. As before, we introduce conforming finite-element approximation spaces $V_1^{N_1} \subset V$ and $Q_\mu^{N_\mu} \subset Q$ defined in (10) and (11). We then seek the Galerkin finite element approximation of the eigenproblem (16): given $\mu \in D$, find $(z_1^N(\mu), \lambda_1^N(\mu)) \in V_1^{N_1} \times \mathbb{R}$ such that

$$a(z_1^N(\mu), v; \mu) = \lambda_1^N(\mu)(z_1^N(\mu), v)_W \quad \forall v \in V_1^{N_1}.$$  

We then compute a minimum-residual approximation $y^N(\mu) \in Q_\mu^{N_\mu}$ which minimizes the dual norm bound:

$$y^N(\mu) = \text{arg} \inf_{q \in Q_\mu^{N_\mu}} F_{\text{eig}}(z_1^N(\mu), \lambda_1^N(\mu), q; \mu).$$

Finally, assuming $|\lambda_1^N(\mu) - \lambda_1(\mu)| < |\lambda_1^N(\mu) - \lambda_2(\mu)|$, we construct a lower bound of $\tau(\mu)$,

$$\tau_{LB}(\mu) = \lambda_1^N(\mu) - (F_{\text{eig}}(z_1^N(\mu), \lambda_1^N(\mu), y^N(\mu); \mu) + 1)^{1/2} \leq \lambda_1(\mu).$$

We emphasize that the superscript $N$ on $\tau_{LB}(\mu)$ signifies that the lower bound approximation is computed in the $O(N)$-dimensional finite element space, and it does not signify that the lower bound is with respect to the finite element eigenproblem; the lower bound is indeed with respect to the infinite-dimensional eigenproblem (16).

4.7. Exact-space SCM: offline-online computation of $\tau_{LB,M}(\mu)$. We can summarize the exact-space SCM as follows. In the offline stage, we prepare the space

$$\hat{Y}_{LB}(\mu; \Xi_{con}) = \left\{ y \in \hat{B}_{Q_K} \mid \sum_{q=1}^Q \Theta^K_q(\mu') \geq \tau_{LB}^N(\mu'), \ \forall \mu' \in \Xi_{con} \right\},$$

where $\tau_{LB}(\mu')$, $\mu' \in \Xi_{con}$, are computed by (25) and $\hat{B}_{Q_K} = \prod_{q=1}^Q [\hat{\gamma}_q^-, \hat{\gamma}_q^+]$, where $\hat{\gamma}_q^\pm$, $q = 1, \ldots, Q_K$, are computed by (20) and (21).

In the online stage, we solve the linear programming problem

$$\tau_{LB,M}(\mu) = \inf_{y \in \hat{Y}_{LB}(\mu; \Xi_{con})} J(\mu; y),$$

which provide a lower bound of $\tau(\mu)$. We then compute a lower bound of the stability constant $\alpha(\mu; \delta)$,

$$\alpha_{LB,M}(\mu; \delta) = 1 + \frac{\delta}{\tau_{LB,M}(\mu)} \leq \alpha(\mu; \delta).$$  

The subscript $M$ on $\tau_{LB,M}(\mu)$ and $\alpha_{LB,M}(\mu; \delta)$ signifies the cardinality of the constraint set: $M \equiv |\Xi_{con}|$.

4.8. Galerkin reduced basis method: offline-online computation of $\tau_{UB,M}(\mu)$. While the lower bound of $\tau(\mu)$ is needed to construct error bounds, an upper bound of $\tau(\mu)$ is useful to assess the sharpness of our lower bound, especially in the context of adaptive mesh refinement and parameter sampling. The evaluation of an upper bound is significantly simpler than the evaluation of a lower bound; we may simply appeal to Galerkin projection onto a subspace.

At the finite element level, the $\lambda_1^N(\mu)$ associated with the eigenproblem (24) is an upper bound of the exact first eigenvalue $\lambda(\mu)$. The upper bound property follows from $\lambda_1(\mu) = \inf_{v \in V} \|v\|_W \leq \inf_{v \in V} \|v\|_{\mu} \leq \lambda_1^N(\mu)$ for $V_1^N \subset V$. We hence set $\tau_{UB}(\mu) = \lambda_1^N(\mu)$. We then readily compute the (relative) bound gap $|\tau_{UB}(\mu) - \tau_{LB}(\mu)|/\tau_{UB}(\mu)$, which serves as an error estimate.

At the reduced basis level, we first construct a reduced basis space

$$V_1^{\text{eig}} = \text{span}\{z_1^N(\mu')\}_{\mu' \in \Xi_{con}};$$

here in principle the snapshot parameters need not be the same as the SCM constraint set $\Xi_{con}$; however, in practice, we use the same set of parameters because the finite element snapshots $(z_1^N(\mu'), \lambda_1^N(\mu'))$, $\mu' \in \Xi_{con}$,
are generated as a byproduct of evaluating $\tau_{LB}^N(\mu')$, $\mu' \in \Xi_{\text{con}}$, as described in Section 4.6. We then solve a reduced basis eigenproblem: given $\mu \in \mathcal{D}$, find $(z_{M,1}(\mu), \lambda_{M,1}(\mu)) \in \mathcal{V}_M^{\text{eig}} \times \mathbb{R}$ such that

$$a(z_{M,1}(\mu), v; \mu) = \lambda_{M,1}(\mu) (z_{M,1}(\mu), v)_W \quad \forall v \in \mathcal{V}_M^{\text{eig}};$$

the Galerkin approximation is an upper bound of the exact first eigenvalue: we hence set $\tau_{UB,M}(\mu) = \lambda_{M,1}(\mu) > \lambda_1(\mu)$. We note that this reduced basis eigenproblem admits offline-online computational decomposition. Hence, the combination of $\tau_{UB,M}(\mu)$ from the reduced basis eigenproblem and $\tau_{LB,M}(\mu)$ from the SCM allows us to rapidly evaluate for any $\mu \in \mathcal{D}$ the (relative) bound gap $|\tau_{UB,M}(\mu) - \tau_{LB,M}(\mu)|/\tau_{UB,M}(\mu)$, which serves as an error estimate.

5. Error bounds.

5.1. Energy norm. Given our reduced basis approximation (13) to the PDE (1), our computable, online-efficient bound is given by,

$$\|u(\mu) - u_N(\mu)\| \leq \Delta_N(\mu) = \frac{1}{(\alpha_{LB,M}(\mu; \delta))^{1/2}} (F(u_N(\mu), p_N(\mu); \mu; \delta))^{1/2},$$

where $F(\cdot, \cdot; \cdot)$ is the bound form (5), and $\alpha_{LB,M}(\cdot; \cdot)$ is the lower bound of the stability constant associated with the SCM approximation (26).

5.2. Linear functional output. As discussed in Section 2.4, the energy-norm error bound may be readily extended to provide an error bound for linear functional outputs. To bound an output error in an offline-online efficient manner, we first compute a reduced basis approximation of the adjoint solution. We readily extend to provide an error bound for linear functional outputs. To bound an output error in an offline-online efficient manner, we first compute a reduced basis approximation of the adjoint solution. We next bound the dual norm of the adjoint residual by adopting the techniques described in Section 3 for the primal equation to the adjoint equation. We can then compute the output and provide an associated error bound — with respect to the exact output and not the finite element approximation — by appealing to Proposition 4.


6.1. Adaptive finite element for the stability eigenproblem. Although we employ the finite element approximation simply as a means to evaluate $\tau_{LB}^N(\mu')$, $\mu' \in \Xi_{\text{con}}$, required in the online stage, we nevertheless wish to compute $\tau_{LB}^N(\mu')$ efficiently and minimize the computational effort to achieve a given accuracy. Towards this end, we solve the eigenproblem using an adaptive finite element method.

In order to drive adaptive mesh refinement, we must define a elemental error indicator. We use the following elemental residual indicator for the finite element approximation of the eigenpair $(z_1^N(\mu), \lambda_1^N(\mu)) \in \mathcal{V}^{N_\kappa} \times \mathbb{R}$ and the associated dual field $y^N(\mu) \in Q^{N_\kappa}$:

$$\eta_{\text{eig,}\kappa} = (\lambda_1^N(\mu))^2 (\|\nabla \cdot y^N(\mu) + z_1^N(\mu)\|_{L_2(\kappa)}^2 + \|y^N(\mu) - (\lambda_1^N(\mu))^{-1} K(\mu) \nabla z_1^N(\mu) + \nabla z_1^N(\mu)\|_{L_2(\kappa)}^2) + \|y^N(\mu) \cdot n - z_1^N(\mu)\|_{L_2(\partial \kappa \cap \Gamma_N)}^2), \quad \kappa \in \mathcal{T}_h.$$ 

Note that $\eta_{\text{eig,}\kappa}$ is an elemental contribution to the eigenvalue bound gap in the sense that

$$\sum_{\kappa \in \mathcal{T}_h} \eta_{\text{eig,}\kappa} = F_{\text{eig}}(z_1^N(\mu), \lambda_1^N(\mu), y^N(\mu); \mu) = \tau_{UB}^N(\mu) - \tau_{LB}^N(\mu).$$

Once we identify elements with biggest contributions to the eigenvalue bound gap, we employ a fixed-fraction marking strategy to mark top 5% of the elements for refinement. We repeat the solve-mark-refine process until the desired bound-gap tolerance is met.

6.2. Spatio-parameter adaptive SCM. We select the SCM constraint points $\Xi_{\text{con}}$ and evaluate the associated constraints $(\tau_{LB}^N(\mu'))_{\mu' \in \Xi_{\text{con}}}$ using the adaptive algorithm described in Algorithm 1. As in the original SCM [7], we construct the constraint set $\Xi_{\text{con}} \subset \Xi_{\text{train}}$ by selecting the constraint parameters in a greedy manner based on the bound gap; however, we emphasize that, unlike the original SCM, this bound gap is associated with the stability constant with respect to the infinite-dimesional space $\mathcal{V}$. Once we identify
This choice ensures that the next constraint point, we then solve the eigenproblem using the adaptive finite element method described in Section (6.1). The finite element approximation that meets the tolerance requirement is then included in the updated reduced basis set $\mathcal{V}^\text{eig}_M$. The greedy procedure continues until the relative bound gap is smaller than the specified tolerance at all training points.

We make a few remarks regarding the choice of the inputs. First, as in the original SCM [7], we select a sufficiently rich training set $\Xi_{\text{train}}$ to cover the parameter domain $\mathcal{D}$. Second, in theory, we require that $\varepsilon_{\text{SCM}} \in (0, 1)$ and $\varepsilon_{\text{SCM, FE}} \leq \varepsilon_{\text{SCM}}$; in practice, we select $\varepsilon_{\text{SCM}} \in [0.5, 0.9]$ and set $\varepsilon_{\text{SCM, FE}} \approx \varepsilon_{\text{SCM}}/10$. Note that the bound gap tolerance $\varepsilon_{\text{SCM}}$ needs not be very tight thanks to the transformation of the stability constant described in Section 4.1, which desensitizes the effectivity of $\alpha_{\text{LB}, M}(\mu; \delta)$ from the effectivity of $\tau_{\text{LB}, M}(\mu)$ for $\delta$ sufficiently small.

6.3. Selection of $\delta$. While the residual bound introduced in Proposition 5 and the $\alpha(\mu; \delta)$-$\tau(\mu)$ relationship described in Proposition 7 hold for any $\delta \in \mathbb{R}_{\geq 0}$, we in practice select $\delta$ sufficiently small such that the effectivity of $\alpha_{\text{LB}, M}(\mu; \delta)$ is desensitized from the effectivity of $\tau_{\text{LB}, M}(\mu)$, which can be quite poor for the SCM. In practice, we set

$$\delta = \frac{1}{10} \inf_{\mu \in \Xi_{\text{train}}} \tau_{\text{LB}, M}(\mu).$$

This choice ensures that

$$\inf_{\mu \in \Xi_{\text{train}}} \alpha_{\text{LB}, M}(\mu; \delta) = \inf_{\mu \in \Xi_{\text{train}}} \left(1 + \frac{\delta}{\tau_{\text{LB}, M}(\mu)}\right)^{-1} \geq \frac{10}{11}.$$ 

The stability constant $\alpha_{\text{LB}, M}(\mu; \delta)$ is guaranteed to be close to unity for all $\mu \in \Xi_{\text{train}}$; in practice, $\alpha_{\text{LB}, M}(\mu; \delta)$ is close to unity for all $\mu \in \mathcal{D}$ given $\Xi_{\text{train}} \subset \mathcal{D}$ is sufficiently rich.

6.4. Adaptive finite element for the solution. Similar to the approximation of the stability eigenproblem considered in Section 6.1, we also employ an adaptive finite element method to approximate the solution $u(\mu)$. An element residual indicator associated with our finite element approximation $v^N(\mu) \in \mathcal{V}^N$
and \( p^N(\mu) \in Q^N \) is
\[
\eta_\kappa \equiv \| \nabla u^N(\mu) - K^{-1}(\mu) p^N(\mu) \|_{L^2(\kappa)}^2 + \delta^{-1} \| f(\mu) + \nabla \cdot p^N(\mu) \|_{L^2(\kappa)}^2
+ \delta^{-1} \| g(\mu) - p^N(\mu) \cdot n \|_{L^2(\Gamma_N)}^2, \quad \kappa \in \mathcal{T}_h.
\]
Note that \( \eta_\kappa \) is an element contribution to the residual in the sense that
\[
\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa = F(u^N(\mu), p^N(\mu); \mu; \delta).
\]
As in the adaptive procedure for the stability eigenproblem, we use a fixed-fraction marking strategy with a threshold of 5\% for our adaptive mesh refinement.

6.5. Spatio-parameter adaptive weak Greedy algorithm. We choose the reduced basis sampling points using the adaptive algorithm described in Algorithm 2. As in the standard weak Greedy algorithm [15], we choose the sampling point \( \mu_N \) in a greedy manner based on the error bound \( \Delta_N(\mu) \); however, unlike in the standard Greedy algorithm, this error bound is associated with the exact solution \( u(\mu) \in V \) as opposed to the finite element “truth” solution. Once we identify the next sampling parameter, we find the associated solution using the adaptive finite element method described in Section 6.4. We then augment our reduced basis spaces \( V_N \) and \( Q_N \) with the new snapshot. The greedy procedure is repeated until the error bound \( \Delta_N(\mu) \) is smaller than the specified at all training points.

We make a few remarks regarding the choice of the inputs. First, as in the standard weak Greedy [15], we select a sufficiently rich training set \( \Xi_{\text{train}} \) to cover the parameter domain \( D \); in practice we use the same training set as that used for the SCM. Second, in theory, we require that \( \epsilon_{\text{RB,FE}} \leq \epsilon_{\text{RB}} \); in practice, we set \( \epsilon_{\text{RB,FE}} \approx \epsilon_{\text{RB}} / 10 \). Finally, we note that \( \epsilon_{\text{RB}} \), and not \( \epsilon_{\text{RB,FE}} \), ultimately controls the accuracy of the reduced basis model.

6.6. Mesh refinement mechanics: working and common spaces. Our spatio-parameter adaptive algorithms require the computation of the inner product of two finite element fields associated with two different spaces. In order to compute various inner products, we employ two different types of meshes: a “working mesh” which is used to compute the current finite element solution \( u^N(\mu) \) and \( p^N(\mu) \) or the current finite element eigenfunction \( z^1_N(\mu) \) and \( y^N(\mu) \); a “common mesh” which is a superset of all working meshes.
used in the greedy algorithm. Whenever we need to compute the inner product of two fields associated with two different spaces, we re-represent the fields on the common space and then perform the inner product algebraically.

**Remark 11.** In our current implementation, we keep a single common mesh and perform inner product between any two finite element fields on the common mesh. Alternatively, we could compute a common mesh that is a superset of just two fields whose inner product we wish to evaluate [6]. This latter approach could reduce the computational cost, especially if the finite element mesh required for various parameter values differ considerably; on the other hand, the approach would require multiple solutions of the mesh intersection problem.

6.7. Remarks on the spatio-parameter adaptive algorithm. We make two important remarks about the reduced basis model constructed by our spatio-parameter adaptive algorithm. First, for any parameter \( \mu \in \mathcal{D} \), regardless of whether it belongs to \( \Xi_{\text{train}} \), our reduced model provides an error bound \( \Delta_N(\mu) \) with respect to the exact solution \( u(\mu) \in \mathcal{V} \), and not the typical finite element “truth” solution, in complexity independent of the size of the finite element spaces. Second, if the parameter \( \mu \) belongs to the training set \( \Xi_{\text{train}} \), then the error bound \( \Delta_N(\mu) \) is guaranteed to be less than the training tolerance: \( \Delta_N(\mu) \leq \epsilon_{\text{RB}}, \forall \mu \in \Xi_{\text{train}} \). In other words, we certify the reduced basis approximation for any \( \mu \in \mathcal{D} \), and the certificate is less than \( \epsilon_{\text{RB}} \) for \( \mu \in \Xi_{\text{train}} \).

We also note that the proposed algorithm is fundamentally different from the following two-step approach: i) we invoke the standard reduced-basis Greedy algorithm with an error bound with respect to the finite element “truth” to identify the worst case; ii) we compute the reduced basis snapshot using an adaptive finite element method. In the online stage, this two step approach would only provide an error bound with respect to the finite element “truth” for a general \( \mu \in \mathcal{D} \), and an error bound with respect to the exact solution for snapshot parameters. Hence, the two-step approach would not certify the reduced basis approximation with respect to the exact solution for an arbitrary \( \mu \in \mathcal{D} \).


7.1. Problem description. We demonstrate the proposed reduced basis method for the thermal block problem [15]. The problem is defined on a unit square domain, \( \Omega \equiv [0, 1]^2 \), shown in Figure 1, and a \( P = 9 \)-dimensional parameter domain \( \mathcal{D} = [10^{-1/2}, 10^{1/2}]^9 \subset \mathbb{R}^9 \). The weak statement for the problem is the following: given \( \mu \in \mathcal{D} \), find \( u(\mu) \in \mathcal{V} \equiv \{ v \in H^1(\Omega) \mid v|_{\Gamma_{\text{top}}} = 0 \} \) such that

\[
\sum_{i=1}^{P} \int_{\Omega} \nabla v \cdot \mu_i \nabla w \, dx = \int_{\Gamma_{\text{bottom}}} v \, ds \quad \forall v \in \mathcal{V},
\]

and then evaluate the compliance output

\[
s(\mu) = \int_{\Gamma_{\text{bottom}}} u(\mu) \, ds.
\]

Note that we impose homogeneous Dirichlet boundary condition on \( \Gamma_{\text{top}} \), homogeneous Neumann boundary condition on \( \Gamma_{\text{side}} \), and inhomogeneous Neumann condition on \( \Gamma_{\text{bottom}} \). The equation satisfies all assumptions described in Section 2.1: it is coercive for all \( \mu \in \mathcal{D} \), the diffusivity field admits affine decomposition, the inverse of the diffusivity admits affine decomposition, and all fields are piecewise polynomial.

7.2. Spatio-parameter greedy training of the stability constant. We first note that, for the thermal block problem, we readily obtain \( \gamma_q^- = 0 \) and \( \gamma_q^+ = 1 \) for \( q = 1, \ldots, Q_K \). To select the SCM constraint set \( \Xi_{\text{con}} \), we invoke the spatio-parameter adaptive SCM algorithm, Algorithm 1, with the following inputs. We choose a training set \( \Xi_{\text{train}} \) that contains \( 2^P = 512 \) corner points of the nine-dimesional parameter domain \( \mathcal{D} \), 2000 random points in \( \mathcal{D} \), and the midpoint of \( \mathcal{D} \); the total size of the training set is 2513. We choose the SCM and finite element relative bound gap tolerance of \( \epsilon_{\text{SCM}} = 0.8 \) and \( \epsilon_{\text{SCM, FE}} = 0.002 \), respectively. We choose a \( 6 \times 6 \) \( P \times P \) finite element space as the initial approximation space. We note that this initial mesh is somewhat finer than the initial mesh that will be used for the reduced basis training; this initial refinement is intended to ensure that \( \lambda_N(\mu) \) even on the initial mesh is closer to \( \lambda_1(\mu) \) than to \( \lambda_2(\mu) \), though again we have no means to verify if the condition is met.
The result of the training is summarized in Figures 2. We require $M = 10$ SCM constraints to meet the relative bound gap threshold of $\epsilon_{SCM} = 0.8$. Figure 2(a) shows the size of working meshes used for the $M = 10$ eigenproblems solved; the size of the approximation space varies from $N = 216$ for the first parameter to $N = 1066$ for the second parameter. Figure 2(b) shows the final common mesh for $M = 10$. The common space is of size $N = 2568$ and is refined towards singularities that are present for some of the training parameters; it is also worth noting that the smallest element has the edge length of $2^{-8}/3$, which implies that a uniform mesh with the same resolution at the singularity would require over $2 \times 10^6$ degrees of freedom. Figure 2(c) shows the convergence of the bound gap with the cardinality of the SCM constraint set $M$; the bound gap decreases slowly initially, but for $M = 10$ the relative bound gap is $\approx 0.68 < \epsilon_{SCM}$. Figure 2(d) shows the distribution of the lower and upper bound of $\tau(\mu)$ for the 2513 training points; the minimum $\tau(\mu)$ over the training set is $\approx 0.1$. We will hence set $\delta = 0.09$ in our minimum-residual reduced basis method.

Figure 3 summarizes the behavior of the adaptive finite element method for the stability eigenproblem for two of the cases. The first case, shown in Figures 3(a)–3(c), is associated with the second SCM constraint $\mu^{(2)}$, which is the most difficult case in terms of $N$ required to meet the bound gap tolerance at $N = 1066$. The diffusivity is $10^{-1/2}$ for bottom middle block ($\Omega_2$) and is $10^{1/2}$ for the remaining blocks. Figure 3(a) shows the first eigenfunction. Figure 3(b) shows the associated final adapted mesh. We observe that the adaptive finite element targets the bottom middle boundary as well as the two singularities at two of the corners of $\Omega_2$; we also note that the smallest element has the edge length of $2^{-8}/3$, implying that a uniform refinement that achieves the same resolution at the singularity would require over $2 \times 10^6$ as opposed to $N = 1066$ for the adaptive method. Figure 3(c) shows the evolution of the upper and lower bound of the first eigenvalue, $\tau^{SCM}_{UB}(\mu^{(2)})$ and $\tau^{SCM}_{LB}(\mu^{(2)})$, with finite element adaptation; we meet the desired relative bound gap of $\epsilon_{SCM,FE} = 0.002$ after 11 adaptation iterations.

The second case, shown in Figures 3(d)–3(f), is associated with the fifth SCM constraint $\mu^{(5)}$, which is one of the easier cases in terms of $N$ required to meet the bound gap tolerance at $N = 236$. The diffusivity is $10^{-1/2}$ for the middle block ($\Omega_2$) and is $10^{1/2}$ for the remaining blocks. Figure 3(d) shows the first eigenfunction. Figure 3(e) shows the associated final adapted mesh; we observe that only the middle block is refined, and the refinement is not as aggressive as that observed for $\mu^{(2)}$ in Figure 3(b). Figure 3(f) shows the evolution of the upper and lower bound of the first eigenvalue; we need only three adaptation iterations to meet the target relative bound gap tolerance.

### 7.3. Spatio-parameter greedy training of reduced basis

We now train our reduced basis using the spatio-parameter adaptive Greedy algorithm, Algorithm 2, with the following inputs. We choose a training set $\Xi_{train}$ that is identical to that used for the SCM training described in Section 7.2. We choose the target relative output error tolerance of $\epsilon_{RB} = 0.01$. We choose the finite element relative output error tolerance of $\epsilon_{RB,FE} = 0.002$. We choose a $3 \times 3$ $p^2$ finite element space as the initial approximation space. The weight parameter, as discussed in Section 7.2, is $\delta = 0.09 < (1/10) \min_{\mu \in \Xi_{train}, N} \tau_{LB,M}(\mu)$.

Figure 4 summarizes the behavior of the spatio-parameter Greedy algorithm. The total of $N = 26$
reduced basis functions are required to meet the relative error tolerance of 0.01. Figure 4(a) shows the variation in the size of the working mesh used to approximate the solution for the 26 parameter values; the size of the problem varies from $\mathcal{N} = 126$ for the first parameter to $\mathcal{N} = 4868$ for the ninth parameter. Figure 4(b) shows the convergence of the relative error bound with the size of the reduced basis space $N$; we observe that the error bound decreases exponentially with $N$. Figure 4(c) shows the final common mesh for $N = 26$. The common mesh is of size $\mathcal{N} = 8264$ and is aggressively refined towards singularities that are present for some of the training parameters; the edge length of the smallest element is $2^{-10/3}$, which implies that a uniform mesh with the same resolution at the singularity would require $\approx 4 \times 10^7$ degrees of freedom.

Figure 5 summarizes the behavior of the adaptive finite element method for two cases. The first case, shown in Figure 5(a)–5(c), is associated with the fourth reduced basis function $\mu^{(4)}$, which is one of the easiest cases requiring only $N = 230$ to meet the error tolerance. Figure 5(a) shows the solution. Figure 5(b) shows the associated final adapted mesh; the mesh is largely unchanged from the initial coarse mesh aside from some refinement in the bottom right region. Figure 5(c) shows the error convergence; we need only two adaptation iterations to meet the error tolerance. We also observe that the effectivity of the error bound is $\approx 2$, which is quite tight; here, for comparison purposes, the reference solution is computed using the adaptive finite element with an error tolerance 100 times smaller than the target error tolerance of 0.002.

The second case, shown in Figure 5(d)–5(f), is associated with the fourth reduced basis function $\mu^{(9)}$, which is the most difficult case in terms of $\mathcal{N}$ required to meet the error tolerance at $\mathcal{N} = 4868$. Figure 5(d)
8. Summary. We present a reduced basis method for parametrized coercive equations with two objectives: providing error bounds with respect to the exact weak solution in an infinite-dimensional space; providing reliable and efficient construction of a reduced basis model through adaptivity in both physical and parameter spaces. The proposed method builds on two key ingredients: a minimum-residual mixed formulation; an extension of the SCM to the infinite-dimensional function space. Both ingredients build
Fig. 4. Spatio-parameter greedy reduced-basis training.

REFERENCES


Fig. 5. Adaptive finite element approximation of the solution.