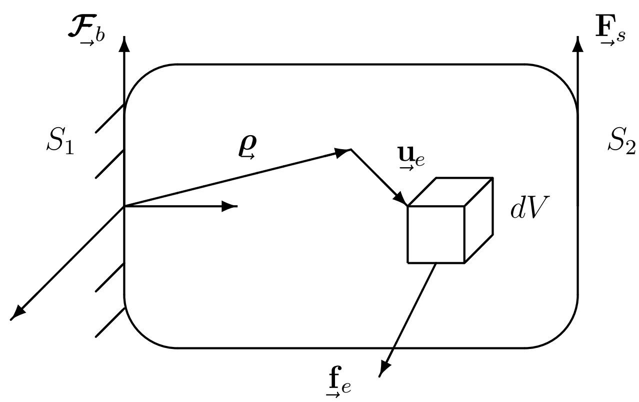
6 Flexible Spacecraft Dynamics

6.1 Summary of Classical Linear Elasticity

<u>**Setting**</u>



We consider a contrained elastic body with volume V. Impressed on V is a body force distribution (per unit volume)

$$\underset{\to}{\mathbf{f}}_e(\underset{\to}{\boldsymbol{\rho}}) = \underset{\to}{\mathcal{F}}_b^T \mathbf{f}_e, \quad \mathbf{f}_e = [f_1 \ f_2 \ f_3]^T \tag{1}$$

The deformation experienced at $\underset{\to}{\boldsymbol{\rho}} = \underset{\to}{\mathcal{F}}_b^T \boldsymbol{\rho}$, $\boldsymbol{\rho} = [x_1 \ x_2 \ x_3]^T$ is

$$\underset{\to}{\mathbf{u}}_e = \underset{\to}{\mathcal{F}}_b^T \mathbf{u}_e(\boldsymbol{\rho}), \quad \mathbf{u}_e = [u_1 \ u_2 \ u_3]^T$$

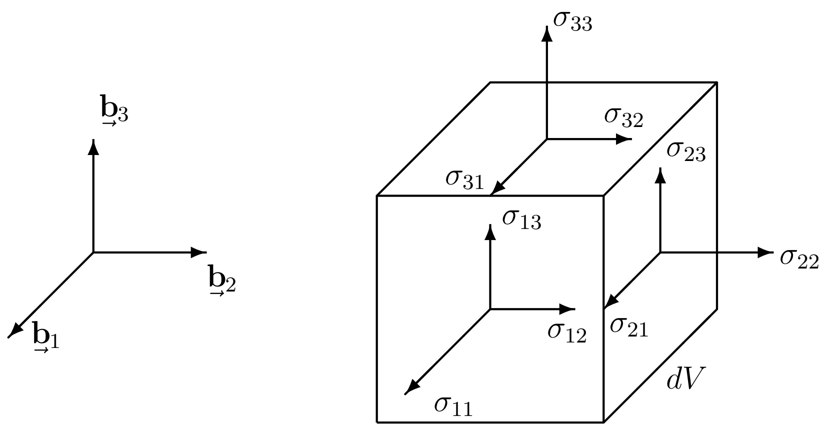
The surface of the body is decomposed as $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$. On S_1, we assume that $\mathbf{u}_e = \mathbf{0}$ and on S_2 there is a surface distribution of forces (per unit surface)

$$\underset{\to}{\mathbf{F}}_s = \underset{\to}{\mathcal{F}}_b^T \mathbf{F}_s, \quad \mathbf{F}_s = [F_1 \ F_2 \ F_3]^T \tag{2}$$

It is assumed that $\mathbf{u}_e$ is small.

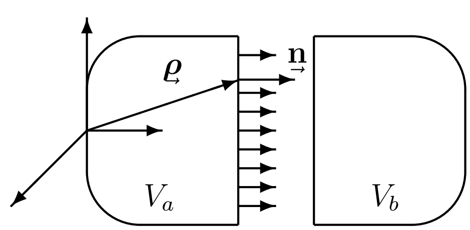
Stress Tensor

The application of $\underset{\rightarrow}{\mathbf{f}}_e$ and $\underset{\rightarrow}{\mathbf{F}}_s$ creates a state of stress in V:



The notation $\sigma_{ij}(\boldsymbol{\rho})$, $i, j = 1, 2, 3$ denotes the stress tensor. σ_{11}, σ_{22}, and σ_{33} are normal stresses and $\tau_{ij} = \sigma_{ij}$, $i \neq j$, are shear stresses. It can be shown that $\sigma_{ij} = \sigma_{ji}$.

The interpretation of the stress tensor is best seen by dividing V into V_a and V_b. Let $\underset{\rightarrow}{\mathbf{n}}(\underset{\rightarrow}{\boldsymbol{\rho}}) = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \mathbf{n}$, $\mathbf{n} = [n_1 \; n_2 \; n_3]^T$, denote the outward normal to V_a along the dividing surface.



The effect of V_b on V_a is a force distribution (per unit surface) $\underset{\rightarrow}{\mathbf{F}}_{ba} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T [F_{a1} \; F_{a2} \; F_{a3}]^T$ where

$$F_{ai} = \sum_{j=1}^{3} \sigma_{ij} n_j = \sigma_{ij} n_j$$

where we have used the summation convention (sum over repeated indices). Hence on S_2 we have

$$\sigma_{ij} n_j = F_i$$

Strain Tensor

The strain tensor is defined by

$$\varepsilon_{ij}(\mathbf{u}_e) = \tfrac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = \varepsilon_{ji}$$

The diagonal entries $\varepsilon_{ii} = \partial u_i/\partial x_i$, $i = 1, 2, 3$, are normal strains and

$$\varepsilon_{ij} = \tfrac{1}{2}\gamma_{ij}, i \neq j$$

are shearing strains.

Hooke's Law

Assuming an elastic body, we write

$$\sigma_{ij} = E_{ijkl}(\boldsymbol{\rho})\varepsilon_{kl} \tag{3}$$

where E_{ijkl} is the tensor of elastic moduli. It possesses the symmetries

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}$$

It is possible to write (3) using a contracted notation. Define

$$\begin{aligned} \boldsymbol{\sigma} &= [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \tau_{23}\ \tau_{31}\ \tau_{12}]^T \\ \boldsymbol{\varepsilon}(\mathbf{u}_e) &= [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{23}\ \gamma_{31}\ \gamma_{12}]^T \end{aligned}$$

Then

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}, \quad \mathbf{E} = \mathbf{E}^T$$

where $\mathbf{E} = \text{matrix}\{E_{ij}\}$, $i, j = 1, \ldots, 6$ contains the E_{ijkl}.

For a homogeneous body, $E_{ijkl}(\boldsymbol{\rho}) \equiv E_{ijkl}$ (*i.e.*, E_{ijkl} is independent of position) and for an isotropic body, E_{ijkl} is independent of the choice of $\boldsymbol{\mathcal{F}}_b$ (*i.e.*, arbitrary orientation). In this case, it can be shown that the 21 independent constants in E_{ijkl} degenerate to two:

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

and λ, μ are the Lamé parameters. They may be expressed in terms of the more familiar Young's modulus E and Poisson's ratio ν using

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

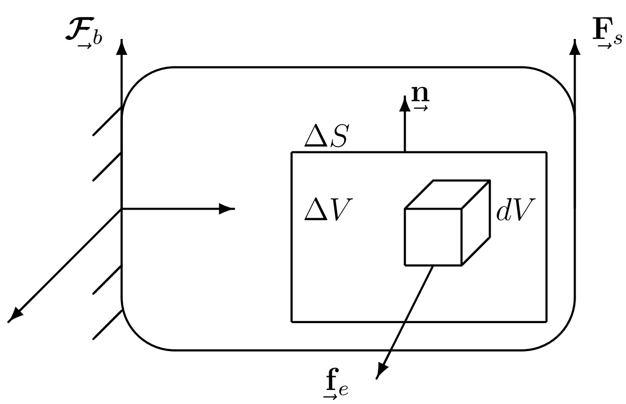
In this case, we have

$$\mathbf{E} = \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)}$$

Note that $G = E/[2(1+\nu)]$ is the shear modulus.

Equilibrium Equation

Initially, we assume the body to be in static equilibrium. Consider a portion of V, say ΔV, with boundary ΔS:



The balance of volume and surface forces yields

$$\int_{\Delta V} f_i dV = -\int_{\Delta S} \sigma_{ij} n_j dS, \; i = 1, 2, 3 \tag{4}$$

Recall Gauss's Law:

$$\int_{\Delta V} \underset{\to}{\boldsymbol{\nabla}} \cdot \underset{\to}{\mathbf{F}} dV = \int_{\Delta S} \underset{\to}{\mathbf{F}} \cdot \underset{\to}{\mathbf{n}} \, dS$$

or

$$\int_{\Delta V} \frac{\partial F_j}{\partial x_j}\, dV = \int_{\Delta S} F_j n_j \, dS$$

Applying this to the right side of (4) gives

$$\int_{\Delta V} f_i \, dV = -\int_{\Delta V} \frac{\partial \sigma_{ij}}{\partial x_j} dV, \;\; i = 1, 2, 3$$

Since ΔV is arbitrary, we conclude that

$$-\frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1, 2, 3$$

Introducing Hooke's Law gives

$$-\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial E_{ijkl}\varepsilon_{kl}}{\partial x_j} = -\frac{\partial}{\partial x_j}\left[E_{ijkl}\frac{\partial u_k}{\partial x_l}\right] = f_i \tag{5}$$

We can write this symbolically as

$$\boldsymbol{\mathcal{K}}\mathbf{u}_e = \mathbf{f}_e(\boldsymbol{\rho}) \tag{6}$$

where

$$\mathcal{K}_{ik}(\,\cdot\,) = -\frac{\partial}{\partial x_j}\left[E_{ijkl}\frac{(\,\cdot\,)}{\partial x_l}\right]$$

is the (3×3) stiffness operator. In the homogeneous isotropic case, we can write

$$\boldsymbol{\mathcal{K}}\mathbf{u}_e = -(\lambda + 2\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}^T\mathbf{u}_e + \mu\boldsymbol{\nabla}^\times\boldsymbol{\nabla}^\times\mathbf{u}_e$$

where $\boldsymbol{\nabla}^T = [\partial/\partial x_1 \;\; \partial/\partial x_2 \;\; \partial/\partial x_3]$. In general, (6) is subject to the boundary conditions

$$u_i = 0, \;\; i = 1, 2, 3, \text{ on } S_1 \tag{7}$$

$$\sigma_{ij} n_j = F_i, \;\; i = 1, 2, 3, \text{ on } S_2 \tag{8}$$

where σ_{ij} is given by (3).

6.2 Variational Formulation

Gauss's theorem can be applied to the product of a scalar field $\phi(\boldsymbol{\rho})$ and a vector one $\boldsymbol{\psi}(\boldsymbol{\rho})$ to give

$$\int_V \boldsymbol{\nabla}^T(\phi\boldsymbol{\psi})\, dV = \int_S \phi\boldsymbol{\psi}^T\mathbf{n}\, dS$$

or

$$\int_V \boldsymbol{\psi}^T \boldsymbol{\nabla}\phi \, dV = -\int_V (\boldsymbol{\nabla}^T \boldsymbol{\psi})\phi \, dV + \int_S \phi \boldsymbol{\psi}^T \mathbf{n} \, dS$$

where $\mathbf{n} = [n_1 \; n_2 \; n_3]^T$ is the outward normal to S. Expressing the above in component form gives

$$\int_V \psi_j \frac{\partial \phi}{\partial x_j} \, dV = -\int_V \frac{\partial \psi_j}{\partial x_j} \phi \, dV + \int_S \phi \psi_j n_j \, dS \tag{9}$$

Now, define the strain energy associated with a distribution of strain by

$$\begin{aligned} U &= \tfrac{1}{2} \int_V \varepsilon_{ij} E_{ijkl} \varepsilon_{kl} \, dV = \tfrac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} \, dV && (10) \\ &= \tfrac{1}{2} \int_V \boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} \, dV, \quad \mathbf{E} = \mathbf{E}^T && (11) \end{aligned}$$

Consider a small change in the configuration of the system in the form of a kinematically admissible virtual displacement ($\mathbf{u}_e = \mathbf{0}$ on S_1), given by $\delta \mathbf{u}_e = [\delta u_1 \; \delta u_2 \; \delta u_3]^T$. The corresponding variation of U is

$$\begin{aligned} \delta U &= \tfrac{1}{2} \int_V [\delta \varepsilon_{ij} E_{ijkl} \varepsilon_{kl} + \varepsilon_{ij} E_{ijkl} \delta \varepsilon_{kl}] \, dV \\ &= \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV \\ &= \int_V \sigma_{ij} \left[\tfrac{1}{2} \left(\delta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \right] dV \\ &= \int_V \sigma_{ij} \delta \left(\frac{\partial u_i}{\partial x_j} \right) dV \\ &= \int_V \sigma_{ij} \frac{\partial (\delta u_i)}{\partial x_j} \, dV \end{aligned}$$

Now apply the form of Gauss's law given above in (9) to the last form of δU to get:

$$\delta U = -\int_V \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i \, dV + \int_S \sigma_{ij} n_j \delta u_i \, dS \tag{12}$$

Using the equilibrium equation in (5) and the boundary conditions (7) and (8) gives

$$\begin{aligned} -\frac{\partial \sigma_{ij}}{\partial x_j} &= f_i, \quad \delta u_i = 0 \text{ on } S_1 \\ \sigma_{ij} n_j &= F_i \text{ on } S_2 \end{aligned}$$

With these in hand, (12) can be written as

$$\delta U = \int_V f_i \delta u_i \, dV + \int_{S_2} F_i \delta u_i \, dS$$

This can be written as

$$\delta U = \delta W_e \tag{13}$$

where the virtual work performed by the external forces is

$$\delta W_e = \int_V \mathbf{f}_e^T \delta \mathbf{u}_e \, dV + \int_{S_2} \mathbf{F}_s^T \delta \mathbf{u}_e \, dS \tag{14}$$

Hence, the first variation of the strain energy equals the virtual work of the external influences. If we interpret U as a type of potential energy, Eq. (13) is entirely consistent with Hamilton's (extended) principle:

$$\delta \int_{t_1}^{t_2} L \, dt + \int_{t_1}^{t_2} \delta W_e \, dt = 0 \tag{15}$$

where $L = T - U$ is the Lagrangian and T is the kinetic energy. Since we are dealing with statics, $T = 0$ and the temporal integration is irrelevant. Eq. (15) then reduces to Eq. (13).

6.3 Dynamics

We can readily extend the treatment to the dynamic case by using Eq. (15) in conjunction with δW_e in (14), U in (11), and introducing the kinetic energy

$$T = \frac{1}{2} \int_V \dot{\mathbf{u}}_e^T \dot{\mathbf{u}}_e \sigma \, dV \tag{16}$$

where $\sigma(\boldsymbol{\rho})$ is the mass density (per unit volume). Alternatively, the equilibrium equation in Eq. (6) becomes

$$\boldsymbol{\mathcal{K}} \mathbf{u}_e = \mathbf{f}_e + \mathbf{f}_I, \quad \mathbf{f}_I = -\sigma \ddot{\mathbf{u}}_e$$

where using d'Alembert's principle, we have introduced the inertial force distribution $\mathbf{f}_I$. Hence the equation of motion becomes

$$\sigma \ddot{\mathbf{u}}_e + \boldsymbol{\mathcal{K}} \mathbf{u}_e(\boldsymbol{\rho}, t) = \mathbf{f}_e(\boldsymbol{\rho}, t) \tag{17}$$

6.4 The Rayleigh-Ritz Method

Consider the statics problem

$$\boldsymbol{\mathcal{K}}\mathbf{u}_e = \mathbf{f}_e, \begin{cases} \mathbf{u}_e = \mathbf{0} \text{ on } S_1 \\ \sigma_{ij}n_j = F_j \text{ on } S_2 \end{cases}$$

As an alternative to solving this PDE we can use $\delta U = \delta W_e$ or

$$\delta(U - W_e) = 0$$

which is called the principle of minimum total potential energy. Here,

$$\begin{aligned} U &= \int_V \boldsymbol{\varepsilon}^T(\mathbf{u}_e)\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u}_e)\,dV \\ W_e &= \int_V \mathbf{f}_e^T\mathbf{u}_e\,dV + \int_{S_2} \mathbf{F}_s^T\mathbf{u}_e\,dS \end{aligned}$$

It is understood that the minimization is respect to those $\mathbf{u}_e$ satisfying $\mathbf{u}_e = \mathbf{0}$ on S_1. However, if $\mathbf{F}_s = \mathbf{0}$, we need not explicitly enforce $\sigma_{ij}n_j = 0$ on S_2.

In the Rayleigh-Ritz method, we assume a solution of the form

$$\mathbf{u}_e(\boldsymbol{\rho}) = \sum_{i=1}^{n} \boldsymbol{\psi}_i(\boldsymbol{\rho})q_i \tag{18}$$

where $\boldsymbol{\psi}_i(\boldsymbol{\rho}) = \mathbf{0}$ on S_1 and the $\boldsymbol{\psi}_i$ are independent. The q_i are determined by minimizing $U - W_e$. Let us take $\mathbf{F}_s = \mathbf{0}$ and note that

$$\boldsymbol{\varepsilon}(\mathbf{u}_e) = \sum_{i=1}^{n} \boldsymbol{\varepsilon}(\boldsymbol{\psi}_i)q_i$$

since $\boldsymbol{\varepsilon}$ is linear in $\mathbf{u}_e$. Therefore,

$$\begin{aligned} U &= \tfrac{1}{2}\int_V \boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}\,dV \\ &= \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_V \boldsymbol{\varepsilon}(\boldsymbol{\psi}_i)^T\mathbf{E}\boldsymbol{\varepsilon}(\boldsymbol{\psi}_j)\,dV\,q_iq_j \\ &= \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \end{aligned}$$

where

$$K_{ij} = \int_V \boldsymbol{\varepsilon}(\boldsymbol{\psi}_i)^T\mathbf{E}\boldsymbol{\varepsilon}(\boldsymbol{\psi}_j)\,dV$$

Here, $\mathbf{K} = \mathbf{K}^T > \mathbf{O}$ is the stiffness matrix. Also,

$$\begin{aligned} W_e &= \int_V \mathbf{f}_e^T \mathbf{u}_e \, dV \\ &= \sum_{i=1}^{n} \int_V \mathbf{f}_e^T \boldsymbol{\psi}_i \, dV q_i \\ &= \mathbf{F}_e^T \mathbf{q}, \quad F_{ei} = \int_V \mathbf{f}_e^T \boldsymbol{\psi}_i \, dV \end{aligned}$$

where $\mathbf{F}_e$ is the generalized force vector. Hence,

$$U - W_e = \tfrac{1}{2}\mathbf{q}^T \mathbf{K} \mathbf{q} - \mathbf{F}_e^T \mathbf{q}$$

Minimizing $U - W_e$, *i.e.*,

$$\frac{\partial (U - W_e)}{\partial \mathbf{q}} = \mathbf{K}\mathbf{q} - \mathbf{F}_e = \mathbf{0}$$

leads to

$$\mathbf{K}\mathbf{q} = \mathbf{F}_e$$

This determines $\mathbf{q}$, hence $\mathbf{u}_e(\boldsymbol{\rho})$ using (18).

Example. Longitudinal Extension of a Rod

6.5 Rayleigh-Ritz in Dynamics Problems

Now we let

$$\mathbf{u}_e(\boldsymbol{\rho}, t) = \sum_{i=1}^{n} \boldsymbol{\psi}_i(\boldsymbol{\rho}) q_i(t)$$

where the generalized coordinates $q_i(t)$ are time varying and introduce the kinetic energy

$$\begin{aligned} T &= \tfrac{1}{2}\int_V \dot{\mathbf{u}}_e^T \dot{\mathbf{u}}_e \sigma(\boldsymbol{\rho})\, dV \\ &= \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_V \boldsymbol{\psi}_i^T \boldsymbol{\psi}_j \sigma\, dV\, \dot{q}_i \dot{q}_j \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} \end{aligned}$$

where

$$M_{ij} = \int_V \boldsymbol{\psi}_i^T \boldsymbol{\psi}_j \sigma\, dV \tag{19}$$

Here, $\mathbf{M} = \mathbf{M}^T > \mathbf{O}$ is the mass matrix.

Recall that

$$U = \tfrac{1}{2}\mathbf{q}^T \mathbf{K} \mathbf{q}, \quad \delta W_e = \mathbf{F}_e^T \delta \mathbf{q}$$

Lagrange's equations corresponding to the variational principle in (15) are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{F}_e, \quad L = T - U \tag{20}$$

Using the energy expressions above in these equations leads to

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}_e(t) \tag{21}$$

This is the discrete parameter form of the PDE

$$\sigma \ddot{\mathbf{u}}_e + \boldsymbol{\mathcal{K}} \mathbf{u}_e = \mathbf{f}_e(\boldsymbol{\rho}, t)$$

6.6 Constrained Modal Analysis

Let us begin with the discrete-parameter motion equation for a constrained elastic body, Eq. (21). Initially, we look for solutions of the form

$$\mathbf{q}(t) = \Re e\{\mathbf{q}_\alpha e^{\lambda_\alpha t}\}$$

Substituting this into the homogeneous form of Eq. (21) gives

$$\lambda_\alpha^2 \mathbf{M}\mathbf{q}_\alpha + \mathbf{K}\mathbf{q}_\alpha = \mathbf{0}, \quad \alpha = 1, 2, 3, \ldots \tag{22}$$

Premultiplying by $\mathbf{q}_\alpha^H = [\mathbf{q}_\alpha^*]^T$ gives

$$\lambda_\alpha^2 = -\frac{\mathbf{q}_\alpha^H \mathbf{K}\mathbf{q}_\alpha}{\mathbf{q}_\alpha^H \mathbf{M}\mathbf{q}_\alpha} < 0$$

Hence $\lambda_\alpha = \pm j\omega_\alpha$ where $\omega_\alpha > 0$ are the vibration frequencies. We can, without loss in generality take the eigencolumn $\mathbf{q}_\alpha$ to be real. The vibration frequencies can be determined from the characteristic equation

$$\det\left[-\omega_\alpha^2 \mathbf{M} + \mathbf{K}\right] = 0$$

which yields n values for ω_α assuming $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$.

Now, Eq. (22) can be written as

$$-\omega_\alpha^2 \mathbf{M}\mathbf{q}_\alpha + \mathbf{K}\mathbf{q}_\alpha = \mathbf{0}$$

Premultiplying by $\mathbf{q}_\beta^T$ gives

$$-\omega_\alpha^2 \mathbf{q}_\beta^T \mathbf{M}\mathbf{q}_\alpha + \mathbf{q}_\beta^T \mathbf{K}\mathbf{q}_\alpha = 0 \tag{23}$$

Interchanging α and β and rewriting produces

$$-\omega_\beta^2 \mathbf{q}_\alpha^T \mathbf{M}\mathbf{q}_\beta + \mathbf{q}_\alpha^T \mathbf{K}\mathbf{q}_\beta = 0 \tag{24}$$

Subtracting (24) from (23) and noting the symmetry of $\mathbf{M}$ and $\mathbf{K}$ gives

$$(\omega_\alpha^2 - \omega_\beta^2)\mathbf{q}_\alpha^T \mathbf{M}\mathbf{q}_\beta = 0$$

For $\omega_\alpha \neq \omega_\beta$, we must have

$$\mathbf{q}_\alpha^T \mathbf{M}\mathbf{q}_\beta = 0 \tag{25}$$

and hence

$$\mathbf{q}_\alpha^T \mathbf{K}\mathbf{q}_\beta = 0 \tag{26}$$

When $\alpha = \beta$, we can set

$$\mathbf{q}_\alpha^T \mathbf{M}\mathbf{q}_\alpha = 1 \tag{27}$$

$$\mathbf{q}_\alpha^T \mathbf{K}\mathbf{q}_\alpha = \omega_\alpha^2 \tag{28}$$

Combining Eqs. (25)-(28) yields

$$\mathbf{q}_\alpha^T \mathbf{M}\mathbf{q}_\beta = \delta_{\alpha\beta} \tag{29}$$

$$\mathbf{q}_\alpha^T \mathbf{K}\mathbf{q}_\beta = \omega_\alpha^2 \delta_{\alpha\beta} \tag{30}$$

If the frequencies are not distinct, we can still write these relations because the eigencolumns can be orthogonalized using a Gram-Schmidt procedure.

6.7 Approximate Mode Shapes

Since the assumed expansion using the Rayleigh-Ritz procedure is

$$\mathbf{u}_e(\boldsymbol{\rho}, t) = \sum_{i=1}^{n} \boldsymbol{\psi}_i(\boldsymbol{\rho}) q_i(t) = \boldsymbol{\Psi}_e(\boldsymbol{\rho})\mathbf{q}(t)$$

where $\boldsymbol{\Psi}_e = \text{row}\{\boldsymbol{\psi}_i\}$, the approximate *mode shape* corresponding to ω_α is

$$\mathbf{u}_{e\alpha}(\boldsymbol{\rho}) = \boldsymbol{\Psi}_e(\boldsymbol{\rho})\mathbf{q}_\alpha$$

The *mode* is described by

$$\mathbf{u}_e(\boldsymbol{\rho}, t) = \Re e\{\mathbf{u}_{e\alpha}(\boldsymbol{\rho}) \exp(\lambda_\alpha t)\} = \mathbf{u}_{e\alpha} \cos \omega_\alpha t$$

6.8 Modal Equations of Motion

The N eigenvectors $\mathbf{q}_\alpha$ are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$ and hence constitute a basis for $\Re^N$. Let us represent the solution of (21) by

$$\mathbf{q}(t) = \sum_{\beta=1}^{N} \mathbf{q}_\beta \eta_\beta(t) \tag{31}$$

where the η_β are *modal coordinates*. Substituting this expansion into (21) and premultiplying by $\mathbf{q}_\alpha^T$ gives

$$\mathbf{q}_\alpha^T \mathbf{M} \sum_{\beta=1}^{N} \mathbf{q}_\beta \ddot{\eta}_\beta(t) + \mathbf{q}_\alpha^T \mathbf{K} \sum_{\beta=1}^{N} \mathbf{q}_\beta \eta_\beta(t) = \mathbf{q}_\alpha^T \mathbf{F}_e$$

Using the orthonormality relations in (29)-(30) gives

$$\ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha = \hat{f}_\alpha = \mathbf{q}_\alpha^T \mathbf{F}_e, \quad \alpha = 1, \ldots, N \tag{32}$$

which represents N decoupled equations for the modal coordinates. Collectively they can be written as

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2 \boldsymbol{\eta} = \hat{\mathbf{f}} \tag{33}$$

where

$$\boldsymbol{\eta} = \text{col}\{\eta_\alpha\}, \quad \hat{\mathbf{f}} = \text{col}\{\hat{f}_\alpha\}, \quad \boldsymbol{\Omega} = \text{diag}\{\omega_\alpha\}$$

Integration of the modal equations for the $\eta_\alpha(t)$ allows us to express the motion of the body as

$$\mathbf{u}_e(\boldsymbol{\rho}, t) = \boldsymbol{\Psi}_e(\boldsymbol{\rho})\mathbf{q}(t) = \sum_{\alpha=1}^{N} \boldsymbol{\Psi}_e \mathbf{q}_\alpha \eta_\alpha(t) = \sum_{\alpha=1}^{N} \mathbf{u}_{e\alpha}(\boldsymbol{\rho})\eta_\alpha(t) \tag{34}$$

Also note that the modal forces can be expressed as

$$\hat{f}_\alpha(t) = \mathbf{q}_\alpha^T \mathbf{F}_e = \mathbf{q}_\alpha^T \int_V \boldsymbol{\Psi}_e^T(\boldsymbol{\rho})\mathbf{f}_e(\boldsymbol{\rho}, t)\, dV = \int_V \mathbf{u}_{e\alpha}^T(\boldsymbol{\rho})\mathbf{f}_e(\boldsymbol{\rho}, t)\, dV \tag{35}$$

6.9 Sensors and Actuators

Let us assume that there are m rate measurements, y_i, in the direction $\mathbf{a}_i$ ($\mathbf{a}_i^T\mathbf{a}_i = 1$) at the location $\boldsymbol{\rho} = \boldsymbol{\rho}_i$. Therefore

$$\begin{aligned} y_i &= \mathbf{a}_i^T \dot{\mathbf{u}}_e(\boldsymbol{\rho}_i, t) \\ &= \sum_{\alpha=1}^{N} \mathbf{a}_i^T \mathbf{u}_{e\alpha}(\boldsymbol{\rho}_i)\dot{\eta}_\alpha(t) \\ &= \hat{\mathbf{c}}_i^T \dot{\boldsymbol{\eta}}(t) \end{aligned}$$

where

$$\hat{\mathbf{c}}_i = \mathrm{col}_\alpha\{\mathbf{a}_i^T \mathbf{u}_{e\alpha}(\boldsymbol{\rho}_i)\}$$

If we define

$$\mathbf{y}(t) = \mathrm{col}\{y_i\}, \quad \widehat{\mathbf{C}} = \mathrm{col}\{\hat{\mathbf{c}}_i^T\}$$

then

$$\mathbf{y} = \widehat{\mathbf{C}}\dot{\boldsymbol{\eta}} \tag{36}$$

Now, assume that there are m force actuators with applied force $u_j(t)$ in the direction $\bar{\mathbf{a}}_j$ ($\bar{\mathbf{a}}_j^T\bar{\mathbf{a}}_j = 1$) at the location $\boldsymbol{\rho} = \bar{\boldsymbol{\rho}}_j$. Therefore the force per unit volume distribution can be written as

$$\mathbf{f}_e(\boldsymbol{\rho}, t) = \sum_{j=1}^{m} u_j(t)\bar{\mathbf{a}}_j \delta(\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}_j) \tag{37}$$

where $\delta(\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}_j)$ is the Dirac delta function located at $\boldsymbol{\rho} = \bar{\boldsymbol{\rho}}_j$. Therefore, using (35)

$$\begin{aligned} \hat{f}_\alpha &= \int_V \mathbf{u}_{e\alpha}^T(\boldsymbol{\rho}) \sum_{j=1}^{m} u_j(t)\bar{\mathbf{a}}_j \delta(\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}_j)\, dV \\ &= \sum_{j=1}^{m} \mathbf{u}_{e\alpha}^T(\bar{\boldsymbol{\rho}}_j)\bar{\mathbf{a}}_j u_j(t) \end{aligned}$$

Forming $\widehat{\mathbf{f}} = \mathrm{col}\{\widehat{f}_\alpha\}$, we have

$$\widehat{\mathbf{f}} = \sum_{j=1}^{m} \widehat{\mathbf{b}}_j u_j(t)$$

where

$$\widehat{\mathbf{b}}_j = \mathrm{col}_\alpha\{\bar{\mathbf{a}}_j^T \mathbf{u}_{e\alpha}(\bar{\boldsymbol{\rho}}_j)\}$$

If we define

$$\mathbf{u}(t) = \mathrm{col}\{u_j\}, \quad \widehat{\mathbf{B}} = \mathrm{row}\{\widehat{\mathbf{b}}_j\}$$

then

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} = \widehat{\mathbf{f}} = \widehat{\mathbf{B}}\mathbf{u} \tag{38}$$

If $\boldsymbol{\rho}_i = \bar{\boldsymbol{\rho}}_i$ and $\mathbf{a}_i = \bar{\mathbf{a}}_i$, we say that the y_i and the u_i are *collocated*. In this case $\widehat{\mathbf{c}}_i = \widehat{\mathbf{b}}_i$, so that $\widehat{\mathbf{B}} = \widehat{\mathbf{C}}^T$.

Claim. If $\mathbf{u}$ and $\mathbf{y}$ correspond to collocated force actuators and rate sensors, then the mapping relating $\mathbf{u}$ to $\mathbf{y}$ is passive.

Proof. Consider the energy of the system in modal coordinates:

$$H(t) = \tfrac{1}{2}\dot{\boldsymbol{\eta}}^T\dot{\boldsymbol{\eta}} + \tfrac{1}{2}\boldsymbol{\eta}^T\boldsymbol{\Omega}^2\boldsymbol{\eta} \geq 0$$

Taking its time derivative, we have

$$\begin{aligned} \dot{H} &= \dot{\boldsymbol{\eta}}^T(\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta}) \\ &= \dot{\boldsymbol{\eta}}^T\widehat{\mathbf{B}}\mathbf{u} \\ &= \mathbf{u}^T\widehat{\mathbf{B}}^T\dot{\boldsymbol{\eta}} \\ &= \mathbf{u}^T\widehat{\mathbf{C}}\dot{\boldsymbol{\eta}} = \mathbf{u}^T\mathbf{y} \end{aligned}$$

Integrating both sides with respect to time and taking $\boldsymbol{\eta}(0) = \dot{\boldsymbol{\eta}}(0) = \mathbf{0}$ gives

$$\int_0^T \mathbf{y}^T\mathbf{u}\,dt = H(T) - H(0) = H(T) \geq 0$$

which establishes the claim. □