4. Method of Moments for 1D Kinetic Theory

Coverage of this section:

- Representative One-Dimensional Kinetic Equation
- Fluid-Limit Solutions
- Grad's Method of Moments
- Maximum-Entropy Method of Moments
- Quadrature-Based Method of Moments
- Other Moment Approximations
- Applications

4.1 Representative One-Dimensional Kinetic Equation

Here we will consider a univariate kinetic model describing a "1D gas" where the NDF, $\mathcal{F} = \mathcal{F}(x, v, t)$, is dependent on a single random velocity variable, v, and a relaxation-time or BGK approximation (Bhatnagar *et al.*, 1954) for the collision operator:

4.1.1 Univariate BGK Kinetic Equation

$$\frac{\partial \mathcal{F}}{\partial t} + v\frac{\partial \mathcal{F}}{\partial x} = \frac{\delta \mathcal{F}}{\delta t} = -\frac{\mathcal{F} - \mathcal{M}}{\tau},$$

BGK Collision Operator

$$\frac{\delta \mathcal{F}}{\delta t} = -\frac{\mathcal{F} - \mathcal{M}}{\tau}, \qquad \tau = \text{constant}$$

4.1 Representative One-Dimensional Kinetic Equation

4.1.2 Moments of NDF

The macroscopic moment, $M(x,t)$, for velocity weight, V, is then defined as

$$M(x,t) = \int_{-\infty}^{\infty} V(v)\mathcal{F}(x,v,t)\,\mathrm{d}v$$
$$= \int_{\infty} V(v)\mathcal{F}(x,v,t)\,\mathrm{d}v = \langle\, V(v)\mathcal{F}\,\rangle\ ,$$

where, in this case, $V(v)$ is a velocity-dependent weight which in general is a polynomial (usually a monomial) in v.

4.1 Representative One-Dimensional Kinetic Equation

4.1.3 Maxwell's Equation of Change (Conservation Form)

By evaluating the appropriate total velocity moments of the univariate kinetic equation, the conservation form of Maxwell's equation of change for the macroscopic moment, $M(x,t)$, in the univariate case can then be derived and written as

$$\frac{\partial}{\partial t}(M) + \frac{\partial}{\partial x}\langle\, v\, V(v)\mathcal{F}\,\rangle = \frac{\partial}{\partial t}(M) + \frac{\partial}{\partial x}(F)$$
$$= -\frac{1}{\tau}\left[M - \langle\, V(v)\mathcal{M}\,\rangle\right]\ ,$$

and the moment flux, F, which will require closure, is given by

$$F = \langle\, v\, V(v)\mathcal{F}\,\rangle\ .$$

4.1 Representative One-Dimensional Kinetic Equation

4.1.4 Total Velocity Moments

Some common total velocity moments:

$$V = m : \quad m \langle \mathcal{F} \rangle = mn = \rho \,,$$

$$V = mv : \quad m \langle v\mathcal{F} \rangle = \rho u \,,$$

$$V = mv^2 : \quad m \left\langle v^2\mathcal{F} \right\rangle = \rho u^2 + p = \rho u^2 + \rho a^2 \,,$$

$$V = mv^3 : \quad m \left\langle v^3\mathcal{F} \right\rangle = \rho u^3 + 3up + q \,,$$

$$V = mv^4 : \quad m \left\langle v^4\mathcal{F} \right\rangle = \rho u^4 + 6u^2 p + 4uq + r \,,$$

$$V = mv^5 : \quad m \left\langle v^5\mathcal{F} \right\rangle = \rho u^5 + 10u^3 p + 10u^2 q + 5ur + s \,,$$

where $a^2 = p/\rho$.

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4.1 Representative One-Dimensional Kinetic Equation

4.1.5 Random Velocity Moments

Using the definition of the mean velocity

$$u = \frac{m \langle v\mathcal{F} \rangle}{m \langle \mathcal{F} \rangle} \,,$$

and letting

$$c = v - u \,,$$

where c is the random gas velocity, the following random velocity or central moments can be defined:

$$V = m : \quad m \langle \mathcal{F} \rangle = mn = \rho \,,$$

$$V = mc : \quad m \langle c\mathcal{F} \rangle = 0 \,,$$

$$V = mc^2 : \quad m \left\langle c^2\mathcal{F} \right\rangle = p = \rho a^2 \,,$$

4.1 Representative One-Dimensional Kinetic Equation

$$V = mc^3 : \quad m\left\langle c^3\mathcal{F} \right\rangle = q \,,$$
$$V = mc^4 : \quad m\left\langle c^4\mathcal{F} \right\rangle = r \,,$$
$$V = mc^5 : \quad m\left\langle c^5\mathcal{F} \right\rangle = s \,.$$

Can then define the corresponding kinetic equation in terms of c:

Univariate BGK Kinetic Equation

$$\frac{\partial \mathcal{F}}{\partial t} + (u+c)\frac{\partial \mathcal{F}}{\partial x} - \left[\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x}\right]\frac{\partial \mathcal{F}}{\partial c} = \frac{\delta \mathcal{F}}{\delta t} = -\frac{\mathcal{F}-\mathcal{M}}{\tau} \,.$$

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4.1 Representative One-Dimensional Kinetic Equation

4.1.6 Maxwell's Equation of Change (Non-Conservation Form)

By evaluating the appropriate random velocity moments of the univariate kinetic equation, the non-conservation form of Maxwell's equation of change for the macroscopic moment, $M_\circ(x,t)$, correspond to velocity-dependent weight, $V(c)$, can also be derived and written as

$$\frac{\partial}{\partial t}(M_\circ) + \frac{\partial}{\partial x}(uM_\circ) + \frac{\partial}{\partial x}\left[\langle\, cV(c)\mathcal{F}\,\rangle\right] + \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\left[\left\langle \frac{\partial V}{\partial c}\mathcal{F} \right\rangle\right] + \frac{\partial u}{\partial x}\left[\left\langle c\frac{\partial V}{\partial c}\mathcal{F} \right\rangle\right] = -\frac{1}{\tau}\left[M_\circ - \langle\, V(c)\mathcal{M}\,\rangle\right] ,$$

where an expression is required for $\langle\, cV(c)\mathcal{F}\,\rangle$ for closure.

4.1 Representative One-Dimensional Kinetic Equation

4.1.7 Equilibrium Solution for NDF

The equilibrium solution for the NDF corresponding to thermodynamic equilibrium in this case is

$$\begin{aligned}\mathcal{F}(x,c,t) &= \mathcal{M}(x,c,t)\\ &= \frac{\rho}{m\sqrt{2\pi}a}\exp\left[-\frac{(v-u)^2}{2a^2}\right]\\ &= \frac{\rho}{m\sqrt{2\pi}a}\exp\left(-\frac{c^2}{2a^2}\right)\\ &= \frac{\rho}{m\sqrt{2\pi\theta}}\exp\left(-\frac{c^2}{2\theta}\right),\end{aligned}$$

where $\theta = a^2 = p/\rho$.

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4.1.7 Equilibrium Solution for NDF

The collisional invariants are again associated with mass, momentum and energy:

Collisional Invariants

$$m\left\langle \frac{\delta\mathcal{F}}{\delta t}\right\rangle = -\frac{m}{\tau}\langle\,(\mathcal{F}-\mathcal{M})\,\rangle = 0\,,$$

$$m\left\langle v\frac{\delta\mathcal{F}}{\delta t}\right\rangle = -\frac{m}{\tau}\langle\, v\,(\mathcal{F}-\mathcal{M})\,\rangle = 0\,,$$

$$m\left\langle v^2\frac{\delta\mathcal{F}}{\delta t}\right\rangle = -\frac{m}{\tau}\langle\, v^2\,(\mathcal{F}-\mathcal{M})\,\rangle = 0\,,$$

where $\mathcal{F}$ and $\mathcal{M}$ share the same moment up to order two.

4.1.7 Equilibrium Solution for NDF

Other Collision Terms

Other collision terms up to order five include

$$m\left\langle v^3 \frac{\delta \mathcal{F}}{\delta t} \right\rangle = -\frac{m}{\tau} \langle v^3 (\mathcal{F} - \mathcal{M}) \rangle = -\frac{q}{\tau} ,$$

$$m\left\langle v^4 \frac{\delta \mathcal{F}}{\delta t} \right\rangle = -\frac{m}{\tau} \langle v^4 (\mathcal{F} - \mathcal{M}) \rangle = -\frac{1}{\tau} \left[4uq + \left(r - 3\frac{p^2}{\rho}\right)\right] ,$$

$$m\left\langle v^5 \frac{\delta \mathcal{F}}{\delta t} \right\rangle = -\frac{m}{\tau} \langle v^5 (\mathcal{F} - \mathcal{M}) \rangle$$

$$= -\frac{1}{\tau} \left[10u^2 q + 5u \left(r - 3\frac{p^2}{\rho}\right) + s\right] .$$

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4.1 Representative One-Dimensional Kinetic Equation

4.1.8 Entropy Balance Equation

As with its 3D counterpart, the 1D kinetic theory also satisfies an equivalent H theorem which states that

$$-K\left\langle \ln \mathcal{F} \frac{\delta \mathcal{F}}{\delta t} \right\rangle \geq 0 ,$$

is positive semi-definite (K here is an appropriately chosen constant). Defining the so-called physical entropy as

$$s(\mathcal{F}) = -\frac{K}{\rho} \langle \mathcal{F} \ln \mathcal{F} \rangle ,$$

we then also have

$$\frac{\partial}{\partial t}(\rho s) - K\frac{\partial}{\partial x}\langle v\mathcal{F} \ln \mathcal{F}\rangle = -K\left\langle K \ln \mathcal{F} \frac{\delta \mathcal{F}}{\delta t} \right\rangle \geq 0 ,$$

showing that the entropy of the 1D system is a monotonically increasing function of time.

4.2 Fluid-Limit Solutions

4.2.1 Local Equilibrium Solution and "Euler" Equations

Assuming the gas is everywhere in local thermodynamic equilibrium, the NDF can be approximated as

$$\mathcal{F}(x, v, t) \approx \mathcal{M}(x, v, t; \rho, u, p)\,.$$

4.2.1 Local Equilibrium Solution and "Euler" Equations

The equilibrium NDF, $\mathcal{M}(x, v, t; \rho, u, p)$, is determined by solving the one-dimensional equivalent of the Euler equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0\,,$$

$$\frac{\partial}{\partial t}\left(\rho u^2 + p\right) + \frac{\partial}{\partial x}\left(\rho u^3 + 3up\right) = 0\,,$$

where the 1D "heat transfer" is taken to vanish:

$$q = m \left\langle c^3 \mathcal{F} \right\rangle \approx m \left\langle c^3 \mathcal{M} \right\rangle = 0\,.$$

4.2.1 Local Equilibrium Solution and "Euler" Equations

Note that the random velocity moments of the 1D Maxwell-Boltzmann distribution, $\mathcal{M}$, are

$$m\langle \mathcal{M} \rangle = \rho\,,$$
$$m\langle c\mathcal{M} \rangle = 0\,,$$
$$m\left\langle c^2\mathcal{M} \right\rangle = p = \rho a^2\,,$$
$$m\left\langle c^3\mathcal{M} \right\rangle = 0\,,$$
$$m\left\langle c^4\mathcal{M} \right\rangle = 3\frac{p^2}{\rho} = 3\rho a^4\,,$$
$$m\left\langle c^5\mathcal{M} \right\rangle = 0\,,$$
$$m\left\langle c^6\mathcal{M} \right\rangle = 15\frac{p^3}{\rho^2} = 15\rho a^6\,,$$
$$m\left\langle c^7\mathcal{M} \right\rangle = 0\,.$$
$$m\left\langle c^8\mathcal{M} \right\rangle = 105\frac{p^4}{\rho^3} = 15\rho a^8\,,$$
$$m\left\langle c^9\mathcal{M} \right\rangle = 0\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The one-dimensional equivalent to the Navier-Stokes equations can again be derived via a Chapman-Enskog expansion applied to the univariate kinetic equation. As in the fully three-dimensional case, here solutions of the unscaled kinetic equation given by

$$\frac{\partial \mathcal{F}}{\partial t} + v\frac{\partial \mathcal{F}}{\partial x} = -\frac{\mathcal{F}-\mathcal{M}}{\tau}\,,$$

are sought having the form

$$\mathcal{F} = \mathcal{M}\left(f^{(0)} + f^{(1)} + f^{(2)} + f^{(3)} + \dots\right)\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The corresponding scaled kinetic equation and scaled solution are then given by

$$\frac{\partial \mathcal{F}}{\partial t} + v\frac{\partial \mathcal{F}}{\partial x} = -\frac{\mathcal{F} - \mathcal{M}}{\epsilon \tau},$$

and

$$\mathcal{F} = \mathcal{M}\left(f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} + \ldots\right).$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Substituting the scaled solution into the scaled kinetic equation and collecting terms of equal order in ϵ yields

$$\begin{aligned}
\mathcal{M}\frac{\left(f^{(0)} - 1\right)}{\tau} &+ \epsilon\left[\frac{\partial}{\partial t}\left(f^{(0)}\mathcal{M}\right) + v\frac{\partial}{\partial x}\left(f^{(0)}\mathcal{M}\right) + \frac{f^{(1)}\mathcal{M}}{\tau}\right] \\
&+ \epsilon^2\left[\frac{\partial}{\partial t}\left(f^{(1)}\mathcal{M}\right) + v\frac{\partial}{\partial x}\left(f^{(1)}\mathcal{M}\right) + \frac{f^{(2)}\mathcal{M}}{\tau}\right] \\
&+ \epsilon^3\left[\frac{\partial}{\partial t}\left(f^{(2)}\mathcal{M}\right) + v\frac{\partial}{\partial x}\left(f^{(2)}\mathcal{M}\right) + \frac{f^{(3)}\mathcal{M}}{\tau}\right] \\
&+ \cdots \\
&= 0\,.
\end{aligned}$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

As with the fully three-dimensional kinetic system, to zeroth order in the small parameter, ϵ, the solution of the univariate kinetic equation must satisfy

$$\mathcal{M}\frac{\left(f^{(0)}-1\right)}{\tau}=0\,,$$

yielding

$$f^{(0)}=1\,,$$

and

$$\mathcal{F}\approx\mathcal{M}\,,$$

and the equivalent Euler solution is recovered as described above.

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The first-order correction, $f^{(1)}$, to the zeroth-order solution must satisfy

$$\frac{\partial\mathcal{M}}{\partial t}+v\frac{\partial\mathcal{M}}{\partial x}+\frac{f^{(1)}\mathcal{M}}{\tau}=0\,,$$

yielding

$$f^{(1)}=-\frac{\tau}{\mathcal{M}}\left[\frac{\partial\mathcal{M}}{\partial t}+v\frac{\partial\mathcal{M}}{\partial x}\right]\,,$$

where to first-order in ϵ the NDF, $\mathcal{F}$, is approximated by

$$\mathcal{F}\approx\mathcal{M}\left(1+f^{(1)}\right)\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

For consistency with the zeroth-order solution, it is required that

$$m\left\langle f^{(1)}\mathcal{M}\right\rangle = 0\,,$$

$$m\left\langle vf^{(1)}\mathcal{M}\right\rangle = 0\,, \quad m\left\langle cf^{(1)}\mathcal{M}\right\rangle = 0\,,$$

$$m\left\langle v^2f^{(1)}\mathcal{M}\right\rangle = 0\,, \quad m\left\langle c^2f^{(1)}\mathcal{M}\right\rangle = 0\,.$$

such that

$$m\left\langle (1+f^{(1)})\mathcal{M}\right\rangle = \rho\,,$$

$$m\left\langle v(1+f^{(1)})\mathcal{M}\right\rangle = \rho u\,, \quad m\left\langle c(1+f^{(1)})\mathcal{M}\right\rangle = 0\,,$$

$$m\left\langle v^2(1+f^{(1)})\mathcal{M}\right\rangle = \rho u^2 + p\,, \quad m\left\langle c^2(1+f^{(1)})\mathcal{M}\right\rangle = p\,.$$

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4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Substitution of the first-order approximation for the NDF into the unscaled univariate kinetic equation yields

$$\frac{\partial}{\partial t}\left[(1+f^{(1)})\mathcal{M}\right] + v\frac{\partial}{\partial x}\left[(1+f^{(1)})\mathcal{M}\right] = -\frac{f^{(1)}\mathcal{M}}{\tau}\,,$$

or

$$\frac{\partial}{\partial t}\left[(1+f^{(1)})\mathcal{M}\right] + (u+c)\frac{\partial}{\partial x}\left[(1+f^{(1)})\mathcal{M}\right] - \left[\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x}\right]\frac{\partial}{\partial c}\left[(1+f^{(1)})\mathcal{M}\right] = -\frac{f^{(1)}\mathcal{M}}{\tau}\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Using the former, the continuity equation for the first-order solution can be derived by evaluating the zeroth-order velocity moment, $V = m$, to obtain

$$\left\langle m\frac{\partial}{\partial t}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle + \left\langle mv\frac{\partial}{\partial x}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle = -\left\langle m\frac{f^{(1)}\mathcal{M}}{\tau}\right\rangle,$$

or

$$\frac{\partial}{\partial t}\left[m\left\langle (1+f^{(1)})\mathcal{M}\right\rangle\right] + \frac{\partial}{\partial x}\left[m\left\langle v(1+f^{(1)})\mathcal{M}\right\rangle\right] = -\frac{m}{\tau}\left\langle f^{(1)}\mathcal{M}\right\rangle,$$

yielding the 1D form of the continuity equation

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) = 0\,.$$

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4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

A similar procedure can be applied for the momentum equation of the first-order solution. Evaluating the first-order velocity moment, $V = mv$, of the unscaled kinetic equation, we can write

$$\left\langle mv\frac{\partial}{\partial t}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle + \left\langle mv^2\frac{\partial}{\partial x}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle = -\left\langle mv\frac{f^{(1)}\mathcal{M}}{\tau}\right\rangle,$$

or

$$\frac{\partial}{\partial t}\left[m\left\langle v(1+f^{(1)})\mathcal{M}\right\rangle\right] + \frac{\partial}{\partial x}\left[m\left\langle v^2(1+f^{(1)})\mathcal{M}\right\rangle\right] = -\frac{m}{\tau}\left\langle vf^{(1)}\mathcal{M}\right\rangle,$$

and this yields the 1D form of the momentum equation

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Finally, by evaluating the second-order velocity moment, $V = mv^2$, for the unscaled kinetic equation, we can write

$$\left\langle mv^2 \frac{\partial}{\partial t}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle + \left\langle mv^3 \frac{\partial}{\partial x}\left[(1+f^{(1)})\mathcal{M}\right]\right\rangle = -\left\langle mv^2 \frac{f^{(1)}\mathcal{M}}{\tau}\right\rangle,$$

or

$$\frac{\partial}{\partial t}\left[m\left\langle v^2(1+f^{(1)})\mathcal{M}\right\rangle\right] + \frac{\partial}{\partial x}\left[m\left\langle v^3(1+f^{(1)})\mathcal{M}\right\rangle\right] = -\frac{m}{\tau}\left\langle v^2 f^{(1)}\mathcal{M}\right\rangle,$$

and this yields the 1D form of the momentum equation

$$\frac{\partial}{\partial t}\left(\rho u^2 + p\right) + \frac{\partial}{\partial x}\left[m\left\langle v^3(1+f^{(1)})\mathcal{M}\right\rangle\right] = 0\,,$$

where the third moment $m\left\langle v^3(1+f^{(1)})\mathcal{M}\right\rangle$ requires evaluation.

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Letting $v = u + c$ have

$$v^3 = (u+c)^3 = u^3 + 3u^2c + 3uc^2 + c^3\,,$$

and thus

$$\begin{aligned} m\left\langle v^3(1+f^{(1)})\mathcal{M}\right\rangle &= m\left\langle \left(u^3 + 3u^2c + 3uc^2 + c^3\right)\left(1+f^{(1)}\right)\mathcal{M}\right\rangle \\ &= u^3 m\left\langle \left(1+f^{(1)}\right)\mathcal{M}\right\rangle + 3u^2 m\left\langle c\left(1+f^{(1)}\right)\mathcal{M}\right\rangle \\ &\quad + 3um\left\langle c^2\left(1+f^{(1)}\right)\mathcal{M}\right\rangle + m\left\langle c^3\left(1+f^{(1)}\right)\mathcal{M}\right\rangle \\ &= \rho u^2 + 3up + m\left\langle c^3\left(1+f^{(1)}\right)\mathcal{M}\right\rangle \\ &= \rho u^2 + 3up + m\left\langle c^3\mathcal{M}\right\rangle + m\left\langle c^3 f^{(1)}\mathcal{M}\right\rangle \\ &= \rho u^2 + 3up + m\left\langle c^3 f^{(1)}\mathcal{M}\right\rangle\,. \end{aligned}$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Defining the heat flux, q, as

$$q = m \left\langle \, c^3 f^{(1)} \mathcal{M} \, \right\rangle \, ,$$

the 1D form of the energy equation can finally be expressed as

$$\frac{\partial}{\partial t}\left(\rho u^2 + p\right) + \frac{\partial}{\partial x}\left(\rho u^2 + 3up + q\right) = 0 \, ,$$

where a knowledge of the form for the first-order correction to the NDF, $f^{(1)}$, is required to evaluate q.

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4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Returning to

$$f^{(1)} = -\frac{\tau}{\mathcal{M}}\left[\frac{\partial \mathcal{M}}{\partial t} + v\frac{\partial \mathcal{M}}{\partial x}\right] \, ,$$

and noting that $\mathcal{M} = \mathcal{M}(\rho, u, \theta)$, we can write

$$\begin{aligned}
f^{(1)} &= -\frac{\tau}{\mathcal{M}}\left[\frac{\partial \mathcal{M}}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial u}\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial \theta}\left(\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial x}\right)\right] \\
&= -\frac{\tau}{\mathcal{M}}\left[\frac{\partial \mathcal{M}}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + [u+c]\frac{\partial \rho}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial u}\left(\frac{\partial u}{\partial t} + [u+c]\frac{\partial u}{\partial x}\right)\right. \\
&\qquad \left. + \frac{\partial \mathcal{M}}{\partial \theta}\left(\frac{\partial \theta}{\partial t} + [u+c]\frac{\partial \theta}{\partial x}\right)\right] \\
&= -\frac{\tau}{\mathcal{M}}\left[\frac{\partial \mathcal{M}}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + [u+c]\frac{\partial \rho}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial u}\left(\frac{\partial u}{\partial t} + [u+c]\frac{\partial u}{\partial x}\right)\right. \\
&\qquad \left. + \frac{1}{\rho}\frac{\partial \mathcal{M}}{\partial \theta}\left(\frac{\partial p}{\partial t} + [u+c]\frac{\partial p}{\partial x}\right) - \frac{p}{\rho^2}\frac{\partial \mathcal{M}}{\partial \theta}\left(\frac{\partial \rho}{\partial t} + [u+c]\frac{\partial \rho}{\partial x}\right)\right] \, ,
\end{aligned}$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The preceding can be re-organized as

$$f^{(1)} = -\frac{\tau}{\mathcal{M}}\left[\left(\frac{\partial \mathcal{M}}{\partial \rho} - \frac{p}{\rho^2}\frac{\partial \mathcal{M}}{\partial \theta}\right)\left(\frac{\partial \rho}{\partial t} + [u+c]\frac{\partial \rho}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial u}\left(\frac{\partial u}{\partial t} + [u+c]\frac{\partial u}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \mathcal{M}}{\partial \theta}\left(\frac{\partial p}{\partial t} + [u+c]\frac{\partial p}{\partial x}\right)\right].$$

The time derivatives in the convective derivatives of ρ, u, and p above can then be replaced using the expressions from the zeroth-order moment equations (i.e., the equivalent Euler equations) to express $f^{(1)}$ as

$$f^{(1)} = -\frac{\tau}{\mathcal{M}}\left[\left(\frac{\partial \mathcal{M}}{\partial \rho} - \frac{p}{\rho^2}\frac{\partial \mathcal{M}}{\partial \theta}\right)\left(-\rho\frac{\partial u}{\partial x} + c\frac{\partial \rho}{\partial x}\right) + \frac{\partial \mathcal{M}}{\partial u}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x} + c\frac{\partial u}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \mathcal{M}}{\partial \theta}\left(-3p\frac{\partial u}{\partial x} + c\frac{\partial p}{\partial x}\right)\right].$$

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4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

From the definition of the equilibrium Maxwell-Boltzmann NDF, $\mathcal{M}$, in the univariate case, one can write

$$\ln \mathcal{M} = \ln \rho - \frac{1}{2}\ln \theta - \frac{1}{2}\frac{c^2}{\theta} - \ln\left[m\,(2\pi)^{1/2}\right],$$

and thus

$$\frac{1}{\mathcal{M}}\frac{\partial \mathcal{M}}{\partial \rho} = \frac{1}{\rho},$$

$$\frac{1}{\mathcal{M}}\frac{\partial \mathcal{M}}{\partial u} = -\frac{c}{\theta}\frac{\partial c}{\partial u} = \frac{c}{\theta} = \frac{\rho c}{p},$$

$$\frac{1}{\mathcal{M}}\frac{\partial \mathcal{M}}{\partial \theta} = -\frac{1}{2\theta} + \frac{c^2}{2\theta^2} = -\frac{\rho}{2p} + \frac{\rho^2 c^2}{2p^2}.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Substituting the expressions for the derivatives of $\mathcal{M}$ with respect to ρ, u, and θ into the preceding equation for $f^{(1)}$ can then write

$$
\begin{aligned}
f^{(1)} &= -\tau\left[\frac{1}{\rho}\left(\frac{3}{2}-\frac{\rho c^2}{2p}\right)\left(-\rho\frac{\partial u}{\partial x}+c\frac{\partial \rho}{\partial x}\right)+\frac{\rho c}{p}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}+c\frac{\partial u}{\partial x}\right)\right.\\
&\qquad\left.+\frac{1}{\rho}\left(-\frac{\rho}{2p}+\frac{\rho^2 c^2}{2p^2}\right)\left(-3p\frac{\partial u}{\partial x}+c\frac{\partial p}{\partial x}\right)\right]\\
&= -\tau\left(\frac{\rho^2}{2p^2}c^3-\frac{3\rho}{2p}c\right)\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).
\end{aligned}
$$

Thus, to first-order in ϵ,

$$
\mathcal{F}\approx\mathcal{M}\left[1+f^{(1)}\right]=\mathcal{M}\left[1-\tau\left(\frac{\rho^2}{2p^2}c^3-\frac{3\rho}{2p}c\right)\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\right].
$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Finally, can now evaluate q as follows:

$$
\begin{aligned}
q &= m\left\langle c^3 f^{(1)}\mathcal{M}\right\rangle\\
&= -m\tau\left\langle c^3\left[\left(\frac{\rho^2}{2p^2}c^3-\frac{3\rho}{2p}c\right)\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\right]\mathcal{M}\right\rangle\\
&= -\frac{\tau\rho^2}{2p^2}m\left\langle c^6\mathcal{M}\right\rangle\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)+\frac{3\tau\rho}{2p}m\left\langle c^4\mathcal{M}\right\rangle\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\\
&= -15\frac{\tau\rho^2}{2p^2}\frac{p^3}{\rho^2}\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)+9\frac{\tau\rho}{2p}\frac{p^2}{\rho}\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\\
&= -\tau p\left(\frac{15}{2}-\frac{9}{2}\right)\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\\
&= -3\tau p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).
\end{aligned}
$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Thus, to first-order in ϵ, the following approximation is obtained for the heat-flux moment, q:

$$q^{(1)} = -3p\tau \frac{\partial}{\partial x} \left(\frac{p}{\rho} \right) .$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The one-dimensional equivalent to the Navier-Stokes equations can also be derived through a Chapman-Enskog-like expansion applied to the moment equations. To do this, it is convenient to define the fourth moment, k, given by

$$k = m \left\langle\, c^4 \mathcal{F} \,\right\rangle - m \left\langle\, c^4 \mathcal{M} \,\right\rangle = m \left\langle\, c^4 \mathcal{F} \,\right\rangle - \frac{3p^2}{\rho} = r - \frac{3p^2}{\rho} ,$$

which is the deviation of the random-velocity fourth moment, r, from its value in thermodynamic equilibrium.

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Next, the third-order random-velocity heat-transfer moment, $q = m\left\langle\, c^3\mathcal{F}\,\right\rangle$ and k are written as perturbative expansions about their equilibrium values such that the scaled solution are given by

$$q = q^{(0)} + \epsilon q^{(1)} + \epsilon^2 q^{(2)} + \epsilon^3 q^{(3)} + \cdots\,,$$

$$k = k^{(0)} + \epsilon k^{(1)} + \epsilon^2 k^{(2)} + \epsilon^3 k^{(3)} + \cdots\,,$$

and the unscaled solutions are then

$$q = q^{(0)} + q^{(1)} + q^{(2)} + q^{(3)} + \cdots\,,$$

$$k = k^{(0)} + k^{(1)} + k^{(2)} + k^{(3)} + \cdots\,,$$

where $\epsilon \ll 1$ and is a measure of the nearness to equilibrium and that any deviation from equilibrium is attenuated very rapidly by collisional processes.

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4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The third-order moment equation describing q can then be derived from the kinetic equation in terms of the random variable via direct integration. The result is

$$\frac{\partial q}{\partial t} - \frac{3p^2}{\rho^2}\frac{\partial \rho}{\partial x} + 4q\frac{\partial u}{\partial x} + \frac{3p}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial q}{\partial x} + \frac{\partial k}{\partial x}$$
$$= \frac{\partial q}{\partial t} + 4q\frac{\partial u}{\partial x} + 3p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + u\frac{\partial q}{\partial x} + \frac{\partial k}{\partial x} = -\frac{q}{\tau}\,.$$

The scaled version of this transport equation is given by:

$$\frac{\partial q}{\partial t} + 4q\frac{\partial u}{\partial x} + 3p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + u\frac{\partial q}{\partial x} + \frac{\partial k}{\partial x} = -\frac{q}{\epsilon\tau}\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Additionally, the fourth-order moment equation for r can also be derived from the kinetic equation in terms of the random variable via direct integration. After subsequently substituting $r = 3p^2/\rho + k$, one can arrive at the following unscaled transport equation for k:

$$\frac{\partial k}{\partial t} + 5k\frac{\partial u}{\partial x} - \frac{4q}{\rho}\frac{\partial p}{\partial x} - \frac{6p}{\rho}\frac{\partial q}{\partial x} + u\frac{\partial k}{\partial x} + \frac{\partial s}{\partial x} = -\frac{k}{\tau}\,.$$

The scaled version of this equation is

$$\frac{\partial k}{\partial t} + 5k\frac{\partial u}{\partial x} - \frac{4q}{\rho}\frac{\partial p}{\partial x} - \frac{6p}{\rho}\frac{\partial q}{\partial x} + u\frac{\partial k}{\partial x} + \frac{\partial s}{\partial x} = -\frac{k}{\epsilon\tau}\,.$$

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

The scaled solutions for q and k are then inserted into the scaled moment equations for q and k and terms of equal order in ϵ are gathered. Once this is done, it is easy to show that to zeroth-order in ϵ, we have

$$q^{(0)} = 0\,,$$

and

$$k^{(0)} = 0\,,$$

as should be expected.

4.2 Fluid-Limit Solutions

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

Additionally, collecting terms to first-order in ϵ yields the following first-order approximation for the heat-flux moment, q:

$$q^{(1)} = -3p\tau \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) .$$

The preceding is identical to the expression obtained via the Chapman-Enskog method applied to the univariate kinetic equation derived previously. Note that it can also be shown that the corresponding first-order approximation for k is

$$k^{(1)} = 0 ,$$

which is also in agreement with the result that $k = m\left\langle c^4 f^{(1)} \mathcal{M} \right\rangle = 0$.

4.2.2 First-Order Chapman-Enskog Solution & "Navier-Stokes" Equations

In summary, the one-dimensional equivalent of the Navier-Stokes equations are then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 ,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 ,$$

$$\frac{\partial}{\partial t}(\rho u^2 + p) + \frac{\partial}{\partial x}(\rho u^3 + 3up) - \frac{\partial}{\partial x}\left[3p\tau \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\right] = 0 ,$$

where the following constitutive relation is applicable for q:

$$q = -3p\tau \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) .$$

4.3 Grad's Method of Moments

4.3.1 Assumed Form for the 4-Moment NDF

Similar to the three-dimensional case, a third-order 4-moment Grad-type closure can be derived for approximating general non-equilibrium solutions to the one-dimensional or univariate system by assuming that

$$\mathcal{F} \approx \mathcal{F}^{(4)} = \mathcal{M}\left[1 + \alpha_0 + \alpha_1\left(\frac{c}{a}\right) + \alpha_2\left(\frac{c}{a}\right)^2 + \alpha_3\left(\frac{c}{a}\right)^3\right],$$

where α_i are the coefficients for this Grad-type expansion.

4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

The coefficients, α_i, of this 4-moment Grad-type expansion can be determined by satisfying the following moment constraints:

$$m\left\langle \mathcal{F}^{(4)} \right\rangle = \rho\,,$$

$$m\left\langle c\mathcal{F}^{(4)} \right\rangle = 0\,,$$

$$m\left\langle c^2\mathcal{F}^{(4)} \right\rangle = p = \rho a^2\,,$$

$$m\left\langle c^3\mathcal{F}^{(4)} \right\rangle = q\,.$$

4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

Thus we have for the first constraint

$$\begin{aligned}
\rho &= m \left\langle \mathcal{F}^{(4)} \right\rangle \\
&= m \left\langle \mathcal{M} \left[1 + \alpha_0 + \alpha_1 \left(\frac{c}{a}\right) + \alpha_2 \left(\frac{c}{a}\right)^2 + \alpha_3 \left(\frac{c}{a}\right)^3 \right] \right\rangle \\
&= m \left(1 + \alpha_0\right) \langle \mathcal{M} \rangle + m\frac{\alpha_1}{a} \langle c\mathcal{M} \rangle + m\frac{\alpha_2}{a^2} \langle c^2\mathcal{M} \rangle + m\frac{\alpha_3}{a^3} \langle c^3\mathcal{M} \rangle \\
&= \left(1 + \alpha_0\right) \rho + \frac{\alpha_2}{a^2} \rho a^2 \\
0 &= \alpha_0 \rho + \alpha_2 \rho \\
0 &= \alpha_0 + \alpha_2 \,.
\end{aligned}$$

4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

For the second constraint:

$$\begin{aligned}
0 &= m \left\langle c\mathcal{F}^{(4)} \right\rangle \\
&= m \left\langle c\mathcal{M} \left[1 + \alpha_0 + \alpha_1 \left(\frac{c}{a}\right) + \alpha_2 \left(\frac{c}{a}\right)^2 + \alpha_3 \left(\frac{c}{a}\right)^3 \right] \right\rangle \\
&= m\frac{\alpha_1}{a} \langle c^2\mathcal{M} \rangle + m\frac{\alpha_3}{a^3} \langle c^4\mathcal{M} \rangle \\
&= \frac{\alpha_1}{a} \rho a^2 + 3\frac{\alpha_3}{a^3} \rho a^4 \\
&= \alpha_1 \rho a + 3\alpha_3 \rho a \\
0 &= \alpha_1 + 3\alpha_3 \,.
\end{aligned}$$

4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

The third constraint yields

$$\begin{aligned}
\rho a^2 &= m\left\langle\, c^2 \mathcal{F}^{(4)} \,\right\rangle \\
&= m\left\langle\, c^2 \mathcal{M}\left[1+\alpha_0+\alpha_1\left(\frac{c}{a}\right)+\alpha_2\left(\frac{c}{a}\right)^2+\alpha_3\left(\frac{c}{a}\right)^3\right]\,\right\rangle \\
&= m\left(1+\alpha_0\right)\left\langle\, c^2\mathcal{M}\,\right\rangle + m\frac{\alpha_2}{a^2}\left\langle\, c^4\mathcal{M}\,\right\rangle \\
&= \left(1+\alpha_0\right)\rho a^2 + 3\frac{\alpha_2}{a^2}\rho a^4 \\
1 &= 1+\alpha_0+3\alpha_2 \\
0 &= \alpha_0+3\alpha_2\,.
\end{aligned}$$

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4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

Lastly, the final and fourth constraint yields

$$\begin{aligned}
q &= m\left\langle\, c^3 \mathcal{F}^{(4)} \,\right\rangle \\
&= m\left\langle\, c^3 \mathcal{M}\left[1+\alpha_0+\alpha_1\left(\frac{c}{a}\right)+\alpha_2\left(\frac{c}{a}\right)^2+\alpha_3\left(\frac{c}{a}\right)^3\right]\,\right\rangle \\
&= m\frac{\alpha_1}{a}\left\langle\, c^4\mathcal{M}\,\right\rangle + m\frac{\alpha_3}{a^3}\left\langle\, c^6\mathcal{M}\,\right\rangle \\
&= 3\frac{\alpha_1}{a}\rho a^4 + 15\frac{\alpha_3}{a^3}\rho a^6 \\
&= 3\alpha_1\rho a^3 + 15\alpha_3\rho a^3 \\
\frac{q}{\rho a^3} &= 3\alpha_1+15\alpha_3\,.
\end{aligned}$$

4.3.1 Assumed Form for the 4-Moment NDF

Random Velocity Moment Contraints

Thus the four moment constraints on the expansion coefficients are

$$0 = \alpha_0 + \alpha_2 \,,$$

$$0 = \alpha_1 + 3\alpha_3 \,,$$

$$0 = \alpha_0 + 3\alpha_2 \,,$$

$$\frac{q}{\rho a^3} = 3\alpha_1 + 15\alpha_3 \,.$$

Combining the first and third constraints, we have

$$\alpha_0 = \alpha_2 = 0 \,,$$

and combining the second and fourth constraints yields

$$\alpha_1 = -3\alpha_3 = -\frac{1}{2}\frac{q}{\rho a^3} \,, \qquad \alpha_3 = \frac{1}{6}\frac{q}{\rho a^3} \,,$$

4.3 Grad's Method of Moments

4.3.1 Assumed Form for the 4-Moment NDF

Substitution of the preceding expressions for α_i back into the assumed form for $\mathcal{F}$ then yields

$$\mathcal{F}^{(4)} = \mathcal{M}\left\{1 - \frac{q}{6\rho a^3}\left[3\left(\frac{c}{a}\right) - \left(\frac{c}{a}\right)^3\right]\right\} \,,$$

where, in this univariate case, q is the sole measure of the 1D gases departure from equilibrium.

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

The macroscopic moment equations for the moment vector of the 4-moment Grad-type closure, $\mathbf{M}^{(4)}(x,t)$, corresponding to velocity-dependent weight vector, $\mathbf{V}^{(4)} = m[1, v, v^2, v^3, v^4]^{\mathrm{T}}$, can then defined as

$$\frac{\partial}{\partial t}\left[\mathbf{M}^{(4)}\right] + \frac{\partial}{\partial x}\left[\mathbf{F}^{(4)}\right] = -\frac{1}{\tau}\left[\mathbf{M}^{(4)} - \left\langle\, \mathbf{V}^{(4)}\mathcal{M}\,\right\rangle\right],$$

where

$$\mathbf{M}^{(4)} = \left\langle\, \mathbf{V}^{(4)}\mathcal{F}^{(4)}\,\right\rangle, \quad \mathbf{F}^{(4)} = \left\langle\, v\mathbf{V}^{(4)}\mathcal{F}^{(4)}\,\right\rangle.$$

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

The moment equations of the 4-moment closure are then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0,$$

$$\frac{\partial}{\partial t}(\rho u^2 + p) + \frac{\partial}{\partial x}(\rho u^3 + 3up + q) = 0,$$

$$\frac{\partial}{\partial t}(\rho u^3 + 3up + q) + \frac{\partial}{\partial x}(\rho u^4 + 6u^2 p + 4uq + r) = -\frac{q}{\tau}.$$

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

Introducing $r = 3p^2/\rho + k$, can also write

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0\,,$$

$$\frac{\partial}{\partial t}\left(\rho u^2 + p\right) + \frac{\partial}{\partial x}\left(\rho u^3 + 3up + q\right) = 0\,,$$

$$\frac{\partial}{\partial t}\left(\rho u^3 + 3up + q\right) + \frac{\partial}{\partial x}\left(\rho u^4 + 6u^2 p + 4uq + 3\frac{p^2}{\rho} + k\right) = -\frac{q}{\tau}\,,$$

where an expression for k is needed for closure.

4.3.2 Moment Equations of 4-Moment Grad Closure

Using matrix-vector notation, can also write

$$\frac{\partial}{\partial t}\left[\mathbf{M}^{(4)}\right] + \frac{\partial}{\partial x}\left[\mathbf{F}^{(4)}\right] = -\frac{1}{\tau}\mathbf{S}^{(4)}\,,$$

where

$$\mathbf{M}^{(4)} = \begin{bmatrix} \rho \\ \rho u \\ \rho u^2 + p \\ \rho u^3 + 3up + q \end{bmatrix}, \quad \mathbf{S}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{q}{\tau} \end{bmatrix},$$

$$\mathbf{F}^{(4)} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u^3 + 3up + q \\ u^4 + 6u^2 p + 4uq + 3\frac{p^2}{\rho} + k \end{bmatrix}.$$

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

Related transport equations for the macroscopic moment equations of the random velocity, $\mathbf{M}_{\circ}^{(4)}(x,t)$, corresponding to velocity-dependent weight vector, $\mathbf{V}^{(4)} = m[1, c, c^2, c^3, c^4]^{\mathrm{T}}$, can also be defined as

$$\frac{\partial}{\partial t}\left[\mathbf{M}_{\circ}^{(4)}\right] + \frac{\partial}{\partial x}\left[u\mathbf{M}_{\circ}^{(4)}\right] + \frac{\partial}{\partial x}\left[\left\langle c\mathbf{V}^{(4)}\mathcal{F}^{(4)}\right\rangle\right] + \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\left[\left\langle \frac{\partial \mathbf{V}^{(4)}}{\partial c}\mathcal{F}^{(4)}\right\rangle\right] + \frac{\partial u}{\partial x}\left[\left\langle c\frac{\partial \mathbf{V}^{(4)}}{\partial c}\mathcal{F}^{(4)}\right\rangle\right] = -\frac{1}{\tau}\left[\mathbf{M}_{\circ}^{(4)} - \left\langle \mathbf{V}^{(4)}\mathcal{M}\right\rangle\right],$$

where the vector of random velocity moments is given by

$$\mathbf{M}_{\circ}^{(4)} = \left\langle \mathbf{V}^{(4)}\mathcal{F}^{(4)}\right\rangle .$$

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

In this case, the non-conservation form of the moment equations are

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + 3p\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0,$$

$$\frac{\partial q}{\partial t} + 4q\frac{\partial u}{\partial x} - \frac{3p}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial q}{\partial x} + \frac{\partial r}{\partial x} = -\frac{q}{\tau}$$

4.3 Grad's Method of Moments

4.3.2 Moment Equations of 4-Moment Grad Closure

Again, introducing $r = 3p^2/\rho + k$, can write

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0\,,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,,$$

$$\frac{\partial p}{\partial t} + 3p\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0\,,$$

$$\frac{\partial q}{\partial t} - \frac{3p^2}{\rho^2}\frac{\partial \rho}{\partial x} + 4q\frac{\partial u}{\partial x} + \frac{3p}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial q}{\partial x} + \frac{\partial k}{\partial x} = -\frac{q}{\tau}$$

4.3.2 Moment Equations of 4-Moment Grad Closure

Using matrix-vector notation, can also express the primitive form of the equations as

$$\frac{\partial}{\partial t}\left[\mathbf{W}^{(4)}\right] + \mathbf{A}^{(4)}\frac{\partial}{\partial x}\left[\mathbf{W}^{(4)}\right] = -\frac{1}{\tau}\mathbf{S}^{(4)}\,,$$

where

$$\mathbf{W}^{(4)} = \begin{bmatrix} \rho \\ u \\ p \\ q \end{bmatrix}\,, \quad \mathbf{S}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{q}{\tau} \end{bmatrix}\,,$$

$$\mathbf{A}^{(4)} = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & 0 \\ 0 & 3p & u & 1 \\ -3\frac{p^2}{\rho^2} + \frac{\partial k}{\partial \rho} & 4q + \frac{\partial k}{\partial u} & 3\frac{p}{\rho} + \frac{\partial k}{\partial p} & u + \frac{\partial k}{\partial q} \end{bmatrix}\,,$$

where, prior to determining the closing flux, it has been assumed generally that $k = k(\rho, u, p, q)$.

4.3 Grad's Method of Moments

4.3.3 Closing Flux

Finally, the expression for the closing flux, k, of the 4-moment Grad-type closure is given by

$$k = r - 3\frac{p^2}{\rho} = m\left\langle\, c^4 \mathcal{F}^{(4)} \,\right\rangle - 3\frac{p^2}{\rho}$$
$$= m\left\langle\, c^4 \mathcal{M}\left\{1 - \frac{q}{6\rho a^3}\left[3\left(\frac{c}{a}\right) - \left(\frac{c}{a}\right)^3\right]\right\} \,\right\rangle - 3\frac{p^2}{\rho}$$
$$= m\left\langle\, c^4 \mathcal{M} \,\right\rangle - 3\frac{p^2}{\rho}$$
$$k = 0\,.$$

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4.3.3 Closing Flux

Thus the final weak form of the 1D conservation equations govern the 4-moment closure are

$$\frac{\partial}{\partial t}\left[\mathbf{M}^{(4)}\right] + \frac{\partial}{\partial x}\left[\mathbf{F}^{(4)}\right] = -\frac{1}{\tau}\mathbf{S}^{(4)}\,,$$

where

$$\mathbf{M}^{(4)} = \left[\begin{array}{c} \rho \\ \rho u \\ \rho u^2 + p \\ \rho u^3 + 3up + q \end{array}\right], \quad \mathbf{S}^{(4)} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -\frac{q}{\tau} \end{array}\right],$$

$$\mathbf{F}^{(4)} = \left[\begin{array}{c} \rho u \\ \rho u^2 + p \\ \rho u^3 + 3up + q \\ u^4 + 6u^2 p + 4uq + 3\frac{p^2}{\rho} \end{array}\right].$$

4.3.3 Closing Flux

Additionally, the primitive form of the 4-moment equations take the form

$$\frac{\partial}{\partial t}\left[\mathbf{W}^{(4)}\right] + \mathbf{A}^{(4)}\frac{\partial}{\partial x}\left[\mathbf{W}^{(4)}\right] = -\frac{1}{\tau}\mathbf{S}^{(4)},$$

where

$$\mathbf{W}^{(4)} = \begin{bmatrix} \rho \\ u \\ p \\ q \end{bmatrix}, \quad \mathbf{S}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{q}{\tau} \end{bmatrix},$$

$$\mathbf{A}^{(4)} = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & 0 \\ 0 & 3p & u & 1 \\ -3\frac{p^2}{\rho^2} & 4q & 3\frac{p}{\rho} & u \end{bmatrix}.$$

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4.4 Maximum-Entropy Method of Moments

The

4.5 Quadrature-Based Method of Moments

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4.6 Other Moment Approximations

The

4.7 Applications

The