8 Spillover

A typical model of a flexible spacecraft will be of the form

$$\begin{aligned} \mathbf{M}_{rr}\ddot{\boldsymbol{\eta}}_r &= \mathbf{B}_r\mathbf{u} \\ \ddot{\eta}_\alpha + 2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha &= \mathbf{b}_\alpha^T\mathbf{u}, \quad \alpha = 1, \cdots, N_c, N_c+1, \cdots, N_c+N_u \end{aligned}$$

We have added a structural damping term to the flexible modes parametrized by the damping ratio ζ_α. N_c of the vibration modes are the controlled modes to be used for control system design and N_u of the modes are the residual or uncontrolled modes. We define

$$\begin{aligned} \boldsymbol{\eta}_e &= \mathrm{col}\{\eta_\alpha\}, \alpha = 1, \ldots, N_c \\ \boldsymbol{\eta}_u &= \mathrm{col}\{\eta_\alpha\}, \alpha = N_c+1, \ldots, N_c+N_u \end{aligned}$$

We can then form the controlled and uncontrolled state vectors as

$$\mathbf{x}_c = \begin{bmatrix} \dot{\boldsymbol{\eta}}_r \\ \boldsymbol{\eta}_r \\ \dot{\boldsymbol{\eta}}_e \\ \boldsymbol{\eta}_e \end{bmatrix}, \quad \mathbf{x}_u = \begin{bmatrix} \dot{\boldsymbol{\eta}}_u \\ \boldsymbol{\eta}_u \end{bmatrix} \tag{1}$$

A state-space model will then be of the form

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{u} \\ \dot{\mathbf{x}}_u &= \mathbf{A}_u\mathbf{x}_c + \mathbf{B}_u\mathbf{u} \end{aligned} \tag{2}$$

The term containing $\mathbf{B}_u$ is termed *control spillover* and reflects the fact that the control input $\mathbf{u}$ will effect not only the modes used for its design but also the unmodelled modes.

To keep things simple, assume that there are p measurements of the form

$$\begin{aligned} \mathbf{y} &= \mathbf{C}_r\boldsymbol{\eta}_r + \sum_{\alpha=1}^{N_c+N_u} \mathbf{c}_\alpha\eta_\alpha \\ &= \mathbf{C}_c\mathbf{x}_c + \mathbf{C}_u\mathbf{x}_u \end{aligned} \tag{3}$$

The term containing $\mathbf{C}_u$ is called observation spillover and indicates that the measurements contain contributions from not only the controlled modes.

For control, assume an observer-based compensator has been designed based on the controlled modes. It is of the form

$$\mathbf{u} = \mathbf{F}\hat{\mathbf{x}}, \quad \dot{\hat{\mathbf{x}}} = \mathbf{A}_c\hat{\mathbf{x}} + \mathbf{B}_c\mathbf{u} + \mathbf{L}(\mathbf{C}_c\hat{\mathbf{x}} - \mathbf{y}) \tag{4}$$

It is also assumed that $\lambda\{\mathbf{A}_c + \mathbf{B}_c\mathbf{F}\}$ and $\lambda\{\mathbf{A}_c + \mathbf{L}\mathbf{C}_c\}$ have negative real parts.

The closed-loop system can be formed by combining (1)-(4):

$$\begin{bmatrix} \dot{\mathbf{x}}_c \\ \dot{\hat{\mathbf{x}}} \\ \dot{\mathbf{x}}_u \end{bmatrix} = \begin{bmatrix} \mathbf{A}_c & \mathbf{B}_c\mathbf{F} & \mathbf{O} \\ -\mathbf{L}\mathbf{C}_c & \mathbf{A}_c + \mathbf{B}_c\mathbf{F} + \mathbf{L}\mathbf{C}_c & -\mathbf{L}\mathbf{C}_c \\ \mathbf{O} & \mathbf{B}_u\mathbf{F} & \mathbf{A}_u \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \hat{\mathbf{x}} \\ \mathbf{x}_u \end{bmatrix}$$

We would like to specify the eigenvalues of the system matrix. This is easier if we work with the estimation error $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}_c$ in place of the controller state $\hat{\mathbf{x}}$. Therefore,

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}}_c \\ &= (\mathbf{A}_c + \mathbf{B}_c\mathbf{F} + \mathbf{L}\mathbf{C}_c)\hat{\mathbf{x}} - \mathbf{L}\mathbf{C}_c\mathbf{x}_c - \mathbf{L}\mathbf{C}_u\mathbf{x}_u \\ &\quad -\mathbf{A}_c\mathbf{x}_c - \mathbf{B}_c\mathbf{F}\hat{\mathbf{x}} \\ &= \mathbf{A}_c(\hat{\mathbf{x}} - \mathbf{x}_c) + \mathbf{L}\mathbf{C}_c(\hat{\mathbf{x}} - \mathbf{x}_c) - \mathbf{L}\mathbf{C}_u\mathbf{x}_u \\ &= (\mathbf{A}_c + \mathbf{L}\mathbf{C}_c)\mathbf{e} - \mathbf{L}\mathbf{C}_u\mathbf{x}_u \end{aligned}$$

Also,

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{F}(\mathbf{e} + \mathbf{x}_c) \\ &= (\mathbf{A}_c + \mathbf{B}_c\mathbf{F}_c)\mathbf{x}_c + \mathbf{B}_c\mathbf{F}\mathbf{e} \\ \dot{\mathbf{x}}_u &= \mathbf{A}_u\mathbf{x}_u + \mathbf{B}_u\mathbf{F}(\mathbf{e} + \mathbf{x}_c) \\ &= \mathbf{A}_u\mathbf{x}_u + \mathbf{B}_u\mathbf{F}\mathbf{e} + \mathbf{B}_u\mathbf{F}\mathbf{x}_c \end{aligned}$$

Combining these equations gives

$$\begin{bmatrix} \dot{\mathbf{x}}_c \\ \dot{\mathbf{e}} \\ \dot{\mathbf{x}}_u \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}_c + \mathbf{B}_c\mathbf{F} & \mathbf{B}_c\mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_c + \mathbf{L}\mathbf{C}_c & -\mathbf{L}\mathbf{C}_u \\ \mathbf{B}_u\mathbf{F} & \mathbf{B}_u\mathbf{F} & \mathbf{A}_u \end{bmatrix}}_{\mathbf{A}_{cl}} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{e} \\ \mathbf{x}_u \end{bmatrix}$$

Recall that

$$\lambda\left\{\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{O} & \mathbf{A}_3 \end{bmatrix}\right\} = \lambda\left\{\begin{bmatrix} \mathbf{A}_1 & \mathbf{O} \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}\right\} = \lambda\{\mathbf{A}_1\} \cup \lambda\{\mathbf{A}_3\}$$

Therefore, if $\mathbf{B}_u = \mathbf{O}$ and/or $\mathbf{C}_u = \mathbf{O}$ then

$$\lambda\{\mathbf{A}_{cl}\} = \lambda\{\mathbf{A}_c + \mathbf{B}_c\mathbf{F}\} \cup \lambda\{\mathbf{A}_c + \mathbf{L}\mathbf{C}_c\} \cup \lambda\{\mathbf{A}_u\}$$

Hence, the spillover is not destabilizing and does not effect the eigenvalues designed on the basis of the separation theorem. If $\mathbf{B}_u \neq \mathbf{O}$ and $\mathbf{C}_u \neq \mathbf{O}$, then there is no guarantee that $\mathbf{A}_{cl}$ is stable. The following block diagram gives some insight into the additional feedback loops formed by spillover which are not taken into account by the controller design.

Note that with collocated sensors and actuators forming a passive plant and a SPR controller, spillover is not destabilizing because the plant remains passive in the face of additional modes.