Engineering Notes

Nonlinear $\mathcal{H}_\infty$ Attitude Control Using Modified Rodrigues Parameters

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Nomenclature

a	=	angular velocity gain
$\boldsymbol{a}$	=	axis of rotation
b	=	modified Rodrigues parameter gain
$\boldsymbol{C}$	=	rotation matrix
c	=	coefficient in storage (Lyapunov) function
$\boldsymbol{d}$	=	disturbance torque
$\boldsymbol{f}(\boldsymbol{x})$	=	state function
$\boldsymbol{G}(\boldsymbol{\sigma})$	=	modified Rodrigues parameter kinematics matrix
$\boldsymbol{g}(\boldsymbol{x})$	=	control input function
H_γ	=	left-hand side of Hamilton–Jacobi inequality
$\boldsymbol{h}(\boldsymbol{x})$	=	state weighting function
$\boldsymbol{J}$	=	moment of inertia matrix
$\boldsymbol{k}(\boldsymbol{x})$	=	disturbance input function
q_1, q_2	=	state weighting factors
$\boldsymbol{u}$	=	control torque
$V(\boldsymbol{x})$	=	Lyapunov function
$\boldsymbol{x}$	=	state variables
$\boldsymbol{x}_0$	=	equilibrium state
$\boldsymbol{z}$	=	regulated outputs
$\{\boldsymbol{\epsilon}, \eta\}$	=	Euler parameters
γ	=	upper bound on $\mathcal{L}_2$-gain
$\mathcal{K}(\boldsymbol{x})$	=	state feedback controller
$\mathcal{L}_2[0, T]$	=	space of square-integrable functions on $[0, T]$
$\boldsymbol{\omega}$	=	angular velocity
ϕ	=	angle of rotation
$\boldsymbol{\sigma}$	=	modified Rodrigues parameters
$\boldsymbol{\sigma}_s$	=	shadow modified Rodrigues parameters

I. Introduction

THE attitude control problem for rigid spacecraft has attracted significant attention. The problem is readily formulated using Euler's equation for the evolution of the angular velocity vector and an equation to describe the evolution of the attitude. The latter typically employs the rotation matrix or, more commonly, some parameterization of it. The most common parameterization uses the Euler parameters (or quaternions), which are a singularity-free four-parameter set satisfying a constraint. Another interesting possibility uses the modified Rodrigues parameters (MRPs) either by themselves or in conjunction with their shadow parameters [1]. MRPs are a three-parameter set that possesses a singularity when the principal angle of rotation reaches $\pm 2\pi$. The singularity for the shadow set occurs when the principal angle of rotation is zero. Hence, one can combine the MRPs with their shadow set to obtain a singularity-free parameterization of the rotation matrix. This requires switching between the two sets at an appropriate point.

The attitude control problem can be formulated as a disturbance-free regulation or tracking problem. Alternatively, one can view the problem from the disturbance attenuation point of view. That is the approach that will be examined in this note using the MRPs and their shadow set. In particular, it will be shown that a linear feedback law consisting of a linear combination of the spacecraft angular velocity and the generic MRPs locally solves the (suboptimal) nonlinear $\mathcal{H}_\infty$ state feedback problem [2] (which may also be referred to as the $\mathcal{L}_2$ disturbance attenuation problem). The control solution is globally defined in the presence of MRP switching, and the degradation of the $\mathcal{L}_2$-gain due to switching is examined. This control law is the one that was proposed by Schaub and Junkins [1] as a globally asymptotically stable solution to the attitude regulation problem. A similar control law (although without use of the shadow parameters) was proposed by Tsiotras [3,4] using passivity arguments to motivate it.

It should be emphasized that these previous works on the stability of feedback using MRPs [1,3,4] considered only Lyapunov stability and did not establish the disturbance rejection properties of the linear angular velocity and MRP feedback. The fact that this *linear* feedback can be shown to solve a local version of the *nonlinear* $\mathcal{H}_\infty$ state feedback problem is our major contribution. The proof of this result will be constructive and will establish analytical formulas for the gain parameters in terms of the desired $\mathcal{L}_2$-gain, the control weighting parameters in the performance output, and the moments of inertia.

Other relevant work on the attitude disturbance rejection problem is that of Kang [5], who showed that a linear combination of angular velocity and some function of the rotation matrix locally solved the nonlinear $\mathcal{H}_\infty$ control problem, and Dalsmo and Egeland [6], who showed that a linear combination of angular velocity and quaternion feedback for the control torques solved the problem. The main goal of the present work is to obtain the comparable result using MRPs in lieu of the quaternions.

II. Suboptimal Nonlinear $\mathcal{H}_\infty$ State Feedback Problem

Consider the nonlinear dynamical system described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{k}(\boldsymbol{x})\boldsymbol{d}, \qquad \boldsymbol{f}(\boldsymbol{x}_0) = 0 \tag{1}$$

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ \boldsymbol{u} \end{bmatrix}, \qquad \boldsymbol{h}(\boldsymbol{x}_0) = 0 \tag{2}$$

where $\boldsymbol{x} \in R^n$ is the state vector, $\boldsymbol{u} \in R^m$ is the control vector, $\boldsymbol{d} \in R^p$ is the disturbance vector, and $\boldsymbol{z} \in R^q$ are the regulated outputs. The functions $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{k}$, and $\boldsymbol{h}$ are appropriately dimensioned smooth functions of the state.

The objective of the suboptimal nonlinear $\mathcal{H}_\infty$ state feedback problem is to determine a (possibly nonlinear) state feedback $\boldsymbol{u} = \mathcal{K}(\boldsymbol{x})$ so that for given $\gamma > 1$, the following inequality is satisfied:

$$\int_0^T \|\boldsymbol{z}(t)\|^2 \,\mathrm{d}t \le \gamma^2 \int_0^T \|\boldsymbol{d}(t)\|^2 \,\mathrm{d}t, \quad \forall\, T \ge 0, \quad \forall\, \boldsymbol{d} \in \mathcal{L}_2[0, T] \tag{3}$$

when $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Here, $\|(\cdot)\|$ denotes the Euclidean norm and $\mathcal{L}_2[0, T]$ is the space of functions that are square-integrable on $[0, T]$.

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From [2], there is the following result: the control law

$$\boldsymbol{u} = -\boldsymbol{g}^T(\boldsymbol{x})(\partial V/\partial \boldsymbol{x})^T \tag{4}$$

with $V(\boldsymbol{x}_0) = 0$, $V(\boldsymbol{x}) > 0(\boldsymbol{x} \neq \boldsymbol{x}_0)$, and

$$\begin{aligned} H_\gamma\left(\boldsymbol{x}, \frac{\partial V}{\partial \boldsymbol{x}}\right) &= \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) \\ &+ \frac{1}{2}\frac{\partial V}{\partial \boldsymbol{x}}\left(\frac{1}{\gamma^2}\boldsymbol{k}(\boldsymbol{x})\boldsymbol{k}^T(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}^T(\boldsymbol{x})\right)\left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \\ &+ \frac{1}{2}\boldsymbol{h}^T(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x}) \leq 0 \end{aligned} \tag{5}$$

solves the suboptimal nonlinear $\mathcal{H}_\infty$ control problem. Equation (5) is termed the Hamilton–Jacobi inequality. Note that $\partial V/\partial \boldsymbol{x}$ is a row vector.

Now, relax the assumption that $\boldsymbol{f}(\boldsymbol{x})$ is smooth by permitting jump discontinuities of the state (such will be the case when switching between the MRPs and their shadow parameters). Assume that the state experiences jump discontinuities at $t = t_k$, $k = 1, \ldots, N-1$, and that $t_0 = 0 \leq t_1 < \cdots < t_{N-1} \leq t_N = T$. It will be assumed that $\boldsymbol{f}(\boldsymbol{x})$ is a continuous function of the state, but because the state is possibly a discontinuous function of time, it is assumed that $\boldsymbol{f}(\boldsymbol{x}(t))$ is only a piecewise continuous function of time t.

Now, applying the preceding theory to each time segment $[t_{k-1}^+, t_k^-]$ while using Eqs. (1), (4), and (5), it can be shown using the methods in [2] that

$$\frac{1}{2}\int_{t_{k-1}}^{t_k} \|z(t)\|^2 \,\mathrm{d}t \leq \frac{1}{2}\gamma^2 \int_{t_{k-1}}^{t_k} \|\boldsymbol{d}(t)\|^2 \,\mathrm{d}t + V(\boldsymbol{x}(t_{k-1}^+)) - V(\boldsymbol{x}(t_k^-)) \tag{6}$$

where $\boldsymbol{x}(t_k^\pm)$ denotes the state vector on either side of the discontinuity. Summing both sides of the inequality over each time interval gives

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k} \|z(t)\|^2 \,\mathrm{d}t \\ &\leq \sum_{k=1}^{N}\left[\frac{1}{2}\gamma^2 \int_{t_{k-1}}^{t_k} \|\boldsymbol{d}(t)\|^2 \,\mathrm{d}t + V(\boldsymbol{x}(t_{k-1}^+)) - V(\boldsymbol{x}(t_k^-))\right] \end{aligned} \tag{7}$$

If $V(\boldsymbol{x})$ is continuous across the discontinuities in the state, then the above implies Eq. (3). Otherwise,

$$\begin{aligned} &\frac{1}{2}\int_0^T \|z(t)\|^2 \,\mathrm{d}t \\ &\leq \frac{1}{2}\gamma^2 \int_0^T \|\boldsymbol{d}(t)\|^2 \,\mathrm{d}t + V(\boldsymbol{x}(0)) + \sum_{k=1}^{N-1}[V(\boldsymbol{x}(t_k^+)) - V(\boldsymbol{x}(t_k^-))] \\ &= \frac{1}{2}\gamma^2 \int_0^T \|\boldsymbol{d}(t)\|^2 \,\mathrm{d}t + \sum_{k=1}^{N-1}[V(\boldsymbol{x}(t_k^+)) - V(\boldsymbol{x}(t_k^-))], \\ &(\boldsymbol{x}(0) = \boldsymbol{x}_0) \end{aligned} \tag{8}$$

which makes the impact of the discontinuities clear. In particular, for $0 < T < \infty$, $\boldsymbol{d} \in \mathcal{L}_2[0, T]$ implies that $z \in \mathcal{L}_2[0, T]$. Letting $T \to \infty$, if the number of discontinuities $N-1$ remains finite, then $\boldsymbol{d} \in \mathcal{L}_2[0, \infty]$ implies that $z \in \mathcal{L}_2[0, \infty]$. Because this result is made possible by limiting the disturbances such that the number of discontinuities remains finite, the feedback solution given by Eqs. (4) and (5) will be termed local in this case.

III. Attitude Control Using Modified Rodrigues Parameters

For a rigid spacecraft, the evolution of the angular velocity, $\boldsymbol{\omega} \in R^3$ (expressed in a body-fixed frame), is governed by Euler's equation:

$$\dot{\boldsymbol{\omega}} = -\boldsymbol{J}^{-1}\boldsymbol{\omega}^\times \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{J}^{-1}\boldsymbol{u} + \boldsymbol{J}^{-1}\boldsymbol{d} \tag{9}$$

Here, $\boldsymbol{J}$ is the 3×3 moment of inertia matrix (expressed in a body-fixed frame with origin at the mass center), $\boldsymbol{u} \in R^3$ are the control torques, and $\boldsymbol{d} \in R^3$ are the disturbance torques. The matrix $(\cdot)^\times$ is defined as

$$\boldsymbol{\omega}^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{10}$$

and can be used to implement the cross product operation.

The attitude can be parameterized using the MRPs $\boldsymbol{\sigma}$ according to

$$\boldsymbol{\sigma} = \boldsymbol{a} \tan(\phi/4) \tag{11}$$

where $\boldsymbol{a}$ and ϕ are the principal axis and angle of rotation from Euler's theorem. It is noted here that $\boldsymbol{\sigma}$ becomes infinite when $\phi = \pm 2\pi$. The MRPs can also be defined in terms of the Euler parameters $\{\boldsymbol{\epsilon}, \eta\}$ using $\boldsymbol{\sigma} = \boldsymbol{\epsilon}/(1+\eta)$, where $\boldsymbol{\epsilon} \in R^3$ is the vector part and $\eta \in R$ is the scalar part. The rotation matrix mapping the components of vectors from an inertial frame to the body-fixed frame can be expressed in terms of MRPs using

$$\boldsymbol{C} = 1 - 4\frac{1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\boldsymbol{\sigma}^\times + \frac{8}{(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\boldsymbol{\sigma}^\times\boldsymbol{\sigma}^\times \tag{12}$$

The rotational rate kinematics relating $\boldsymbol{\omega}$ to the MRP rates are given by

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega}, \boldsymbol{G}(\boldsymbol{\sigma}) = \frac{1}{4}[(1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma})1 + 2\boldsymbol{\sigma}^\times + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T] \tag{13}$$

The matrix $\boldsymbol{G}(\boldsymbol{\sigma})$ possesses the following properties:

$$\boldsymbol{\sigma}^T\boldsymbol{G}(\boldsymbol{\sigma}) = \frac{1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{4}\boldsymbol{\sigma}^T \tag{14}$$

$$\boldsymbol{G}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}^\times = \boldsymbol{G}^T(\boldsymbol{\sigma}) \tag{15}$$

$$\|\boldsymbol{G}(\boldsymbol{\sigma})\| = \frac{1}{4}(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}) \tag{16}$$

For square matrices, $\|(\cdot)\|$ denotes the matrix norm induced by the Euclidean norm.

The MRP shadow set $\boldsymbol{\sigma}_s$ is related to the regular set according to

$$\boldsymbol{\sigma}_s = -\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}^T\boldsymbol{\sigma}} = \boldsymbol{a} \tan\left(\frac{\pi - 2\pi}{4}\right) \tag{17}$$

and possesses a singularity when $\phi = 0$ but is singularity-free at $\phi = \pm 2\pi$. By judiciously switching between the regular MRPs and the shadow set, one can obtain a singularity-free three-parameter representation of the spacecraft attitude. The price paid for this is a discontinuity. In this work, it is assumed that the switch is performed on the surface $\|\boldsymbol{\sigma}\| = \|\boldsymbol{\sigma}_s\| = 1$. By switching between the two parameter sets (using $\boldsymbol{\sigma}_s = -\boldsymbol{\sigma}$ on the switching surface), one obtains a composite description satisfying $\|\boldsymbol{\sigma}\| \leq 1$ [7]. Another useful property of the shadow set is that these parameters continue to satisfy Eqs. (12) and (13). Hence, they are satisfied by the composite description. It is also noted that $\dot{\boldsymbol{\sigma}}$ is globally defined, although discontinuous on the switching boundary [8].

For a truly global representation, one needs to determine the parameter set to be used if starting at $\|\boldsymbol{\sigma}\| = 1$. At this point

$$\left.\frac{\mathrm{d}(\boldsymbol{\sigma}^T\boldsymbol{\sigma})}{\mathrm{d}t}\right|_{\|\boldsymbol{\sigma}\|=1} = 2\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} = 2\boldsymbol{\sigma}^T\boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega} = \boldsymbol{\sigma}^T\boldsymbol{\omega}$$

where Eq. (14) has been used. To maintain $\|\boldsymbol{\sigma}\| \le 1$, the regular parameters should be adopted when $\boldsymbol{\sigma}^T\boldsymbol{\omega} < 0$ and the shadow parameters when $\boldsymbol{\sigma}^T\boldsymbol{\omega} = -\boldsymbol{\sigma}_s^T\boldsymbol{\omega} > 0$. When $\boldsymbol{\omega} = 0$, then $\|\boldsymbol{\sigma}\| = 1$ is a local maximum or minimum and one needs to examine

$$\left.\frac{\mathrm{d}^2(\boldsymbol{\sigma}^T\boldsymbol{\sigma})}{\mathrm{d}t^2}\right|_{\omega=0,\|\sigma\|=1} = \boldsymbol{\sigma}^T\dot{\boldsymbol{\omega}} = \boldsymbol{\sigma}^T\boldsymbol{J}^{-1}(\boldsymbol{u}+\boldsymbol{d})$$

If $\boldsymbol{d}$ is small and the control law is given by $\boldsymbol{u} = -a\boldsymbol{\omega} - b\boldsymbol{\sigma}$ ($a > 0, b > 0$), then

$$\left.\frac{\mathrm{d}^2(\boldsymbol{\sigma}^T\boldsymbol{\sigma})}{\mathrm{d}t^2}\right|_{\omega=0} \doteq -b\boldsymbol{\sigma}^T\boldsymbol{J}^{-1}\boldsymbol{\sigma} < 0$$

and either MRP set yields a local maximum for $\|\boldsymbol{\sigma}\|$ and can be adopted.

Combining Eqs. (9) and (13) produces the state-space model

$$\begin{bmatrix}\dot{\boldsymbol{\omega}}\\ \dot{\boldsymbol{\sigma}}\end{bmatrix} = \begin{bmatrix}-\boldsymbol{J}^{-1}\boldsymbol{\omega}^\times\boldsymbol{J}\boldsymbol{\omega}\\ \boldsymbol{G}(\boldsymbol{\sigma})\boldsymbol{\omega}\end{bmatrix} + \begin{bmatrix}\boldsymbol{J}^{-1}\\ 0\end{bmatrix}\boldsymbol{u} + \begin{bmatrix}\boldsymbol{J}^{-1}\\ 0\end{bmatrix}\boldsymbol{d} \tag{18}$$

which upon comparison with Eq. (1) permits the identification $\boldsymbol{x} = \mathrm{col}\{\boldsymbol{\omega},\boldsymbol{\sigma}\}$ with $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{g}(\boldsymbol{x})$, and $\boldsymbol{k}(\boldsymbol{x})$ being obvious. The equilibrium state is $\boldsymbol{x}_0 = 0$.

The regulated output is taken to be

$$\boldsymbol{z} = \begin{bmatrix}\boldsymbol{h}(\boldsymbol{\omega},\boldsymbol{\sigma})\\ \boldsymbol{u}\end{bmatrix}, \qquad \boldsymbol{h}(\boldsymbol{\omega},\boldsymbol{\sigma}) = \left(\frac{1}{2}q_1\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + q_2\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)^{\frac{1}{2}}, \quad q_1 > 0, \qquad q_2 > 0 \tag{19}$$

which clearly penalizes the rotational kinetic energy; the penalization on the MRPs is adopted from [8].

Theorem 1: Let $\gamma > 1$ and

$$a \ge \sqrt{\left(\frac{1}{2}q_1 + b\right)\|\boldsymbol{J}\|\frac{\gamma^2}{\gamma^2 - 1}} \tag{20}$$

$$b \ge \sqrt{q_2\frac{\gamma^2}{\gamma^2 - 1}} \tag{21}$$

Then the linear state feedback

$$\boldsymbol{u} = -a\boldsymbol{\omega} - b\boldsymbol{\sigma} \tag{22}$$

locally solves the state feedback suboptimal $\mathcal{H}_\infty$ control problem.

Proof: Taking the shape of the proof from [6], it suffices to find a positive-definite solution $V(\boldsymbol{\omega},\boldsymbol{\sigma})$ to the Hamilton–Jacobi inequality $H_\gamma(\boldsymbol{x},\partial V/\partial\boldsymbol{x}) \le 0$.

To this end, consider the Lyapunov function candidate $V(\boldsymbol{\omega},\boldsymbol{\sigma}) = \frac{1}{2}a\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\sigma} + c\,\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})$, which has the property $V(0,0) = 0$ and yields the control law in Eq. (22) using Eq. (4). The third term on the right-hand side has been used in [1,3,4] and takes advantage of some peculiarities of the form of the MRP kinematics.

It follows that

$$\begin{aligned} V(\boldsymbol{\omega},\boldsymbol{\sigma}) &= \frac{1}{2}a\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\sigma} + c\,\ln(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})\\ &\ge \frac{1}{2}a\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\sigma} + \ln(2)c\boldsymbol{\sigma}^T\boldsymbol{\sigma}\end{aligned} \tag{23}$$

$$= \frac{1}{2}\begin{bmatrix}\boldsymbol{\omega}^T\boldsymbol{\sigma}^T\end{bmatrix}\begin{bmatrix}a\boldsymbol{J} & b\boldsymbol{J}\\ b\boldsymbol{J} & 2\,\ln(2)c1\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}\\ \boldsymbol{\sigma}\end{bmatrix} \tag{24}$$

for $\|\boldsymbol{\sigma}\|^2 \le 1$. $V(\boldsymbol{\omega},\boldsymbol{\sigma})$ will be positive definite when the central matrix in Eq. (24) is positive definite. Sufficient conditions for this are

$$a > 0 \tag{25}$$

$$ac1 > \frac{1}{2\,\ln(2)}b^2\boldsymbol{J} \tag{26}$$

Substituting the Lyapunov candidate $V(\boldsymbol{\omega},\boldsymbol{\sigma})$ from Eq. (23) into Eq. (5) and simplifying gives

$$\begin{aligned} H_\gamma\left(\boldsymbol{x},\frac{\partial V}{\partial\boldsymbol{x}}\right) &= b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{G}\boldsymbol{\omega} + \frac{2c}{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}\boldsymbol{\sigma}^T\boldsymbol{G}\boldsymbol{\omega} - b\boldsymbol{\sigma}^T\boldsymbol{\omega}^\times\boldsymbol{J}\boldsymbol{\omega}\\ &+ \frac{1}{2}a^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\omega}\|^2 + \frac{1}{2}b^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\sigma}\|^2 + ab\left(\frac{1-\gamma^2}{\gamma^2}\right)\boldsymbol{\omega}^T\boldsymbol{\sigma}\\ &+ \frac{1}{4}q_1\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + \frac{1}{2}q_2\boldsymbol{\sigma}^T\boldsymbol{\sigma}\end{aligned}$$

Noting that $\boldsymbol{\sigma}^T\boldsymbol{\omega}^\times\boldsymbol{J}\boldsymbol{\omega} = \boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\sigma}^\times\boldsymbol{\omega}$, $\boldsymbol{\sigma}^T\boldsymbol{G} = \frac{1}{4}\boldsymbol{\sigma}^T(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})$, and $\boldsymbol{G} - \boldsymbol{\sigma}^\times = \boldsymbol{G}^T$ yields

$$\begin{aligned} H_\gamma\left(\boldsymbol{x},\frac{\partial V}{\partial\boldsymbol{x}}\right) &= b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{G}^T\boldsymbol{\omega} + \frac{1}{2}c\boldsymbol{\omega}^T\boldsymbol{\sigma}\\ &+ \frac{1}{2}a^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\omega}\|^2 + \frac{1}{2}b^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\sigma}\|^2 + ab\left(\frac{1-\gamma^2}{\gamma^2}\right)\boldsymbol{\omega}^T\boldsymbol{\sigma}\\ &+ \frac{1}{4}q_1\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + \frac{1}{2}q_2\boldsymbol{\sigma}^T\boldsymbol{\sigma}\end{aligned}$$

Choosing

$$c = 2ab\left(\frac{\gamma^2 - 1}{\gamma^2}\right) \tag{27}$$

makes it possible to eliminate the two cross terms. Also, noting that $\|\boldsymbol{G}\| = \|\boldsymbol{G}^T\| = \frac{1}{4}(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})$ and $\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} \le \|\boldsymbol{J}\|\cdot\|\boldsymbol{\omega}\|^2$ gives

$$\begin{aligned} H_\gamma\left(\boldsymbol{x},\frac{\partial V}{\partial\boldsymbol{x}}\right) &\le \frac{1}{4}b\|\boldsymbol{J}\|(1+\|\boldsymbol{\sigma}\|^2)\|\boldsymbol{\omega}\|^2\\ &+ \frac{1}{2}a^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\omega}\|^2 + \frac{1}{2}b^2\left(\frac{1-\gamma^2}{\gamma^2}\right)\|\boldsymbol{\sigma}\|^2\\ &+ \frac{1}{4}q_1\|\boldsymbol{J}\|\|\boldsymbol{\omega}\|^2 + \frac{1}{2}q_2\|\boldsymbol{\sigma}\|^2\\ &\le \frac{1}{4}\left[2b\|\boldsymbol{J}\| + 2a^2\left(\frac{1-\gamma^2}{\gamma^2}\right) + q_1\|\boldsymbol{J}\|\right]\|\boldsymbol{\omega}\|^2\\ &+ \frac{1}{2}\left[b^2\left(\frac{1-\gamma^2}{\gamma^2}\right) + q_2\right]\|\boldsymbol{\sigma}\|^2\end{aligned}$$

If a and b satisfy Eqs. (20) and (21), respectively, it follows that

$$H_\gamma(\boldsymbol{x},\partial V/\partial\boldsymbol{x}) \le 0 \tag{28}$$

Also, $a > 0$ [using Eq. (20)] and $ac \ge 2b^2\|\boldsymbol{J}\|$ [using Eqs. (27) and (20)] so that the conditions in Eqs. (25) and (26) are satisfied. The result follows upon noting that the control law in Eq. (22) can be obtained using Eqs. (4) and (23). □

The global asymptotic stability properties of the control law in Eq. (22) have been previously demonstrated [1,3,4]. The Lyapunov functions used there were identical (within a scaling) to the first and third terms used here in Eq. (23). The second term (the cross term) $b\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\sigma}$ is not present in those references. It is noted that the first and third terms are continuous across the MRP-switching boundary (because $\boldsymbol{\sigma}^T\boldsymbol{\sigma} = 1$ on the boundary and $\boldsymbol{\omega}$ is continuous). The second term is not (because $\boldsymbol{\sigma}$ is replaced with $\boldsymbol{\sigma}_s = -\boldsymbol{\sigma}$ on the boundary). This leads to the jump $V(t_k^+) - V(t_k^-)$ in Eq. (8) and, hence, the local nature of the result. It should also be pointed out that the cross term is the only way that one can obtain the linear MRP feedback term $-b\boldsymbol{\sigma}$ using the $\mathcal{H}_\infty$ feedback $\boldsymbol{u} = -\boldsymbol{g}^T(\boldsymbol{x})(\partial V/\partial \boldsymbol{x})^T$.

IV. Conclusions

In this note, the suboptimal nonlinear $\mathcal{H}_\infty$ state feedback problem has been examined for rigid spacecraft attitude control. The attitude parameterization that has been adopted is the modified Rodrigues parameters (MRPs) and their shadow parameters. A linear combination of angular velocity and MRP feedback has been shown to provide a local solution to the problem, that is, one that is valid in the presence of a finite number of MRP switches. The local nature of the solution stems from the discontinuous nature of the Lyapunov function used to solve the Hamilton–Jacobi inequality. The proof of Theorem 1, the main result, has been constructive and establishes bounds on the gain parameters that solve the problem in terms of the desired $\mathcal{L}_2$-gain, the control weighting parameters in the performance output, and the moments of inertia. If a Lyapunov function solution of the Hamilton–Jacobi inequality can be found that is continuous across the MRP switching boundary, then the result can be made a global solution to the problem. It is also worth noting that a linear feedback solution (although with MRP switching and hence a local solution) has been found for a problem with nonlinear dynamics.

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