2 Lyapunov Stability

Whereas I/O stability is concerned with the effect of inputs on outputs, Lyapunov stability deals with unforced systems:

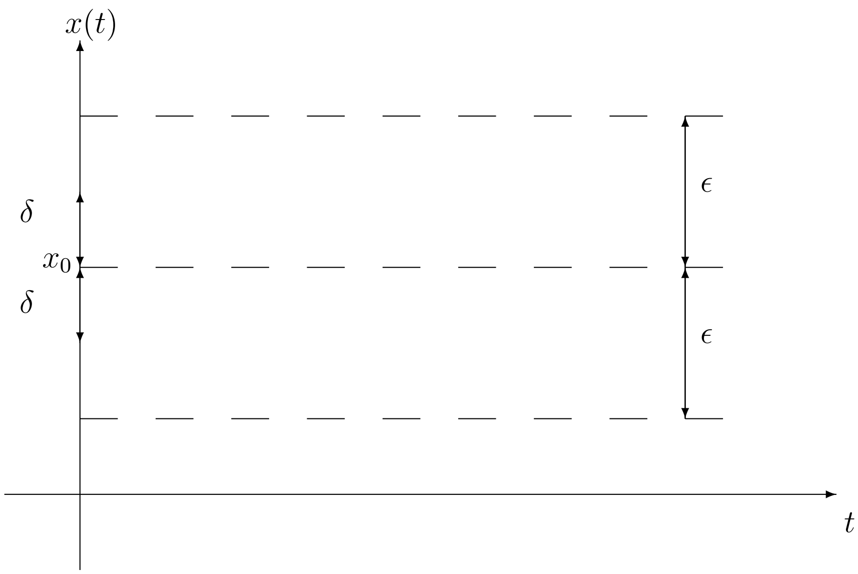
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x} \in R^n$, $t \in R^+$, and $\mathbf{f} : R^n \times R^+ \rightarrow R^n$. The system in (1) is autonomous (time invariant) if $\mathbf{f} = \mathbf{f}(\mathbf{x})$. Otherwise, it is nonautonomous (time varying).

If $\mathbf{f}(\mathbf{x}_0, t) = \mathbf{0}$, then $\mathbf{x}_0$ is an equilibrium. Then, the unique solution of Eq. (1) with $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = \mathbf{x}_0$.

Definition. The equilibrium $\mathbf{x}_0$ is *stable* (or L-stable or stable in the sense of Lyapunov) if for $\epsilon > 0$ there exists $\delta > 0$ such that

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \Rightarrow ||\mathbf{x}(t) - \mathbf{x}_0|| < \epsilon$$



The equilibrium $\mathbf{x}_0$ is *unstable* if it is not stable.

The equilibrium $\mathbf{x}_0$ is *asymptotically stable* if it is stable and there exists $\delta > 0$ such that

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \Rightarrow \mathbf{x}(t) \rightarrow \mathbf{x}_0 \text{ as } t \rightarrow \infty$$

The equilibrium $\mathbf{x}_0$ is *globally asymptotically stable* if it is stable and

$$\mathbf{x}(t) \rightarrow \mathbf{x}_0 \text{ as } t \rightarrow \infty$$

Consider the linear time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{A} \in R^{n\times n} \tag{2}$$

The eigenvalues of $\mathbf{A}$ are λ_i, $i = 1, \cdots, n$. The solution of (2) is

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) \tag{3}$$

where

$$\exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}t)^k = \mathbf{1} + \mathbf{A}t + \tfrac{1}{2}\mathbf{A}^2t^2 + \cdots$$

which converges for $t \in [0, \infty]$. If $\mathbf{A}$ has distinct eigenvalues λ_i, and eigenvectors $\mathbf{e}_i$, then we have the following eigendecomposition:

$$\begin{aligned} \mathbf{A} &= \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1} \\ \mathbf{E} &= \text{row}\{\mathbf{e}_i\} \\ \mathbf{\Lambda} &= \text{diag}\{\lambda_i\} \end{aligned}$$

In this case, it is not hard to show that

$$\begin{aligned} \mathbf{A}^k &= \mathbf{E}\mathbf{\Lambda}^k\mathbf{E}^{-1} \\ \mathbf{\Lambda}^k &= \text{diag}\{\lambda_i^k\} \\ \exp(\mathbf{A}t) &= \mathbf{E}\exp(\mathbf{\Lambda}t)\mathbf{E}^{-1} \\ \exp(\mathbf{\Lambda}t) &= \text{diag}\{\exp(\lambda_i t)\} \end{aligned}$$

We have the following stability results for the system in Eq. (2) and its equilibrium $\mathbf{x}_0 = \mathbf{0}$.

(a) The system is stable if $Re\{\lambda_i\} \leq 0$, $i = 1 \cdots n$, and there are no repeated eigenvalues on the imaginary axis.

(b) The system is unstable if there is at least one λ_i with $Re\{\lambda_i\} > 0$.

(c) The linear system is (globally) asymptotically stable if $Re\{\lambda_i\} < 0$, $i = 1 \cdots n$.

2.1 Lyapunov's Linearization (or First) Method

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. Let $\mathbf{x}(t) = \mathbf{x}_0 + \delta\mathbf{x}(t)$. Then,

$$\begin{aligned} \dot{\mathbf{x}} = \delta\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}_0 + \delta\mathbf{x}) \\ &= \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}^T}\right|_{\mathbf{x}=\mathbf{x}_0} \delta\mathbf{x} + \mathcal{O}(||\delta\mathbf{x}||^2) \end{aligned}$$

Neglecting higher order terms, we have the linearization

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x}, \quad \mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}^T}\right|_{\mathbf{x}=\mathbf{x}_0} \tag{4}$$

If the equilibrium $\delta\mathbf{x} = \mathbf{0}$ of (4) is:

(a) aysmptotically stable then $\mathbf{x}_0$ is an asymptotically stable equilibrium of the nonlinear system;

(b) unstable then $\mathbf{x}_0$ is an unstable equilibrium of the nonlinear system;

(c) stable then no conclusion can be drawn about the nonlinear system.

2.2 Lyapunov's Direct or Second Method

Consider the system in Eq. (1) with $\mathbf{x} = \mathbf{x}_0$ an equilibrium. We assume that $\mathbf{x}_0 = \mathbf{0}$ which can be accomplished with a change of coordinates.

A function $V(\mathbf{x}, t)$ is C^1 if it is continuously differentiable. It is a *locally positive-definite function* [lpdf] if $V(\mathbf{0}, t) \equiv \mathbf{0}$ and there exists $r > 0$ such that $V(\mathbf{x}, t) > 0$, $\forall \mathbf{x} \neq \mathbf{0}$ *s.t.* $||\mathbf{x}|| < r$. It is a *positive-definite function* [pdf] if $V(\mathbf{0}, t) \equiv \mathbf{0}$ and $V(\mathbf{x}, t) > 0$, $\forall \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$.

It is *radially unbounded* if

$$V(\mathbf{x}, t) \to \infty \text{ as } ||\mathbf{x}|| \to \infty$$

It is *locally negative definite* if $-V$ is a lpdf.

It is *negative definite* if $-V$ is a pdf.

Let $B_r = \{\mathbf{x} \in R^n | \quad ||\mathbf{x}|| < r\}$.

Examples

$V(x) = 1 - \cos x$ is a lpdf but it is not radially unbounded.

$V(x) = x^2$ is a pdf and it is radially unbounded.

$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a pdf and radially unbounded if $\mathbf{P}$ is a symmetric positive-definite matrix.

Theorem 1. The equilibrium $\mathbf{x}_0 = \mathbf{0}$ of (1) is stable if there exists a C^1 lpdf $V(\mathbf{x}, t)$ and $r > 0$ such that

$$\dot{V}(\mathbf{x}, t) \leq 0, \ \forall t \geq 0, \ \forall \mathbf{x} \in B_r$$

where $\dot{V}$ is evaluated along the trajectories of (1), *i.e.*,

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}^T}\mathbf{f}(\mathbf{x}, t)$$

V is called a Lyapunov function.

Example. Consider the equation of the simple pendulum

$$\ddot{\theta} + \sin\theta = 0$$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$ so that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 \end{aligned}$$

If we adopt $V(x_1, x_2) = (1 - \cos x_1) + \frac{1}{2}x_2^2$ as a Lyapunov function we have

$$\begin{aligned} \dot{V} &= \dot{x}_1 \sin x_1 + x_2\dot{x}_2 \\ &= x_2 \sin x_1 - x_2 \sin x_1 \\ &= 0 \end{aligned}$$

Hence $\mathbf{x} = \mathbf{0}$ is stable.

Example. Consider Euler's equation $\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times\mathbf{I}\boldsymbol{\omega} = \mathbf{0}$ where $\mathbf{I} = \mathbf{I}^T > \mathbf{0}$. Letting $V = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega}$, we have

$$\begin{aligned} \dot{V} &= \boldsymbol{\omega}^T\mathbf{I}\dot{\boldsymbol{\omega}} \\ &= -\boldsymbol{\omega}^T\boldsymbol{\omega}^\times\mathbf{I}\boldsymbol{\omega} \\ &= 0 \end{aligned}$$

Hence, $\boldsymbol{\omega} = \mathbf{0}$ is a stable equilibrium.

Theorem 2. The equilibrium $\mathbf{x}_0 = \mathbf{0}$ of Eq. (1) is asymptotically stable if there exists a C^1 lpdf V such that $-\dot{V}$ is a lpdf.

Theorem 3. The equilibrium $\mathbf{x}_0 = \mathbf{0}$ of Eq. (1) is globally asymptotically stable if there exists a C^1 pdf V such that V is radially unbounded and $-\dot{V}$ is a pdf.

Note: Theorems 1-3 give sufficient conditions for stability.

Example. Consider the system

$$\dot{x} = -ax, \quad a > 0$$

Letting $V = \frac{1}{2}x^2 > 0,\ \ x \neq 0$, we have $\dot{V} = x\dot{x} = -ax^2 < 0$, $x \neq 0$. Therefore $x = 0$ is a globally asymtotically stable equilibrium.

Example. Consider the system

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} &+ \mathbf{K}\mathbf{q} = \mathbf{u}(t), \ \ \mathbf{u} = -\mathbf{D}\dot{\mathbf{q}} \\ \mathbf{M} &= \mathbf{M}^T > \mathbf{O}, \mathbf{K} = \mathbf{K}^T > \mathbf{O}, \mathbf{D} = \mathbf{D}^T > \mathbf{O} \end{aligned}$$

Letting $\mathbf{x}_1 = \mathbf{q}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$, we have

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\mathbf{M}^{-1}\mathbf{K}\mathbf{x}_1 - \mathbf{M}^{-1}\mathbf{D}\mathbf{x}_2 \end{aligned}$$

Let

$$V = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} = \frac{1}{2}[\mathbf{x}_1^T\ \ \mathbf{x}_2^T]\begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} > 0\ \ (\mathbf{x} \neq \mathbf{0})$$

which is a radially unbounded pdf. Then,

$$\dot{V} = \dot{\mathbf{q}}^T(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = -\dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}} = -[\mathbf{x}_1^T\ \ \mathbf{x}_2^T]\begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq 0$$

Hence, $\mathbf{x} = \mathbf{0}$ is a stable equilibrium. It is in fact globally asymptotically stable. How do we show this when $\dot{V}$ does not contain $\mathbf{q}$?

2.3 Krasovskii-LaSalle Theorem

Consider the autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \ \mathbf{f}(\mathbf{0}) = \mathbf{0} \tag{5}$$

Suppose there exists a C^1 pdf $V(\mathbf{x})$ that is radially unbounded and

$$\dot{V} \leq 0,\ \forall t \geq 0, \forall \mathbf{x} \in R^n$$

Define the *invariant set* according to

$$M = \{\mathbf{x} \in R^n |\ \dot{V}(\mathbf{x}) = 0\}$$

If M contains only $\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable.

In the previous example,

$$\dot{V} = -\dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}} = 0 \Rightarrow \dot{\mathbf{q}} = \mathbf{0}$$

If $\dot{\mathbf{q}} \equiv \mathbf{0}$, then $\ddot{\mathbf{q}} = \mathbf{0}$ and

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \Rightarrow \mathbf{K}\mathbf{q} = \mathbf{0}$$

and hence $\mathbf{q} = \mathbf{0}$. Therefore the invariant set is given by $M = \{\mathbf{0}\}$ and $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable.

Theorem 4. The equilibrium $\mathbf{x} = \mathbf{0}$ is unstable if there exists a C^1 lpdf function V such that $\dot{V}$ is an lpdf.

Example. Consider the system

$$\dot{x} = ax, \quad a > 0$$

Letting $V = \frac{1}{2}x^2 > 0, \quad x \neq 0$, we have $\dot{V} = x\dot{x} = ax^2 > 0$, $x \neq 0$. Therefore $x = 0$ is an unstable equilibrium.

2.4 Stability of Linear Systems

Consider the linear time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

Let us select

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x}, \quad \mathbf{P} = \mathbf{P}^T > \mathbf{O} \tag{7}$$

as a Lyapunov function candidate. Hence

$$\begin{aligned} \dot{V} &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} \\ &= \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} \end{aligned}$$

If,

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}, \quad \mathbf{Q} = \mathbf{Q}^T > \mathbf{O} \tag{8}$$

then

$$\dot{V} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} < 0 \;\; (\mathbf{x} \neq \mathbf{0})$$

Hence, if given $\mathbf{Q} = \mathbf{Q}^T > \mathbf{O}$, the Lyapunov equation (8) has a symmetric positive-definite solution $\mathbf{P}$, then the eigenvalues of $\mathbf{A}$ have negative real parts, *i.e*, the system (6) is asymptotically stable.