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Time-domain floating body dynamics by rational approximation of the radiation impedance and diffraction mapping

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Abstract

The problem of approximating the dynamics of a floating structure in a transient wave environment with a set of constant-coefficient differential equations is explored. It is assumed that the solutions of the corresponding steady-state time-harmonic radiation and diffraction problems are available. It is proposed to fit the frequency responses associated with the 'radiation impedance' and wave-exciting forces with appropriate analytic functions. In the case of the radiation problem, these possess certain properties corresponding to the passivity of the radiation mapping. By choosing rational approximations, the transformation from the frequency to the time domain is facilitated. The method is illustrated for both two-dimensional and three-dimensional problems using a floating cylinder, sphere, and a model of Salter's Duck which exhibits hydrodynamic coupling between sway, heave, and pitch motions. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

This paper is concerned with the motion of a floating body which floats on the surface of an infinitely deep ocean of infinite extent in the presence of surface waves.

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The hydrodynamics is formulated under the assumptions of small body motions and wave heights and the fluid medium is incompressible, of constant density, inviscid, and irrotational. It is highly desirable if the mapping from transient wave motion to body motion can be approximated by a finite-dimensional, linear time-invariant one which corresponds to a system of constant-coefficient differential equations. Such a description is useful for simulation and control system design.

The problem was originally studied in the frequency domain in terms of steady-state harmonic solutions by John (1950) and others. The connection to the transient time-domain problem was noted by Cummins (1962); Wehausen (1971). The latter emphasized that inverse Fourier transformation of harmonic solutions was but one possibility; alternatively, the problem can be formulated directly in the time domain using a time-varying transient Green's function. These functions are known from the work of Finkelstein (1957) and have been used by Yeung (1982); Newman (1985); Beck and Liapis (1987); Pot and Jami (1991), who studied the radiation problem for various cylinders and spheres.

The transient motion of a two-dimensional cylinder has been studied by Ursell (1964); Maskell and Ursell (1970) where properties of the added mass and damping coefficients were used to infer asymptotic properties of the temporal solution. The full solution was obtained numerically using inverse Fourier transformation. None of the above approaches yields an explicit representation or realization of the transient body response to radiation and diffraction forces which is the subject of this work.

The spatially discretized motion equations of a general flexible body under the assumptions of small motions and linear elasticity naturally form a second-order system of constant-coefficient differential equations. These are characterized by symmetric mass and stiffness matrices and the latter can be augmented by the hydrostatic forces. Similar representations are desired for the radiation and diffraction forces. These are the outputs of linear systems driven by, respectively, the body motion (which we take as a velocity) and the wave motion (which is interpreted as the free surface displacement at one spatial location). The first of these is termed the radiation impedance and the second will be called the diffraction mapping. In the time domain, these linear systems manifest themselves as convolution operators and the Fourier transforms of the corresponding impulse responses give rise to the well-known hydrodynamic coefficients.

The radiation impedance and diffraction mapping can be approximated in the time or frequency domains. Recently, Yu and Falnes (1995) pursued the first route and provide a good survey of previous approaches; the work of Jefferys (1980, 1984) is of particular note. Both sets of authors seem unduly pessimistic on the subject of frequency domain approximation citing 'no obvious method.' Although Jefferys (1984) fitted single degree of freedom frequency responses, it was suggested that the technique was not appropriate for the multiple degree of freedom case owing to the number of parameters that are required. It should be noted that time-domain fitting procedures typically require the impulse response which must be obtained by inverse Fourier transformation or experimentally. Furthermore, the resulting least squares optimization is typically nonlinear in the unknown parameters since it involves fitting a matrix exponential to the impulse response. Yu and Falnes (1995), in addition to considering the single degree of freedom heave radiation problem, treat the diffraction problem pointing out the dependency of the causality of this mapping on the wave height datum. The issue was further studied by Falnes (1995).

In this paper, rational approximation of the radiation impedance and diffraction mappings is treated in the general multiple degree of freedom case which potentially exhibits hydrodynamic coupling. By using rational (matrix) functions for the approximation, the transformation from the frequency domain to the time domain is facilitated. The radiation approximation relies heavily on the passive (energy dissipative) nature of the impedance. This imparts the frequency domain transfer function with the positive real property which greatly constrains the form of the rational approximation. The fitting procedures all reduce to linear least squares problems. The diffraction transfer function does not possess the positive real property but good stable (causal) approximations are shown to be possible for suitable choices of the wave datum.

The method is illustrated for both two- and three-dimensional problems using a floating cylinder and sphere. A model of Salter's Duck is used to illustrate the ability to model multiple degrees of freedom which are coupled hydrodynamically.

2. Problem formulation

The motions of a floating body $\mathcal{B}$ will be described by $\mathbf{w}(\mathbf{r},t) = [w_1\ w_2\ w_3]^T$ where $\mathbf{r} = [x\ y\ z]^T$ with the z-axis vertically upwards and the origin lying in the mean free surface. A spatial discretization of the form

$$\mathbf{w}(\mathbf{r},t) = \sum_{\alpha=1}^{N} \psi_\alpha(\mathbf{r})\mathbf{q}_\alpha(t) = \boldsymbol{\Psi}(\mathbf{r})\mathbf{q}(t) \tag{1}$$

is employed where $\boldsymbol{\Psi} = \mathrm{row}\{\psi_\alpha\}$ and $\mathbf{q} = \mathrm{col}\{q_\alpha\}$. Included in $\boldsymbol{\Psi}$ are the six rigid-body motions:

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \psi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \psi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \psi_4 = \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}, \psi_5 = \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}, \psi_6 = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$$

Additional basis functions may be used to discretize elastic deformations in the case of a flexible body. Let $\mathbf{n}(\mathbf{r})$ denote the components of the outward normal to the wetted portion of $\mathcal{B}$, $\mathcal{S}$, and define $n_\alpha(\mathbf{r}) = \mathbf{n}^T\psi_\alpha$.

If $\sigma(\mathbf{r})$ denotes the mass density per unit volume and $p(\mathbf{r}, t)$ is the fluid pressure, the ensuing discrete motion equations are given by

$$\boldsymbol{M}\ddot{\mathbf{q}} + \boldsymbol{K}\mathbf{q} = \boldsymbol{f}(t) \tag{2}$$

where $M_{\alpha\beta} = \int_{\mathcal{B}} \psi_\alpha^T\psi_\beta\sigma\,\mathrm{d}V$, $f_\alpha = -\int_{\mathcal{S}} p(\mathbf{r},t)n_\alpha(r)\,\mathrm{d}S$, and $K_{\alpha\beta} = K_{\beta\alpha}$ are the elements of the stiffness matrix associated with structural flexibility or the elastic restoring effects of moorings. The motion of the fluid is governed by the velocity potential $\Phi(\mathbf{r}, t)$ which, in $\mathcal{V} = \{z \leq 0 \backslash \mathcal{B}\}$ satisfies

$$\nabla^2\Phi = 0, \mathbf{r} \in V; \frac{\partial^2\Phi}{\partial t^2} = -g\frac{\partial\Phi}{\partial z}, \mathbf{r} \in \mathcal{S}_f; \lim_{z \to -\infty}\frac{\partial\Phi}{\partial z} = 0; \tag{3}$$

$$\frac{\partial\Phi}{\partial n} = \mathbf{n}^T\dot{\mathbf{w}} = \sum_{\alpha=1}^{N} n_\alpha(\mathbf{r})\dot{q}_\alpha, \mathbf{r} \in \mathcal{S} \tag{4}$$

Here $\mathcal{S}_f = \{z = 0 \backslash \mathcal{B}\}$ denotes the free surface and g is the acceleration due to gravity.

We make the standard decomposition

$$\Phi = \Phi_I + \Phi_S + \Phi_R, \Phi_R = \sum_{\alpha=1}^{N} \phi_\alpha * \dot{q}_\alpha \tag{5}$$

where Φ_I describes the incident wave field and * is the temporal convolution product. The scattered (Φ_S) and radiated fields (Φ_R) satisfy Eq. (3) and are chosen to satisfy Eq. (4) such that

$$\frac{\partial\Phi_S}{\partial n} = -\frac{\partial\Phi_I}{\partial n}, \frac{\partial\phi_\alpha}{\partial n} = n_\alpha\delta(t), r \in \mathcal{S} \tag{6}$$

In addition, Φ_S and ϕ_α are bounded as are their first derivatives as $R = \sqrt{x^2 + y^2} \to \infty$. A further decomposition of ϕ_α is possible which separates memoryless effects from nonmemoryless ones in the convolution in Eq. (6) as demonstrated by Cummins (1962). Initial conditions are required to complete the problem and are discussed later.

For simplicity, the incident wave field is assumed to consist of transient waves propagating in the positive x-direction, i.e.,

$$\Phi_I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}_I e^{j(k'a' + \omega t)}\tilde{A}(\omega)\,\mathrm{d}\omega, \tilde{\phi}_I = \frac{jg}{\omega}e^{kz - jk'x}, k = \omega^2/g,$$

$$k' = k\,\mathrm{sgn}(\omega) \tag{7}$$

and a' is an appropriately chosen constant whose selection is described below. The free surface elevation satisfies

$$\eta(x,y,t) = -\frac{1}{g}\frac{\partial\Phi_I}{\partial t}\bigg|_{z=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk'(x-a')}\tilde{A}(\omega)e^{j\omega t}\,\mathrm{d}\omega \tag{8}$$

so that we can make the identification $\tilde{A}(\omega) = \mathcal{F}\{A(t)\}$ where $A(t) = \eta(a',y,t)$ and $\mathcal{F}\{(\cdot)\} = \widetilde{(\cdot)}$ denotes the Fourier transform:

$$\tilde{\Phi}(\mathbf{r}) = \int_{-\infty}^{\infty} \Phi(\mathbf{r},t)e^{-j\omega t}\,\mathrm{d}t \tag{9}$$

Using Bernoulli's equation, the (linearized) components of $\boldsymbol{f}(t)$ in Eq. (2) stemming from the fluid forces are

$$\boldsymbol{f}(t) = \mathrm{col}\left\{\iint_{\mathcal{S}}\left(\rho\frac{\partial\Phi}{\partial t} + \rho g z\right)n_\alpha\,\mathrm{d}S\right\} = \boldsymbol{f}_R(t) + \boldsymbol{f}_D(t) - \boldsymbol{K}_s\mathbf{q}(t) \tag{10}$$

Here, ρ is the fluid density and $\boldsymbol{K}_s = \mathrm{matrix}\{K_{s,\alpha\beta}\} = \boldsymbol{K}_s^T \geq \mathbf{O}$ is the matrix of hydrostatic restoring coefficients; see Newman (1977) for the rigid-body case. The radiation and diffraction forces are obtained by substituting Eq. (5) into Eq. (10) and satisfy

$$\boldsymbol{f}_R = \mathrm{col}\{f_{R\alpha}\},\ f_{R\alpha} = \rho\sum_{\beta=1}^{N}\iint_{\mathcal{S}}\left[\frac{\partial}{\partial t}(\phi_\beta * \dot{q}_\beta)n_\alpha\right]\mathrm{d}S \tag{11}$$

$$\boldsymbol{f}_D = \mathrm{col}\{f_{D\alpha}\},\ f_{D\alpha} = \rho\iint_{\mathcal{S}}\left[\frac{\partial\Phi_I}{\partial t} + \frac{\partial\Phi_s}{\partial t}\right]n_\alpha\,\mathrm{d}S \tag{12}$$

Introducing Eq. (10) into the motion Eq. (2) yields the dynamics of a floating body:

$$\boldsymbol{M}\ddot{\mathbf{q}} + (\boldsymbol{K} + \boldsymbol{K}_s)\mathbf{q} = \boldsymbol{f}_R(t) + \boldsymbol{f}_D(t) \tag{13}$$

The problem is now to relate the wave motion $A(t)$ to the body motion $\mathbf{q}(t)$ in an explicit manner.

3. Frequency domain modeling

Taking the Fourier transform of the force expressions in Eq. (11) and Eq. (12) gives

$$\tilde{f}_{R\alpha} = -\sum_{\beta=1}^{N}\tilde{H}_{\alpha\beta}(\omega)\tilde{\dot{q}}_\beta,\ \tilde{H}_{\alpha\beta}(\omega) \stackrel{\Delta}{=} -j\omega\rho\iint_{\mathcal{S}}(\tilde{\phi}_\beta n_\alpha)\mathrm{d}S \tag{14}$$

$$\tilde{f}_{D\alpha} = \tilde{X}_\alpha(\omega)\tilde{A}(\omega),\ \tilde{X}_\alpha(\omega) \stackrel{\Delta}{=} j\omega\rho\iint_{\mathcal{S}}[\tilde{\phi}_I + \tilde{\phi}_S]n_\alpha\,\mathrm{d}S e^{jk'a'} \tag{15}$$

where we have written $\tilde{\Phi}_I = \tilde{\phi}_I e^{jk'a'}\tilde{A}(\omega)$ and $\tilde{\Phi}_S = \tilde{\phi}_S e^{jk'a'}\tilde{A}(\omega)$. Transformation of the boundary value problem in Eq. (3) leads to

$$\nabla^2\tilde{\Phi} = 0,\ r \in \mathcal{V};\ \frac{\partial\tilde{\Phi}}{\partial z} = k\tilde{\Phi},\ r \in \mathcal{S}_f;\ \lim_{z\to-\infty} \frac{\partial\tilde{\Phi}}{\partial z} = 0 \tag{16}$$

Using Eq. (4), the potentials $\tilde{\phi}_\alpha$, $\alpha = 1,\ldots,N$, and $\tilde{\phi}_S$ also satisfy the above with $\tilde{\phi}_{\alpha,n} = n_\alpha$, $\tilde{\phi}_{S,n} = -\tilde{\phi}_{I,n}$ on $\mathcal{S}$ in keeping with the Fourier transform of Eq. (6). These boundary value problems are formally identical to the standard time-harmonic ones; in this light, the frequency domain radiation and scattered potentials also satisfy an outgoing radiation condition as $R\to\infty$.

Both potentials can be obtained in terms of a source distribution $\tilde{\gamma}(\mathbf{r})$ on $\mathcal{S}$ using

$$\tilde{\phi}(\mathbf{r}) = \int_{\mathcal{S}} G(\mathbf{r},\boldsymbol{\xi})\tilde{\gamma}(\boldsymbol{\xi})\mathrm{d}S \tag{17}$$

where $\tilde{\gamma}$ is a solution to a Fredholm integral equation of the form

$$-2\pi\tilde{\gamma}(\mathbf{r}) + \int_{\mathcal{S}} \frac{\partial G(\mathbf{r},\boldsymbol{\xi})}{\partial n_r}\tilde{\gamma}(\boldsymbol{\xi})\mathrm{d}S_\xi = \frac{\partial\tilde{\phi}(\mathbf{r})}{\partial n_r} \tag{18}$$

In two-dimensional problems, the factor -2π is replaced with $+\pi$. Here, $G(\mathbf{r},\boldsymbol{\xi})$ is Green's function which satisfies Laplace's equation and the free surface, bottom, and radiation conditions. Explicit expressions for $G(\mathbf{r},\boldsymbol{\xi})$ are given by Thorne (1953) in the two- and three-dimensional cases. Accurate techniques for calculation have been given by Newman (1984) in the three-dimensional case and are used here. The form of $G(x,z,\xi,\zeta)$ used here in the two-dimensional case has not been widely reported. It is

$$G(x,z,\xi,\zeta) = \log\frac{r}{r'} + 2\pi j e^{k\varrho} - 2\mathrm{Re}\{e^{k\varrho}E_1(k\varrho)\}$$

where $r = \sqrt{(x-\xi)^2+(z-\zeta)^2}$, $r' = \sqrt{(x-\xi)^2+(z+\zeta)^2}$, $\varrho = (z+\zeta) - j|x-\xi|$, and $E_1(\varrho) = -\gamma - \log\varrho - \sum_{n=1}^{\infty}[(-1)^n\varrho^n/(nn!)]$ $(\arg\{\varrho\} < \pi)$ is the exponential integral ($\gamma = 0.57721\ldots$ is Euler's constant). In the sequel, such an approach using constant source panels is employed to determine $\tilde{\phi}_\alpha$ and $\tilde{\phi}_s$ at discrete points on $\mathcal{S}$ for several values of k. Problems associated with irregular frequencies are handled using the modifications to $G(\mathbf{r},\boldsymbol{\xi})$ outlined by Ursell (1981):

$$G(\mathbf{r},\boldsymbol{\xi})\to G(\mathbf{r},\boldsymbol{\xi}) + \alpha_0 G(\mathbf{r},\mathbf{r}_0)G(\boldsymbol{\xi},\mathbf{r}_0)) \tag{19}$$

which effectively adds a source at $\mathbf{r} = \mathbf{r}_0$; we take $\alpha_0 = 1 - j$ and typically $\mathbf{r}_0 = \mathbf{0}$.

It is conventional to describe $\tilde{H}_{\alpha\beta}$ in Eq. (14) in terms of the added damping and mass coefficients: $\lambda_{\alpha\beta}(\omega) = \mathrm{Re}\{\tilde{H}_{\alpha\beta}\}$, $\mu_{\alpha\beta}(\omega) = \omega^{-1}\mathrm{Im}\{\tilde{H}_{\alpha\beta}\}$. It can be shown using energy considerations that $\boldsymbol{\Lambda} = \mathrm{matrix}\{\lambda_{\alpha\beta}\}$ is symmetric and positive-semidefinite and $\boldsymbol{\Lambda}\to\mathbf{O}$ as $\omega\to0$ and $\omega\to\infty$. The added mass matrix $\mathbf{M}_R = \mathrm{matrix}\{\mu_{\alpha\beta}\}$ is positive-definite in both the low- and high-frequency limits. The values of $\lambda_{\alpha\beta}$, $\mu_{\alpha\beta}$, and $\tilde{X}_\alpha$, are readily determined from Eqs. (14) and (15) using numerical approximations to the corresponding potential functions obtained using Eq. (17).

Assuming that the motions $A(t)$ and $\mathbf{q}(t)$ start at rest at $t = 0$, the radiation and diffraction forces can be written in the time domain as

$$f_{D\alpha}(t) = \int_0^t X_\alpha(t-\tau)A(\tau)\mathrm{d}\tau,\ f_{R\alpha}(t) = -\sum_{\beta=1}^{N}\int_0^t H_{\alpha\beta}(t-\tau)\dot{q}_\beta(\tau)\mathrm{d}\tau \tag{20}$$

where $X_\alpha(t)$ and $H_{\alpha\beta}(t)$ are inverse Fourier transforms of the quantities in Eqs. (14) and (15). It is known that $\tilde{H}_{\alpha\beta}(\omega)$ is an analytic function in a region containing the upper half of the complex ω plane thus yielding a causal convolution operator above for the radiation forces. Arguing on physical grounds, we take a' in Eq. (15) to satisfy $a' \leq \min_{\mathbf{r}\in S}[1\ 0\ 0]^T\mathbf{r}$ (the x-coordinate of the wetted portion of the body first exposed to the incoming wave) so that $X_\alpha(t)$ is causal and hence $\tilde{X}_\alpha(\omega)$ is analytic for $\mathrm{Im}\{\omega\} > 0$. Discussion provided by Falnes (1995) suggests that the 'diffraction operator' X_α can be made only approximately causal by such a device.

Our purposes are better served using the Laplace transform $\mathscr{L}\{\cdot\}$. Setting $X_\alpha(s) = \mathscr{L}\{X_\alpha(t)\}$, $H_{\alpha\beta}(s) = \mathscr{L}\{H_{\alpha\beta}(t)\}$, and taking the transform of Eqs. (14) and (15) gives

$$\boldsymbol{f}_D(s) = \mathbf{X}(s)A(s),\ \boldsymbol{f}_R(s) = -\mathbf{H}(s)\dot{\mathbf{q}}(s) \tag{21}$$

Note that $\mathbf{X}(j\omega) = \mathrm{col}\{\tilde{X}_\alpha(\omega)\}$ and $\mathbf{H}(j\omega) = \mathrm{matrix}\ \{\tilde{H}_{\alpha\beta}(\omega)\}$. The block diagram in Fig. 1 illustrates the relationship between the wave height and the body motion. The transfer matrix describing the body is a rational (matrix) function of s and gives rise to the constant-coefficient differential equations in Eq. (2). We seek similar descriptions for $\mathbf{H}(s)$ and $\mathbf{X}(s)$ in Eq. (21).

Consider the energy of the floating body described by Eq. (13):

$$\epsilon(t) = \tfrac{1}{2}\dot{\mathbf{q}}^T\boldsymbol{M}\dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T(\boldsymbol{K}+\boldsymbol{K}_s)\mathbf{q} \tag{22}$$

where the hydrostatic potential energy has been included. Its time derivative in con-

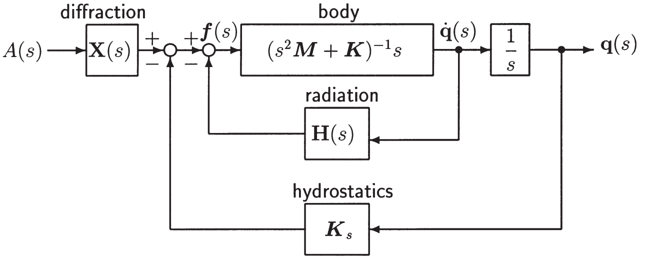


Fig. 1. Block diagram of combined radiation/diffraction problem.

junction with Eq. (13) gives $\dot{\epsilon} = (\boldsymbol{f}_R^T + \boldsymbol{f}_D^T)\dot{\mathbf{q}}$. In the absence of the incident wave field, we have upon integration

$$\epsilon(T) - \epsilon(0) = \int_0^T \boldsymbol{f}_R^T(t)\dot{\mathbf{q}}(t)\,\mathrm{d}t \leq 0,\ \forall T \geq 0 \tag{23}$$

where it has been argued that $\epsilon(T) \leq \epsilon(0)$, $\forall T \geq 0$. The inequality in Eq. (23) expresses the passivity of the mapping from $\dot{\mathbf{q}}$ to $-\boldsymbol{f}_R(t)$ and states that radiation is a dissipative process. Passive systems have been well studied in the context of electrical networks (see, for example, Anderson and Vongpanitlerd (1973)) and are treated in general by Zemanian (1965). There it is shown that a passive linear time-invariant system has a transfer function which is *positive real*; that is, for $\mathrm{Re}\{s\} > 0$: (i) $\mathbf{H}(s)$ is analytic; (ii) $\mathbf{H}(s)$ is real for real s; (iii) $\frac{1}{2}[\mathbf{H}(s) + \mathbf{H}^H(s)] \geq \mathbf{O}$.

It will be further assumed that $\mathbf{H}(s)$ and $\mathbf{X}(s)$ are continuous on the axis $s = j\omega$ which excludes the possibility of isolated poles on the imaginary axis. In this case, condition (iii) above holds on the imaginary axis and implies that $\boldsymbol{\Lambda}(\omega) = (1/2)\,[\mathbf{H}(\mathrm{j}\omega) + \mathbf{H}^{\mathrm{H}}(\mathrm{j}\omega)] \geq \mathbf{O}$. The radiation impedance can be further written as

$$\mathbf{H}(s) = \mathbf{H}_r(s) + s\boldsymbol{M}_\infty,\ \boldsymbol{M}_\infty = \text{matrix}\,\{\lim_{\omega\to\infty} \mu_{\alpha\beta}(\omega)\}$$

$\mathbf{H}_r(s)$ continues to enjoy the positive real property, as noted by Newcomb (1966), but it is strictly proper, i.e., $\mathbf{H}_r(s)\to\mathbf{O}$ as $s\to\infty$ since $\boldsymbol{\Lambda}\to\mathbf{O}$ in this limit. It is assumed that $\boldsymbol{M}_\infty$ is known; this calculation is feasible using a special limiting form of Green's function.

4. Approximation by analytic functions

Since $\mathbf{X}(s)$ and $\mathbf{H}_r(s)$ are analytic in a region containing the open right-half plane, they can be uniformly approximated in compact regions of this region by rational functions with poles in the left-half plane (this is Runge's theorem; see Rudin (1987)). We propose to make such an approximation on a finite part of the imaginary axis then extend the function to the right-half plane (RHP) by analytic continuation. Standard properties of rational positive real functions are given by Tao and Iaonnou (1988). In the scalar case, it is interesting to note that all rational PR functions have necessarily relative degree one so that rational approximations to the added damping and mass coefficients that are positive real satisfy $\lambda_{\alpha\alpha}\propto\omega^{-2}$ and $[\mu_{\alpha\alpha} - \mu_{\alpha\alpha}(\infty)]\propto\omega^{-2}$ as $\omega\to\infty$. This is at odds with known asymptotics for these coefficients but the discrepancy can be pushed to arbitrarily high frequency by increasing the order of the approximation.

The low-frequency asymptotics are similarly mismatched. For example, in the case of a two-dimensional heaving cylinder, Ursell (1964) has shown that near $s = 0$, $H_r(s)$ (suitably nondimensionalized) behaves like

$$H_r(s) = \frac{8}{\pi^2}\left[\frac{-2s\log s + (\frac{3}{2} - 2\log 2 - \gamma)s\ldots}{1 + \frac{4}{\pi}s^2\log s + \ldots}\right] \tag{24}$$

and hence has a logarithmic branch point at $s = 0$. Much of the hydrodynamical literature has emphasized the role of the acceleration to force map ($H_r(s)/s$) which does not possess the positive real property. The impedance $H_r(s)$ is not only positive real but partially avoids the problem at $s = 0$. Unlike $H_r(s)/s$ on the imaginary axis, $H_r(j\omega)$ is continuous at $\omega = 0$; Mergelyan's extension to Runge's theorem (Rudin, 1987) permits uniform approximation by rational functions in a compact region where the function is analytic inside but upon whose boundary the function is merely continuous.

4.1. Single degree of freedom case

To fix ideas, consider a single degree of freedom $q(t)$, say the heaving motion of a cylinder in two dimensions. The simplest rational positive real function, without poles on the imaginary axis, and satisfying $\mathrm{Re}\{\hat{H}_r(j\omega)\} = 0$ in the high- and low-frequency limits is one of second order:

$$\hat{H}_r(s) = \frac{bs}{s^2 + 2\zeta\Omega s + \Omega^2} \tag{25}$$

with $b > 0$, $\zeta > 0$, and $\Omega > 0$. This corresponds to the following relationship between $\dot{q}(t)$ and $f_R(t)$: $f_R(t) = -M_\infty\ddot{q}(t) + f_r(t)$ with $f_r = -\dot{q}_r$, $\ddot{q}_r + 2\zeta\Omega\dot{q}_r + \Omega^2 q_r = b\dot{q}$. In general, we propose to approximate $H_r(s)$ by

$$\hat{H}_r(s) = \frac{b(s)}{a(s)} = \frac{\sum_{i=1}^{n-1} b_i s^{n-i}}{s^n + \sum_{i=1}^{n} a_i s^{n-i}} \tag{26}$$

A sufficient condition for this to be positive real is that it be expressible as a combination of terms of the form of Eq. (25).

Given the values of the added mass and damping coefficients at N_p discrete frequencies ω_i, we propose to minimize the following function with respect to the a_i and b_i:

$$\mathcal{J} = \tfrac{1}{2}\sum_{i=1}^{N_p} |\hat{H}_r(j\omega_i) - H_r(j\omega_i)|^2,\; H_r(j\omega_i) = \lambda(\omega_i) + j\omega_i[\mu(\omega_i) = \mu(\infty)] \tag{27}$$

This is not in the form of a standard least squares problem given the nonquadratic dependence on the a_i but can be approximated by the series of problems

$$\mathcal{J}^{(m)} = \frac{1}{2}\sum_{i=1}^{N_p} |b^{(m)}(j\omega_i) - H_r(j\omega_i)a^{(m)}(j\omega_i)|^2 |W^{(m)}(j\omega_i)|^2, \; m = 1,\ldots,M \tag{28}$$

which are quadratic in the $b_i^{(m)}$ and $a_i^{(m)}$. Here, $W^{(m)}(s)$ is a weighting function which we take as 1 for $m = 1$ and $W^{(m)}(s) = 1/a^{(m-1)}(s)$, $m = 2,3,\ldots,M$. In carrying out the optimization, it may so happen that after M iterations $a^{(M)}(s)$ has poles in the RHP, leading to an unstable approximation. This is remedied by carrying out one further optimization according to Eq. (27) with a prescribed denominator for $\hat{H}_r(s)$ containing the stable poles of $a^{(M)}(s)$. Typically the resulting approximations are positive real since the data possess this property. If not, it is possible to further approximate $\hat{H}_r(s)$ by the closest positive real function with the same poles using the algorithm of Damaren et al. (1996).

The constituents of $\mathbf{X}(s)$, $X_\alpha(s)$, can be approximated in the same way but the positive real and low-frequency conditions are not required. We take

$$\hat{X}_\alpha(s) = \frac{b_{D\alpha}(s)}{a_{D\alpha}(s)} = \frac{\sum_{i=1}^{n} b_{\alpha,i}s^{n-i}}{s^n + \sum_{i=1}^{n} a_{\alpha,i}s^{n-i}} \tag{29}$$

and employ Eq. (28) for determining $b_{D\alpha}(s)$ and $a_{D\alpha}(s)$.

4.2. Multiple degree of freedom case

In this case, it is helpful if the same denominator polynomial $a_D(s) = s^n + a_{d1}s^{n-1} + \ldots + a_{dn}$ is used for each $\hat{X}_\alpha(s)$ in Eq. (29). Then, the first mapping in Eq. (18) can be realized in the time domain as

$$\boldsymbol{f}_D(t) = \mathbf{C}_d\mathbf{x}_d, \; \dot{\mathbf{x}}_d = \mathbf{A}_d\mathbf{x}_d + \mathbf{B}_d A(t) \tag{30}$$

where

$$\mathbf{C}_d = \text{matrix}\,\{b_{\alpha,i}\}, \; \mathbf{B}_d = [1\; 0\ldots 0\; 0]^T,$$

$$\mathbf{A}_d = \begin{bmatrix} -a_{d1} & -a_{d2} & \cdots & -a_{d,n-1} & -a_{dn} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The denominator $a_D(s)$ can be the product of individual ones $a_{D\alpha}(s)$ optimised for each $X_\alpha(s)$ and the $b_{\alpha,i}$ can be obtained using one final least squares optimization for each $\hat{X}_\alpha$ using the composite $a_D(s)$. Each of these problems is a standard least squares problem similar in form to Eq. (27).

In the multivariable case, we propose to represent $\mathbf{H}_r(s)$ as

$$\hat{\mathbf{H}}_r(s) = \sum_{k=1}^{N_2} \mathbf{W}_k \frac{s}{s^2 + 2\zeta_k\Omega_k s + \Omega_k^2}, \ \zeta_k > 0, \ \Omega_k > 0, \ \mathbf{W}_k = \mathbf{W}_k^T \tag{31}$$

This constitutes a rational matrix function without poles on the imaginary axis, which is strictly proper, and has a real part vanishing in the low-frequency limit. A sufficient condition for it to be positive real is that $\mathbf{W}_k$ is positive-definite. The ζ_k and Ω_k are determined by applying the scalar approach outlined above to the diagonal terms $H_{r,\alpha\alpha}(s)$. The totality of stable poles for $\alpha = 1\ldots N$ is forced to be even and used to calculate ζ_k and Ω_k. With these parameters then fixed, the quantities $W_{k,\alpha\beta}$ are determined by approximating $H_{r,\alpha\beta}(s)$ with fixed poles according to Eq. (27). Implicit in this approach is that poles corresponding to $\hat{H}_{r,\alpha\alpha}$ can be used to successfully approximate $H_{r,\alpha\beta}$, $\alpha \neq \beta$.

The resulting matrices $\mathbf{W}_k$ can be factored as $\mathbf{W}_k = \mathbf{C}_{rk}\mathbf{B}_{rk}^T$ where $\mathbf{B}_{rk}$ has full column rank which we denote by m_k. The advantage of Eq. (31) with this factorization lies in its ability to be realized in a matrix second-order form:

$$\boldsymbol{f}_r(t) = -\mathbf{C}_r\dot{\mathbf{q}}_r, \ \ddot{\mathbf{q}}_r + \mathbf{D}_r\dot{\mathbf{q}}_r + \mathbf{K}_r\mathbf{q}_r = \mathbf{B}_r\dot{\mathbf{q}} \tag{32}$$

where $\mathbf{C}_r = \text{row}\{\mathbf{C}_{rk}\}$, $\mathbf{B}_r = \text{col}\{\mathbf{B}_{rk}\}$, $\mathbf{D}_r = \text{block diag}\{2\zeta_k\Omega_k\mathbf{1}_{m_k}\}$, $\mathbf{K}_r = \text{block diag}\{\Omega_k^2\mathbf{1}_{m_k}\}$, and $\mathbf{1}_{m_k}$ is the $m_k \times m_k$ identity matrix. In other words, taking Laplace transforms of Eq. (32) yields $\boldsymbol{f}_r(s) = -\hat{\mathbf{H}}_r(s)\dot{\mathbf{q}}(s)$ where $\hat{\mathbf{H}}_r(s)$ is given by Eq. (31). Note that there are Σm_k coordinates in $\mathbf{q}_r$.

When the radiation forces are written as $\boldsymbol{f}_R(t) = -\boldsymbol{M}_\infty\ddot{\mathbf{q}} + \boldsymbol{f}_r(t)$ and combined with Eqs. (13), (30) and (32), we have

$$\begin{bmatrix} \boldsymbol{M} + \boldsymbol{M}_\infty & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{q}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{O} & \mathbf{C}_r \\ \mathbf{B}_r & \mathbf{D}_r \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} + \boldsymbol{K}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_r \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{q}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{O} \end{bmatrix} \mathbf{x}_d \tag{33}$$

with $\mathbf{x}_d$ given by Eq. (30). By defining $\chi = \text{col}\{\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r, \mathbf{x}_d\}$, the mapping from $A(t)$ to $\mathbf{q}(t)$ can be written in the standard form

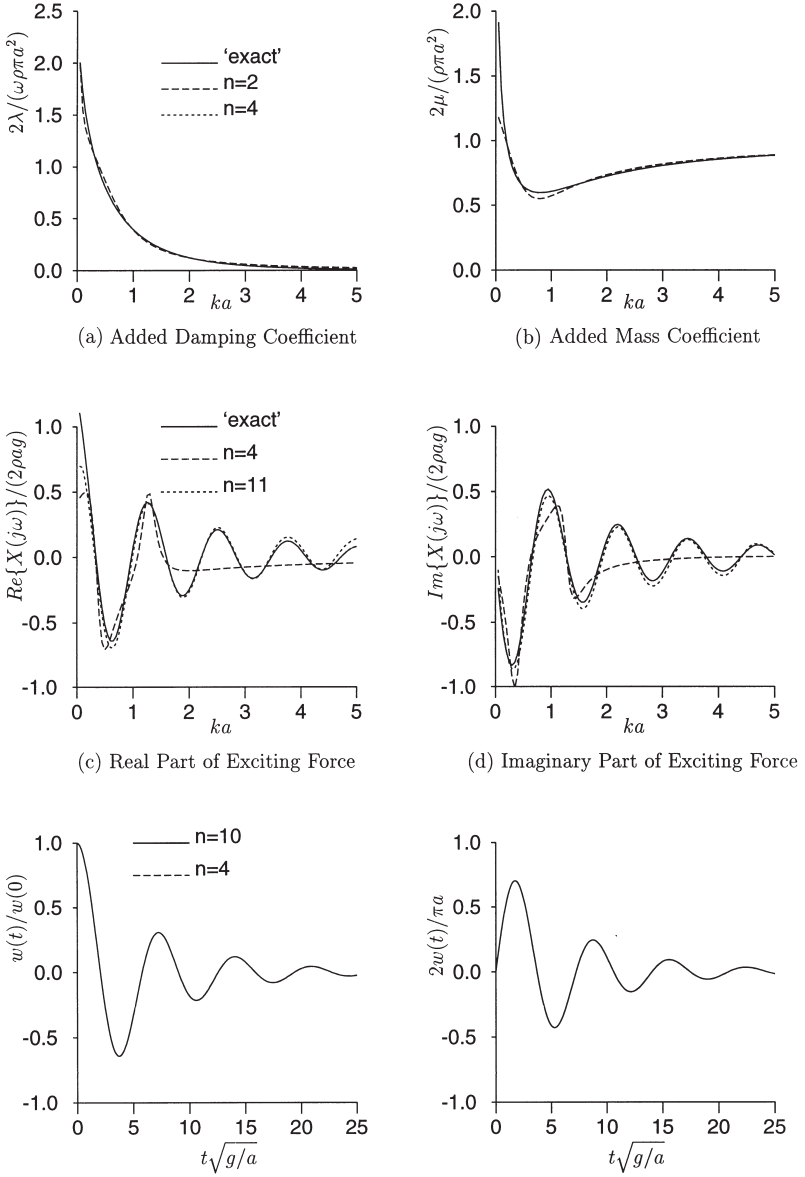
$$\mathbf{q}(t) = \mathscr{C}\chi(t), \ \dot{\chi} = \mathscr{A}\chi + \mathscr{B}A(t) \tag{34}$$

for appropriate $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$. This model is of the desired form and the unforced problem ($A(t) \equiv 0$, $\mathbf{x}_d \equiv \mathbf{0}$) as given by Eq. (33) is characterized by symmetric mass, damping, and stiffness matrices. The corresponding eigenproblem permits the determination of the coupled body/water 'hydrodynamic modes' whose truncation is one way of further reducing the order of the model.

5. Numerical examples

5.1. The heaving cylinder

Consider the two-dimensional heaving motion of a circular cylinder of radius a and draft a. The added damping and mass coefficients obtained with an integral equation approach with 200 constant source panels are shown in Fig. 2(a) and (b). The modification to Green's function indicated by Eq. (19) was employed to reduce the effect of irregular frequencies. A single second-order approximation like Eq. (25) captures the added mass and damping coefficients quite well and for $n = 4$ there is



(a) Added Damping Coefficient (b) Added Mass Coefficient

(c) Real Part of Exciting Force (d) Imaginary Part of Exciting Force

(e) Free Decay (Initial Position) (f) Free Decay (Initial Velocity)

Fig. 2. Results for two-dimensional heave of a cylinder.

little discernible difference. For all fits, $N_p = 100$ at equally spaced values in the range $0 < ka \leq 5$.

The real and imaginary parts of the diffraction force coefficient are shown in Fig. 2(c) and (d) where a value of $a' = -6a$ was used to describe the wave height. Fits to both functions are shown for various values of n. The given value of n corresponds to the effective value after performing $M = 10$ iterations in Eq. (28) and removing unstable poles. For example, for the $n = 11$ curve, the initial approximation was of order 12. It is interesting to note the effect of a' on the ability to determine a good stable approximation. For $a' = 0$, most of the poles in the approximation are typically unstable. As a' is made more negative, the proportion of stable poles tends to increase but the function $X(j\omega)$ becomes more oscillatory and is harder to approximate. The value of $a' = -6a$ represents a satisfactory compromise.

The corresponding time behavior of the free decay from a displaced initial configuration is shown in Fig. 2(e) for $n = 4$ and $n = 10$ in the radiation approximation. This was obtained by solving the homogeneous form of Eq. (34) (without the $\mathbf{x}_d$ states) using the matrix exponential. Our results show excellent agreement with those of Yeung (1982) and slightly less good agreement with those of Maskell and Ursell (1970). The latter authors showed that the cylinder free decay was dominated by a pair of poles located at $(-0.131 \pm j0.919)\sqrt{g/a}$. The dominant eigenvalues of the unforced system in Eq. (34) for $n = 4$ and $n = 10$ are given by $(-0.143 \pm j0.919)\sqrt{g/a}$. The corresponding free decay for an initial velocity $\dot{w}(0) = \sqrt{ag}$ is given in Fig. 2(f) with normalization corresponding to that of Maskell and Ursell. The agreement is quite favorable.

5.2. The heaving sphere

A similar approach was employed for the heaving motion of a (three-dimensional) hemisphere and good agreement was obtained between the panel solutions and the accurate solutions of Hulme (1982) for the added mass and damping coefficients. The latter are shown in Fig. 3(a) and (b) as well as the fit obtained using $n = 4$. The force coefficients are given in Fig. 3(c) and (d) for an array of 40×12 panels (linear circumferential spacing and cosine azimuthal spacing) and $a' = -4a$. The corresponding rational approximation for $n = 11$ ($M = 10$ iterations and $N_p = 200$ points in the range $0 < ka \leq 10$) is also shown.

The free decay for an initial displacement and an initial velocity $\dot{w}(0) = \sqrt{ag}$ is shown in Fig. 3(e) and (f) for $n = 4$ and $n = 10$ in the radiation approximation. They are virtually identical, indicating the converged nature of the radiation approximation and show good agreement with the results of Beck and Liapis (1987).

5.3. A multivariable example: Salter's Duck

Consider the two-dimensional model shown in Fig. 4 representing a partially submerged ‘Salter's Duck.’ The hydrodynamic properties were considered by Mynett et al. (1979) in finite depth water. The hydrodynamic coefficients obtained using 300 equal-sized panels for (coupled) sway (1), heave (3), and pitch (5) motions with

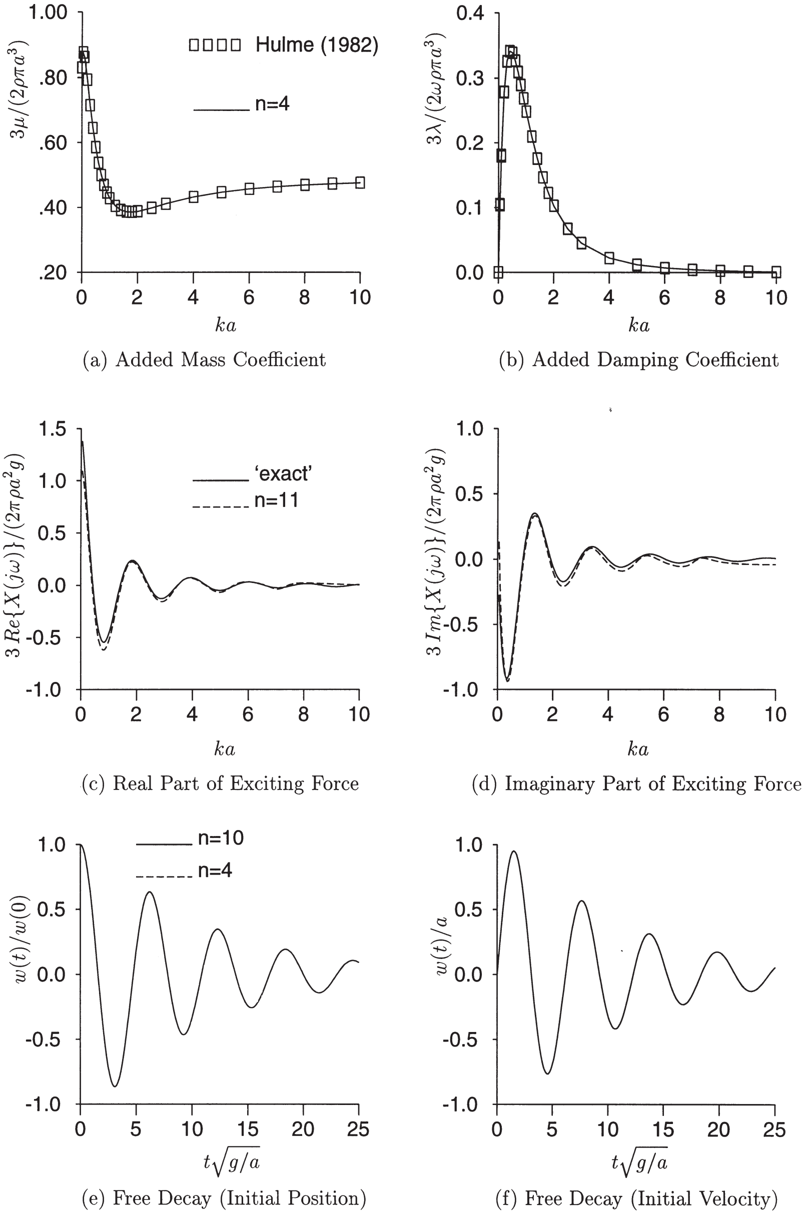


Fig. 3. Results for a heaving sphere.

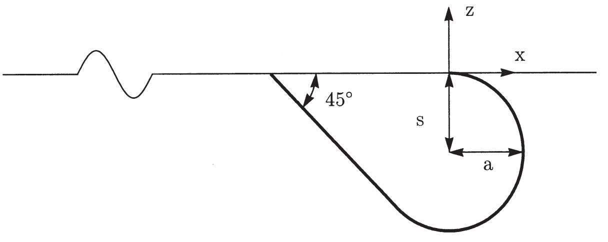


Fig. 4. Model for Salter's Duck.

$s = 0$ are given in Figs. 5 and 6. The 'extra source' in Eq. (19) was located at $\mathbf{r}_0 = [-a(\sqrt{2}-1)/2\ 0\ 0]^T$ (along the line of the center of buoyancy). A tenth-order approximation like Eq. (26) was used to initially fit each $H_{\alpha\alpha}(s)$, $\alpha = 1,3,5$, which furnished 18 appropriate values for the the Ω_k and ζ_k in Eq. (31) after removing

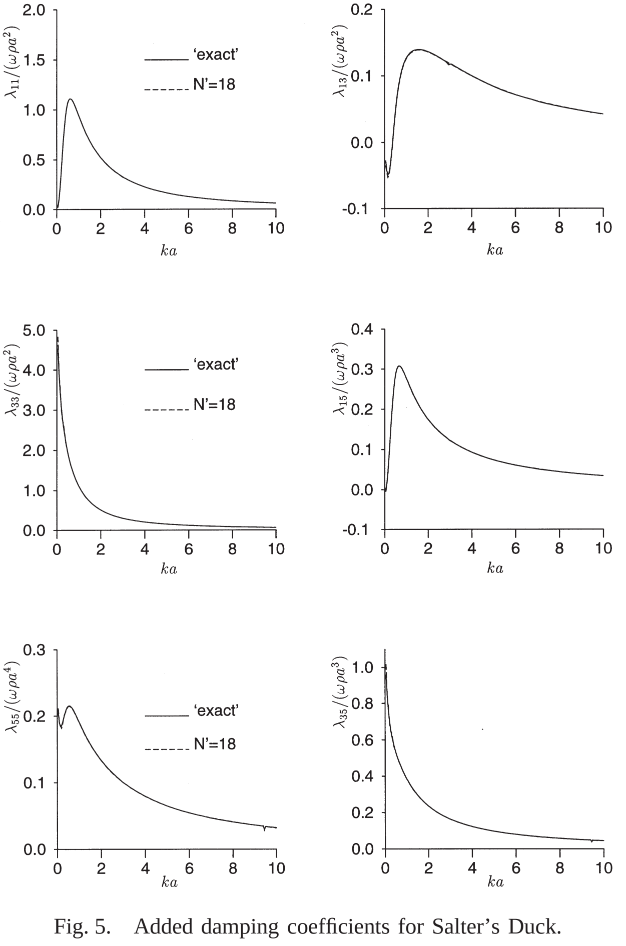


Fig. 5. Added damping coefficients for Salter's Duck.

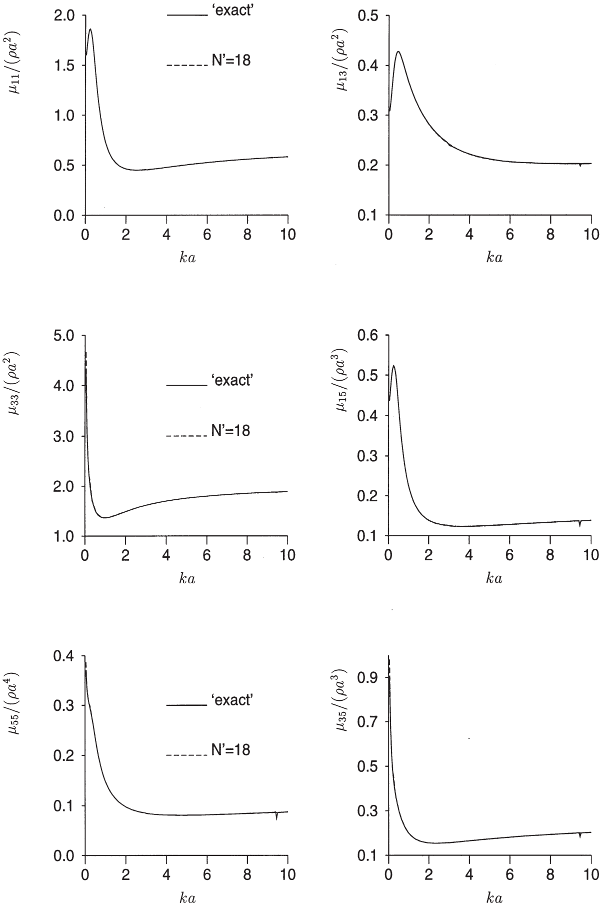


Fig. 6. Added mass coefficients for Salter's Duck.

unstable poles and retaining only complex-conjugate pairs. There were also two stable real poles which were discarded. The entries in each $\mathbf{W}_k$ were then obtained by fitting each $H_{\alpha\beta}(s)$ with the denominators fixed. The approximations for the added damping and mass coefficients are also given in Figs. 5 and 6, respectively. The approximations are very good including those corresponding to the off-diagonal coefficients and show no discernible differences.

The exciting forces corresponding to $a' = 6a$ are shown in Fig. 7. These were validated using the Haskind relations,

$$\tilde{X}_\alpha = j\omega\rho\int_s\left(\tilde{\phi}_I n_\alpha - \frac{\partial\tilde{\phi}_I}{\partial n}\,\tilde{\phi}_\alpha\right)\mathrm{d}S \tag{35}$$

to test their consistency with the radiation potentials. Each $X_\alpha(j\omega)$ was initially fit

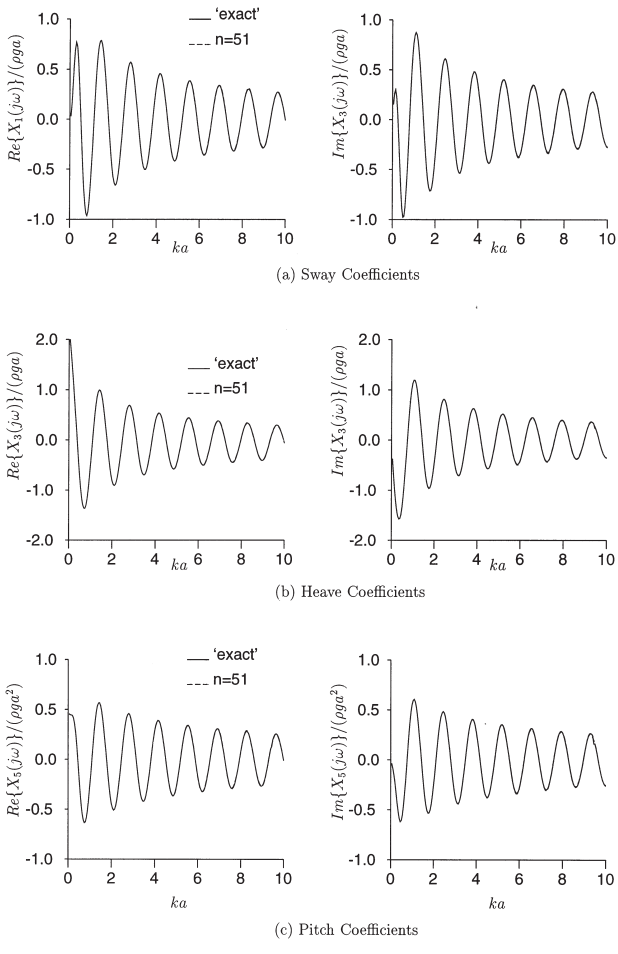


Fig. 7. Diffraction coefficients for Salter's Duck.

with $n = 20$ and $M = 10$ iterations. The resulting approximation in Eq. (25) was of order 51 after combining the individual denominator polynomials and removing unstable poles. Given in Fig. 7 are the ensuing approximations for the real and imaginary parts of $\mathrm{Re}\{\tilde{X}_\alpha\}$ and $\mathrm{Im}\{\tilde{X}_\alpha\}$. The combined first-order transient solution in Eq. (34) is of order 93 and can be used to determine the motion of the body given the wave history of $A(t)$ at $x = -6a$.

6. Concluding remarks

The approach presented here has the ability to approximate transient solutions arbitrarily closely and provides an alternative methodology upon which comparison with other techniques can be made. It is also based on frequency-domain solutions for which well-established procedures already exist. Most important is the explicit characterization of the dynamics relating a transient sea to the body motions in terms of linear constant-coefficient matrix differential equations. This is particularly important when one contemplates the problem of control system design for large floating structures.

The success of the method suggests the possibility of using rational approximations for Green's function itself so that the radiation and diffraction forces may be formed directly with their rational dependence on frequency manifested analytically. The motion equations may then be formulated directly in the time domain without the need to determine the hydrodynamic coefficients at many frequencies.

References

Anderson, B.D.O., Vongpanitlerd, S., 1973. Network Analysis and Synthesis: A Modern Systems Theory Approach. Prentice-Hall, Englewood Cliffs, NJ.

Beck, R.F., Liapis, S., 1987. Transient motions of floating bodies at zero forward speed. Journal of Ship Research 31 (3), 164–176.

Cummins, W.E., 1962. The impulse response function and ship motions. Schiffstechnik 47 (9), 101–109.

Damaren, C.J., Marquez, H.J., Buckley, A.G., 1996. Optimal strictly positive real approximations for stable transfer functions. IEE Proceedings—Control Theory and Applications 143 (6), 537–542.

Falnes, J., 1995. On non-causal impulse response functions related to propagating water waves. Applied Ocean Research 17, 379–389.

Finkelstein, A.B., 1957. The initial value problem for transient water waves. Comm. on Pure and Applied Maths 10, 511–522.

Hulme, A., 1982. The wave forces acting on a floating hemisphere undergoing forced periodic oscillations. J. Fluid Mechanics 121, 443–463.

Jefferys, E.R., 1980. Device characterization. In: Count, B. (Ed.), Power from Sea Waves. Academic Press, New York, pp. 413–438.

Jefferys, E.R., 1984. Simulation of wave power devices. Applied Ocean Research 6, 31–39.

John, F., 1950. On the motion of floating bodies II. Comm. on Pure and Applied Maths 3, 45–101.

Maskell, S.J., Ursell, F., 1970. The transient motion of a floating body. J. Fluid Mechanics 44, 303–313.

Mynett, A.E., Serman, D.D., Mei, C.C., 1979. Characteristics of Salter's cam for extracting energy from ocean waves. Applied Ocean Research 1, 13–20.

Newcomb, R.W., 1966. Linear Multiport Synthesis. McGraw-Hill, New York.

Newman, J.N., 1977. Marine Hydrodynamics. MIT Press, Cambridge, MA.

Newman, J.N., 1984. Double-precision evaluation of the oscillatory source potential. J. Ship Research 28 (3), 151–154.

Newman, J.N., 1985. Transient axisymmetric motion of a floating cylinder. J. Fluid Mechanics 157, 17–33.

Pot, G., Jami, A., 1991. Some numerical results in 3-d transient linear naval hydrodynamics. Journal of Ship Research 35 (4), 295–303.

Rudin, W., 1987. Real and Complex Analysis, 3rd ed. McGraw-Hill, New York.

Tao, G., Ioannou, P.A., 1988. Strictly positive real matrices and the Lefschetz–Kalman–Yakubovich lemma. IEEE Trans. Automatic Control 33 (12), 1183–1185.

Thorne, R.C., 1953. Multipole expansions in the theory of surface waves. Proc. Cambridge Philosophical Society 49, 707–716.

Ursell, F., 1964. The decay of the free motion of a floating body. J. Fluid Mechanics 19, 305–319.

Ursell, F., 1981. Irregular frequencies and the motion of floating bodies. J. Fluid Mechanics 105, 143–156.

Wehausen, J.V., 1971. The motion of floating bodies. Annual Review of Fluid Mechanics 3, 237–268.

Yeung, R.W., 1982. The transient heaving motion of floating cylinders. J. Engineering Mathematics 16, 97–119.
Yu, Z., Falnes, J., 1995. State-space modelling of a vertical cylinder in heave. Applied Ocean Research 17, 265–275.
Zemanian, A.H., 1965. Distribution Theory and Transform Analysis. McGraw-Hill, New York.