Engineering Notes

Quaternion-Based Bang-Bang Attitude Stabilizer for Rotating Rigid Bodies

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Nomenclature

q	=	unit quaternion
q_j^i	=	discrete state of the ith automaton $\mathcal{A}_i$; i, j are equal to 1, 2, 3
ϵ	=	vector component of q
η	=	scalar component of q
$\mathbb{1}$	=	north pole of $\mathbb{S}^3$
Ω	=	angular velocity
$\mathcal{H}$	=	controller supervisor
δ	=	hysteresis parameter for the supervisor $\mathcal{H}$
$\mathcal{A}_i$	=	low-level automaton; i is equal to 1, 2, 3
δ_1^i, δ_2^i	=	hysteresis parameters of the automata $\mathcal{A}_i$; i is equal to 1, 2, 3
$\Gamma_i^\pm$	=	initialization sets; i is equal to 1, 2, 3
$\Lambda_i^\pm$	=	switching sets; i is equal to 1, 2, 3
κ_i	=	concavity of the lower parabolas in the definition of $\Lambda_i^\pm$; i is equal to 1, 2, 3

I. Introduction

THIS paper presents a novel feedback controller that stabilizes the attitude of a rigid body to any degree of accuracy. Its main feature is that it is bang-bang and, as argued below, it constitutes the first solution to the attitude stabilization problem with on–off actuators and bounded switching frequency.

Bang-bang attitude controllers based on the use of pulse width modulation (PWM) and pulse width pulse frequency modulation (PWPFM) have been proposed in the context of single-axis maneuvers [1–3], small-angles maneuvers [4], and large maneuvers [5]. Bang-bang modulation schemes with dead-band regions were proposed in [6]. These techniques, however, introduce a problematic coupling between the switching frequency and the asymptotic bound on the state, even in the absence of external perturbations. Moreover, the stability properties of these controllers for general nonlinear systems have yet to be proven rigorously. A review of some of these techniques can be found in [1,2,7].

Most of the bang-bang attitude controllers presented in the past apply only to the case of planar maneuvers; see, for example, [8–11]. A robust bang-bang feedback controller was proposed in [8] for the simplified case of single-axis maneuvers. The controller is obtained by introducing dead-bands about the parabolic switching curves of the classical bang-bang time-optimal controller for double-integrators. The controller is robust with respect to constant uncertainties in the system's parameters and constant external torques.

Designing bang-bang controllers for general, three-dimensional maneuvers is far more challenging. Most of the results proposed in the literature originate from various attempts to find the solution to the time-optimal attitude control problem in the case of rest-to-rest maneuvers, and result in open-loop controllers. It was shown in [12] that, for an inertially symmetric spacecraft, the time-optimal controller is indeed bang-bang. It was observed through simulations that the controller induces a total of seven switches in the control input value if the reorientation maneuver is smaller than 73 deg., a total of five switches otherwise. The control values and switching times were computed numerically, through continuation techniques. These results were later extended to asymmetric rigid spacecraft in [13], under the assumption that the control input magnitude is significantly larger than the nonlinear gyroscopic term in the dynamics. More recently, new trajectories have been found numerically in [14] that are characterized by six control switches. In [15], the problem of performing a time-optimal reconfiguration for an axisymmetric rigid spacecraft, by using only two control torques, was tackled and solved numerically.

All of the controllers discussed above require complex numerical procedures in order to generate accurate-enough estimates of the optimal control input values and switching times. The very nature of the numerical schemes adopted to solve the optimization problem heavily influences the quality of the solution found. Moreover, initial guesses for states, co-states, and control inputs are often required. As is to be expected, these controllers are inherently nonrobust to external unmodeled perturbations, uncertainty in the system's parameters, or measurement noise.

A control strategy based on a sequence of eight bang-bang maneuvers was proposed in [16]. First, a discontinuous controller is applied to bring the spacecraft to a rest configuration, that is, zero angular velocity. A sequence of seven single-axis bang-bang maneuvers is then performed that exploit the structure of the kinematic and dynamic equations. The authors of [16] point out that this particular control strategy suffers from a lack of robustness to both external perturbations and measurement noise, in that unmodeled perturbations may prevent the controller from successfully completing one of the maneuvers. Moreover, each single-axis bang-bang maneuver may induce sliding modes when external perturbations and measurement noise are considered.

In this paper the hybrid feedback presented in [17] is modified so as to take full advantage of the quaternion parametrization of the spacecraft attitude. The controller achieves *practical stabilization* of the spacecraft attitude assuming the full nonlinear dynamics of the spacecraft, while avoiding chattering and high-frequency switching of the actuators. An extensive Monte Carlo numerical analysis indicates that the result might in fact be *global*. To the best of our knowledge, the attitude controller presented in this paper is the first bang-bang feedback controller with guaranteed stability properties and guaranteed bounded switching frequency, even in the presence of measurement noise. Moreover, the proposed controller is not limited to small-angle or planar maneuvers.

The paper is organized as follows. Section II formulates the problem investigated in this paper. Section III presents the controller

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solving the problem. An extensive Monte Carlo numerical simulation is presented in Sec. IV. The robustness of the proposed controller to measurement noise is investigated in Sec. V.

Notation: We denote $B_\epsilon(0) = \{x \in \mathbb{R}^2: (x^\top x)^{1/2} < \epsilon\}$ and $\bar{B}_\epsilon(0) = \{x \in \mathbb{R}^2: (x^\top x)^{1/2} \leq \epsilon\}$. These definitions imply that the set $B_0(0)$ is empty, while $\bar{B}_0(0) = \{0\}$. The 3×3 identity matrix is denoted by I. Throughout the paper, sets are denoted by capital letters. The boundary of a set A is defined as $\partial A = \bar{A} \setminus \mathrm{Int}A$, where $\bar{A}$ is the closure of A and $\mathrm{Int}A$ is its interior. We denote by $-A$ the set $-A = \{x: -x \in A\}$.

II. Model and Problem Formulation

Consider an inertially symmetric rigid spacecraft with moment of inertia J_0, and let $\Omega \in \mathbb{R}^3$ denote its angular velocity expressed in the principal frame. The spacecraft attitude is parametrized by a unit quaternion $q = (\epsilon, \eta) \in \mathbb{S}^3$, where $\mathbb{S}^3 = \{x \in \mathbb{R}^4: x^\top x = 1\}$. We assume that the spacecraft is equipped with a set of reaction thrusters that generates a bang-bang control acceleration $u \in \mathcal{U}$, where $\mathcal{U} := \{-\bar{u}, 0, +\bar{u}\}^3$ with $\bar{u} > 0$, in the body frame. The rotational dynamics of the spacecraft can be written as follows:

$$\begin{aligned} \dot{\epsilon} &= \frac{1}{2}(\eta I + S(\epsilon))\Omega \\ \dot{\eta} &= -\frac{1}{2}\epsilon^\top \Omega \\ \dot{\Omega} &= u \end{aligned} \tag{1}$$

with state $\chi = (q, \Omega) \in X$, where $X = \mathbb{S}^3 \times \mathbb{R}^3$ denotes the state space of system (1), and control input u. The matrix $S(\epsilon)$ is the 3×3 skew-symmetric matrix defined so that $S(\epsilon)\Omega = \epsilon \times \Omega$.

In this paper we discuss the design of a feedback bang-bang controller $u^\star$ with values in $\mathcal{U}$ that stabilizes a target orientation of the spacecraft to any desired degree of accuracy. The control problem must, however, be formulated with particular care, in that it is a well-known fact that any two distinct, antipodal quaternions, q and $-q$, parametrize the same orientation of the spacecraft. When a controller is designed overlooking this feature, unwinding behavior may be induced in the closed-loop system, forcing the spacecraft to unnecessarily undergo a full rotation before being stabilized. As argued in [18], to avoid unwinding, it is necessary and sufficient to simultaneously stabilize both the quaternions that parametrize the desired orientation. In what follows, we assume, without loss of generality, that the target equilibria to be stabilized are $(q, \Omega) = (\mathbb{1}, 0)$ and $(q, \Omega) = (-\mathbb{1}, 0)$, where $\mathbb{1} = (0, 1)$ and $-\mathbb{1} = (0, -1)$ denote the north and south pole of $\mathbb{S}^3$, respectively.

A precise statement of the problem investigated in this paper is given below.

Bang-Bang Attitude Control Problem: Consider system (1) and the desired attitude $P = \{(q, \Omega) \in X: q = \pm\mathbb{1}, \Omega = 0\}$. Design a bang-bang feedback controller $u^\star$ taking values in set $\mathcal{U}$, such that

1) The set P is *practically stable* for the closed-loop system. In other words, given an arbitrarily small neighborhood U of P in X, there exist controller parameters and a compact set W, with $P \subset \mathrm{Int}W \subset U$, such that W is asymptotically stable for the closed-loop system.

2) The number of controller switches is *uniformly bounded* over compact sets of initial conditions and over compact time intervals: for any compact set $W_0 \subset X$ and for any $T > 0$, there exists $N \in \mathbb{N}$ such that for any $(q(0), \Omega(0)) \in W_0$ the controller switches value at most N times over any time interval of length T.

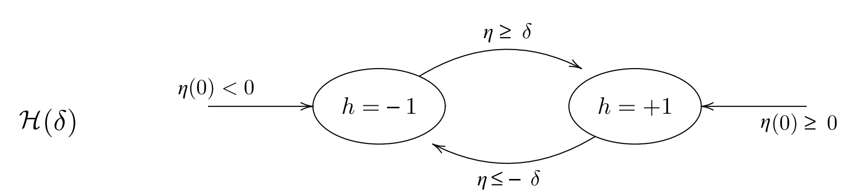
III. Main Results

We propose to solve the bang-bang attitude control problem by designing a hybrid controller that relies on a hierarchical switching logic. The control architecture, depicted in Fig. 1, comprises two hierarchical layers. At the high level, a supervisor automaton $\mathcal{H}(\delta)$ (where δ is a user-defined parameter) is responsible for preventing unwinding and broadcasts an output value $h \in \{-1, 1\}$ to the low level. At the low level, three automata $\mathcal{A}_i(p_i, h)$, $i = 1, 2, 3$ (p_i is a vector of user-defined parameters), are driven by the supervisor through the parameter $h \in \{-1, 1\}$, and are responsible for the assignment of the control values u_i, $i = 1, 2, 3$.

The high-level automaton $\mathcal{H}(\delta)$ selects which equilibrium to stabilize between $(q, \Omega) = (\mathbb{1}, 0)$ and $(q, \Omega) = (-\mathbb{1}, 0)$. It does so by implementing a hysteresis mechanism as in [19], the result of which is the numerical value of parameter $h \in \{-1, +1\}$. Each low-level automaton $\mathcal{A}_i(p_i, h)$ assigns the control input $u_i \in \{-\bar{u}, 0, +\bar{u}\}$ so as to stabilize (ϵ_i, Ω_i) to a neighborhood of $(0, 0)$ whose size depends on the parameters in p_i, and to stabilize η to a neighborhood of $h = \pm 1$, which is decided by the supervisor.

A. Supervisor: Automaton $\mathcal{H}(\delta)$

The supervisor $\mathcal{H}(\delta)$ is depicted below.



(2)

Referring to Fig. 1, the automaton above monitors the scalar part of the quaternion, η, and it chooses the desired value of η to be stabilized, $+1$ or -1, through the choice of the parameter h. The idea is quite simple. If $\eta \geq \delta > 0$, then the supervisor assigns $h = +1$, whereas if $\eta \leq -\delta < 0$, it assigns $h = -1$. If $\eta \in (-\delta, +\delta)$, h is assigned according to an hysteresis mechanism, similarly to what was done in [19], to avoid undesired switching in the presence of measurement noise. As in [19], the hysteresis mechanism presents a trade-off between robustness with respect to measurement noise and unwinding in the hysteresis region $(-\delta, +\delta)$.

B. Low-Level Controller: Automata $\mathcal{A}_i(p_i, h)$ and Control Assignment $u_i^\star$

The low-level controller consists of three automata, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, each driven by h, the output of the supervisor. Each automaton $\mathcal{A}_i$ monitors the variables (ϵ_i, Ω_i) and assigns the control value $u_i \in \{-\bar{u}, 0, +\bar{u}\}$ so

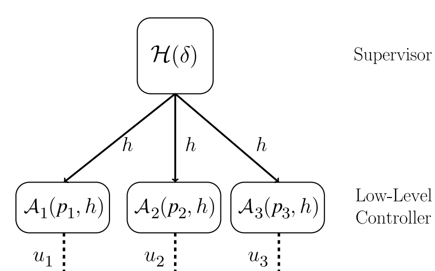
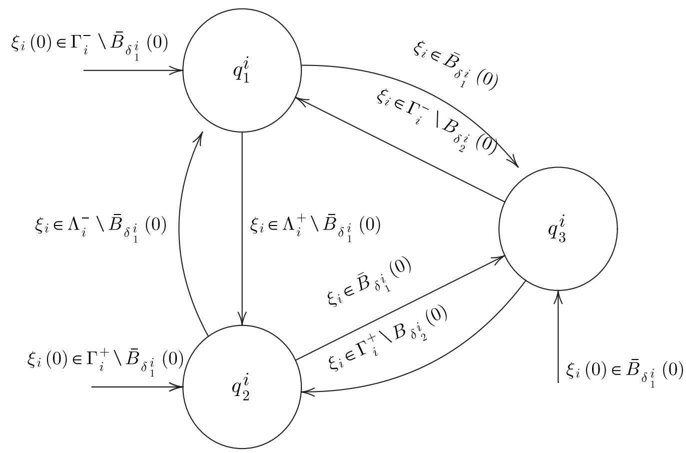


Fig. 1 Pictorial representation of the proposed control structure.

as to drive (ϵ_i, Ω_i) to a neighborhood of (0, 0). The idea for doing so is to view the (ϵ_i, Ω_i) dynamics as a perturbation of a double-integrator, and use the double-integrator stabilizer presented by the authors of this paper in [20,21]. As mentioned earlier, the supervisor output h is used to decide which equilibrium should be stabilized. Let $\xi_i = (h\epsilon_i, \omega_i)$, where h is the output of $\mathcal{H}(\delta)$ currently broadcasted to the low-level controller. The automata $\mathcal{A}_i$ are given below.

The automaton $\mathcal{A}_i(p_i, h)$ is parametrized by the vector $p_i = (\delta_1^i, \delta_2^i, \kappa_i)$ of user-defined parameters, and by h, the output of the supervisor automaton. The parameters δ_1^i, δ_2^i determine the size of the neighborhood of $(\epsilon_i, \Omega_i) = (0, 0)$ being stabilized, whereas the parameter κ_i is useful for proving the theoretical stabilization properties of the attitude controller, but can be set to zero in practice. More comments on the choice of controller parameters are provided in Sec. IV.

$\mathcal{A}_i(p_i, h)$



(3)

In the above, Γ_i^+, Γ_i^- are the *initialization sets* (see Fig. 2a) defined as

$$\Gamma_i^+ = \left\{(\epsilon_i, \Omega_i): \epsilon_i > 0, \Omega_i \le -2\sqrt{\bar{u}\epsilon_i}\right\} \cup \left\{(\epsilon_i, \Omega_i): \epsilon_i \le 0, \Omega_i < 2\sqrt{-\bar{u}\epsilon_i}\right\},$$
$$\Gamma_i^- = -\Gamma_i^+ \tag{4}$$

Λ_i^+, Λ_i^- are the *switching sets* (see Fig. 2b) defined as

$$\Lambda_i^+ = \left\{(\epsilon_i, \Omega_i): \epsilon_i \le 0, \Omega_i \le 2\sqrt{-\kappa_i\bar{u}\epsilon_i}\right\} \cup \left\{(\epsilon_i, \Omega_i): \epsilon_i > 0, \Omega_i \le -2\sqrt{\bar{u}\epsilon_i}\right\},$$
$$\Lambda_i^- = -\Lambda_i^+ \tag{5}$$

with $\kappa_i \in [0, 1)$. Letting set $Q^i = \{q_1^i, q_2^i, q_3^i\}$ denote the set of states of automaton $\mathcal{A}_i(p_i, h)$, each control input u_i is assigned through the feedback $u_i^\star: Q_i \to \mathbb{R}$ defined as

$$u_i^\star(q_1^i) = -\bar{u}$$
$$u_i^\star(q_2^i) = +\bar{u}$$
$$u_i^\star(q_3^i) = 0 \tag{6}$$

Notice that each feedback $u_i^\star$ depends only on the current active state of automaton $\mathcal{A}_i$, whose dynamics is driven by continuous states (ϵ_i, Ω_i).

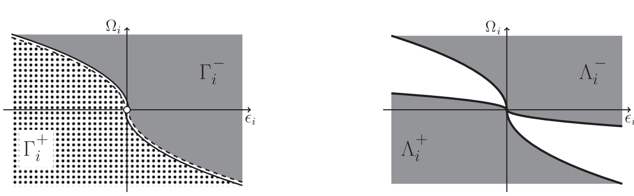


Fig. 2 Initialization sets Γ_i^+, Γ_i^- a), and switching sets Λ_i^+, Λ_i^-, with $\kappa_i \in (0,1)$, b).

The proposed controller practically stabilizes set P: for any open neighborhood U of the target spacecraft configuration P, the controller parameters δ_i, κ_i, $i = 1, 2, 3$, can be chosen so that the proposed controller steers in finite time the state χ inside set U. This result is stated rigorously in the theorem below.

Theorem 1: Consider system (1) with control input $u \in \mathcal{U}$. For any $\bar{u} > 0$, the hybrid feedback controller given by supervisor (2) and three copies of automata (3) with feedback (6) solves the bang-bang attitude control problem. In particular, the following two properties hold for the closed-loop system:

1) For any $\bar{u} > 0$ and for any neighborhood U of the set $P = \{(q, \Omega) \in X: q = \pm\mathbb{1}, \Omega = 0\}$, there exist controller parameters δ_1^i, $\delta_2^i > 0$, $\kappa_i > 0$, $i = 1, 2, 3$, and $\delta \in (0, 1)$ such that U has a compact subset W containing P in its interior, which is asymptotically stable.

2) The number of controller switches is uniformly bounded in the sense stated in part 2 of the problem statement in Sec. II.

Theorem 1 presents a local stabilization result. Specifically, the result of Theorem 1 guarantees only asymptotic stability of the compact set $W \subset U$, not global asymptotic stability. The full proof is omitted due to space limitations. We provide below only an idea of the argument.

C. Idea of the Proof

To simplify the analysis, as proposed in [12], consider a time scaling factor of $\sqrt{\bar{u}}$, and define the nondimensional angular velocity as $\omega = \Omega\sqrt{1/\bar{u}}$. With a slight abuse of notation, we will use the dot notation to denote differentiation with respect to the non-dimensional time $\nu = t\sqrt{\bar{u}}$. The spacecraft rotational dynamics in Eq. (1) can then be rewritten as follows:

$$\dot{\epsilon} = \frac{1}{2}(\eta I + S(\epsilon))\omega$$
$$\dot{\eta} = -\frac{1}{2}\epsilon^T\omega$$
$$\dot{\omega} = u \tag{7}$$

where $u_i \in \{-1, 0, +1\}$ is the nondimensional control input about the ith axis. This coordinate transformation allows us to study the properties of the controller independently of the actual value of the control acceleration $\bar{u}$ generated by the thrusters. Throughout this section we then consider the nondimensional system (7) with controllers (2), (3), and (6).

Since P is the union of two isolated equilibria, one needs to show that each equilibrium, $(q,\Omega)=(\mathbb{1},0)$ and $(q,\Omega)=(-\mathbb{1},0)$, is practically stable. We show the proof with equilibrium $(q,\Omega)=(\mathbb{1},0)$. The proof of practical stability of the other equilibrium is identical. Consider the function $V^+{:}X\to\mathbb{R}$ defined as $V^+(q,\omega)=V_1(\epsilon_1,\omega_1)+V_2(\epsilon_2,\omega_2)+V_3(\epsilon_3,\omega_3)$, where function $V_i{:}\mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\geq 0}$, $i=1,2,3$, is given by $V_i(\epsilon_i,\omega_i)=4\epsilon_i^2\omega_i\sigma(\epsilon_i,\omega_i)+(4/3)\omega_i^3\epsilon_i+(3/20)\omega_i^5\sigma(\epsilon_i,\omega_i)+(4/5)(2\epsilon_i\sigma(\epsilon_i,\omega_i)+(1/2)\omega_i^2)^{5/2}$, with $\sigma{:}\mathbb{R}\times\mathbb{R}\to\{-1,0,+1\}$ given by

$$\sigma(\epsilon_i,\omega_i)=\begin{cases}\operatorname{sign}\Big(2\epsilon_i+\frac{1}{2}|\omega_i|\omega_i\Big), & \text{if } 2\epsilon_i+\frac{1}{2}|\omega_i|\omega_i\neq 0\\ \operatorname{sign}(\omega_i), & \text{if } 2\epsilon_i+\frac{1}{2}|\omega_i|\omega_i=0, \omega_i\neq 0\\ 0, & \text{if } (\epsilon_i,\omega_i)=(0,0)\end{cases}$$

The functions V_i, first proposed in [22], are C^1 and positive definite with respect to the origin. Hence, V^+ is C^1 and $V^+(q,\omega)=0$ if and only if $(\epsilon,\omega)=(0,0)$, so that V^+ is positive definite near the equilibrium $(q,\Omega)=(\mathbb{1},0)$.

Let $V^+_{\epsilon_i}=\partial V^+/\partial\epsilon_i$ and $V^+_{\omega_i}=\partial V^+/\partial\omega_i$. One can easily show that $\dot{V}^+\leq\sum_{i=1}^3-\lambda_i(q,\omega_i,u_i^\star(q^i))$, where $(q,\omega_i)\mapsto-\lambda_i(q,\omega_i,u_i^\star(q^i))$ is continuous and given by $-\lambda_i(q,\omega_i,u_i^\star)=1/2V^+_{\epsilon_i}\eta\omega_i+V^+_{\omega_i}u_i^\star(q^i)+\alpha\sqrt{1-\eta^2}\epsilon_i^2+\beta\sqrt{1-\eta^2}\omega_i^4$, for some $\alpha,\beta>0$, where q^i is the current state of automaton $\mathcal{A}_i(p_i,h)$. Moreover, there exists $\bar{\eta}\in(\delta,1)$ such that if $\eta>\bar{\eta}$ and $q^i\in\{q_1^i,q_2^i\}$, then each $-\lambda_i$ is negative definite with respect to $(\epsilon_i,\omega_i)=(0,0)$. Let U be an arbitrarily small open neighborhood of $(q,\omega)=(\mathbb{1},0)$ and pick $\delta\in(0,1)$. Let $\gamma_0>0$ be such that $\eta>\bar{\eta}$ for any (q,ω) in the set $W_{\gamma_0}\coloneqq\{(q,\omega)\in X{:}\eta>0, V^+(q,\omega)\leq\gamma_0\}$. Pick $\gamma_1\in(0,\gamma_0)$ such that $W_{\gamma_1}\subset U$. We show that W_{γ_1} is made asymptotically stable by an appropriate choice of controller parameters. To do so we apply Theorem 7.8 in [23]. First, pick $\delta_2^i=\delta_2$, $i=1,2,3$, where $\delta_2>0$ is chosen such that $\{(q,\omega)\in X{:}\eta>0, \|(\epsilon_i,\omega_i)\|_2\leq\delta_2$, for all $i=1,2,3\}\subset\operatorname{Int}W_{\gamma_1}$ and pick $\delta_1^i\in(0,\delta_2)$, $i=1,2,3$. By construction, if all the automata are at state q_3^i, $i=1,2,3$, then $(q,\omega)\in W_{\gamma_1}$. We check that $\dot{V}^+<0$ on the open neighborhood $\operatorname{Int}W_{\gamma_0}\setminus W_{\gamma_1}$. On the set $\operatorname{Int}W_{\gamma_0}\setminus W_{\gamma_1}$ at least one of the automata, say $\mathcal{A}_1$ for simplicity, is active; that is, $(\epsilon_1,\omega_1)\notin\bar{B}_{\delta_2}(0)$ and $q^1\in\{q_1^1,q_2^1\}$. One can show that in this case $-\lambda_1\leq-\bar{\lambda}$, where $\bar{\lambda}>0$. Consider the worst-case scenario in which (ϵ_2,ω_2), $(\epsilon_3,\omega_3)\in\bar{B}_{\delta_2}(0)$ and $u_2^\star=u_3^\star=0$. In this case, $-\lambda_2-\lambda_3$ is upper bounded by $w(\delta_2)$, where $w(\delta_2)$ is continuous and positive definite, with $w(0)=0$. It follows that there exists $\delta_2^\star$ such that for any $\delta_2\in(0,\delta_2^\star)$, $\dot{V}^+\leq-\bar{\lambda}+w(\delta_2)<0$. The other scenarios are handled in a similar manner. Because the function V^+ is continuous across state transitions of the automata and the controller does not generate instantaneous Zeno solutions, Theorem 7.8 in [23] (see also [24]) guarantees that set $W_{\gamma_1}\subset U$ is asymptotically stable.

IV. Numerical Estimation of the Controller's Basin of Attraction

This section presents a Monte Carlo numerical study aimed at convincing the reader that the convergence of the proposed controller might, in fact, be global; that is, the controller's basin of attraction is the whole-state space X.

A. Simulations Setup

Consider the nondimensional system (7). To improve the accuracy of the simulation, the coordinate transformation has been modified so to have $u_i\in\{-0.1,0,+0.1\}$ for all $i=1,2,3$. This can be done by replacing $\sqrt{1/\bar{u}}$ with $\sqrt{1/(10\bar{u})}$. Let $\hat{u}=0.1$ and let, for the sake of clarity, $\delta_2^i=\delta_2$, $i=1,2,3$. Pick $\delta_2<2\sqrt{\hat{u}}$ and define $X_C\coloneqq\mathbb{S}^3\times\{(\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3{:}\,|\omega_i|\leq 2\sqrt{\hat{u}}\}$. The set X_C is globally attractive and positively invariant for the closed-loop system. Indeed, for any initial condition, each pair (ϵ_i,ω_i), $i=1,2,3$, reaches the active switching set in finite time. Since $|\epsilon_i|\leq 1$, as the state transition is triggered we must have that $|\omega_i|\leq 2\sqrt{\hat{u}}$. From any point on the boundary of a switching set, the state trajectory will then hit the next switching set in finite time. Since $|\epsilon_i|\leq 1$, $|\omega_i|$ can never become greater then $2\sqrt{\hat{u}}$ between consecutive switches of the automata. The condition $\delta_2<2\sqrt{\hat{u}}$ guarantees that this also holds when the control input is 0. It follows that the state enters set X_C and never leaves. Thanks to this fact, it is enough to pick initial conditions in set X_C to show that the proposed controller yields global convergence. After an initial condition is chosen, the controller is to be initialized. Particular care needs to be applied in performing this operation: any simulation initial conditions (q_0,ω_0) can be seen as the actual initial conditions of the system or as the state of the spacecraft some time after it entered X_C, being initialized outside set X_C. This ambiguity in the interpretation of each simulation initial condition can lead to different initialization of automata $\mathcal{H}(\delta)$ and $\mathcal{A}_i(p_i,h)$. The following strategy is adopted: any time one of the automata could be initialized to multiple different states because of this issue, the automata is initialized randomly.

Control parameters p_i were selected the same for all the automata; that is, $p_i=p=(\delta_1,\delta_2,\kappa)$, for all $i=1,2,3$, with $\delta_1=1\cdot 10^{-4}$, $\delta_2=5\cdot 10^{-4}$, and $\kappa=0$. Supervisor hysteresis parameter has been selected to $\delta=0.04$.

Each simulation is stopped when all three automata are at state q_3^i; that is, $u_i^\star(q_3^i)=0$ and $(\epsilon_i,\omega_i)\in\bar{B}_{\delta_2}(0)$ for all i. If the controller successfully triggers the simulation's stop condition, the initial condition under study is included into the basin of attraction of the controller. A total of 6000 simulations were performed.

Remark 1: In practice one can pick $\delta_2^i=\delta_2>0$, for all $i=1,2,3$, and choose the value of δ_2 knowing that the controller will steer the state (q,ω) into a neighborhood $U=\{(q,\omega){:}|\eta|>\tilde{\eta},\|\omega\|_2<\tilde{\omega}\}$ of $(q,\omega)=(\pm\mathbb{1},0)$, with $\tilde{\eta}\approx\sqrt{1-3(\delta_2)^2}$ and $\tilde{\omega}\approx 2\sqrt{3\bar{u}\delta_2}$. Parameters δ_1^i must be picked such that $\delta_1^i\in(0,\delta_2)$, for all $i=1,2,3$. In practice, the numerical value of δ_1^i affects only the controller switching frequency in a neighborhood of $(\epsilon_i,\omega_i)=(0,0)$. Parameter δ can be chosen arbitrarily small so as to reduce the size of the neighborhood on which the controller might induce unwinding; that is, $\eta\in(-\delta,+\delta)$. Notice, however, that as δ decreases, the controller becomes less robust with respect to sensor noise.

B. Controller Performance

The performance of the proposed controller is evaluated through two main sets of parameters. The first set of parameters is meant to provide an insight into the overall ability of the controller to meet the control specifications and to evaluate the controller's performance during the transient. Each simulation solution is uniformly sampled every $\Delta t=0.001$. A sampling time is denoted by t_k, with $k\in\{0,1,\ldots N\}$, where N is the total number of samples in a simulation. The following performance measures are recorded:

1) Success rate (SR): SR records the percentage of simulations in which the proposed controller successfully triggered the simulation stopping condition.

2) Root mean square of the angular velocity error (e_ω): $e_\omega=\sqrt{(1/N)\sum_{k=0}^N\|\omega(t_k)\|_2^2}$

3) Root mean square of the principal angle error (e_ϕ): $e_\phi=\sqrt{(1/N)\sum_{k=0}^N|\phi(t_k)|^2}$, where $\phi(t_k)$ denotes the principal angle associated to the spacecraft attitude at time t_k

4) Total number of state transitions in automaton $\mathcal{H}(\delta)$

5) Nondimensional simulation time T

The second set of parameters is meant to provide an insight into the performance of the onboard thrusters. For each simulation, the following quantities are recorded:

1) Number of switches of each control torque $\boldsymbol{\tau}_i$ (n_i)

2) Mean switching frequency of each control torque $\boldsymbol{\tau}_i$ (f_i): f_i is computed as the mean of the controller switching frequency at each time step t_k of the simulation, $f_i(t_k)$. $f_i(t_k)$ is computed as the

Table 1 Mean, Max, and Min of e_ω, e_ϕ, T across all the simulations performed

	e_ω	e_ϕ, rad	T
Mean	0.3201	1.2744	12.444
Max	0.4617	2.446	19.8479
Min	0.0673	0.0682	2.7122

Table 2 Mean, Max, and Min of n_i, f_i with $i \in \{1,2,3\}$ across all the simulations performed

	n_1	n_2	n_3	f_1	f_2	f_3
Mean	6.2497	6.125	6.174	0.4942	0.4841	0.4879
Max	50	51	42	4.8917	4.6703	4.2630
Min	2	2	2	0.1321	0.1478	0.1184

number of switches performed by control torque τ_i in the time window $\Delta t_k = [t_k - \Delta, t_k + \Delta]$, with $\Delta = 0.5$ (dimensionless).

C. Results

The success rate obtained across the simulations has been SR = 1.0. Hence, for any initial condition tested in X_C, the controller has successfully steered each (ϵ_i, ω_i) to $\bar{B}_{\delta_2}(0)$. We believe that the large volume of simulations, combined with the randomness in the selection of initial conditions, presents compelling evidence to the claim that the proposed controller solves the bang-bang attitude control problem globally. The mean, maximum, and minimum of quantities e_ω, e_ϕ, T across all the simulations performed (T can be easily converted to seconds by multiplying it by scale factor $\sqrt{1/(10\bar{u})}$) are presented in Table 1. These results will be used as benchmark when analyzing the performance of the controller during the transient when external perturbations and measurement noise are considered.

In Table 2 the statistical analysis of the switching behavior induced by the controller is presented. In particular, we focus on the mean, maximum, and minimum of the number of switches and of the controller switching frequency across all the simulations performed.

Of particular interest is the analysis of the number of switches induced by the controller. One can see from Table 2 that the minimum number of switches performed per channel is 2 as in the case of the time-optimal controller for double-integrators. The maximum number of switches corresponds to situations in which one of the initial conditions $(\epsilon_i(0), \omega_i(0))$ is initialized very close to the origin. In this case, (ϵ_i, ω_i) bounces between switching sets Λ_i^+ and Λ_i^- until the rest of the state has converged sufficiently close to the origin.

It is interesting to observe that only 8.3% of the simulations performed displayed a state transition in automaton $\mathcal{H}(\delta)$. Moreover, automaton $\mathcal{H}(\delta)$ never performed more than one single state transition. This seems to suggest that once $(q, \omega) \in X_C$, the anti-unwinding supervisor $\mathcal{H}(\delta)$ will switch the equilibrium to stabilize at most once.

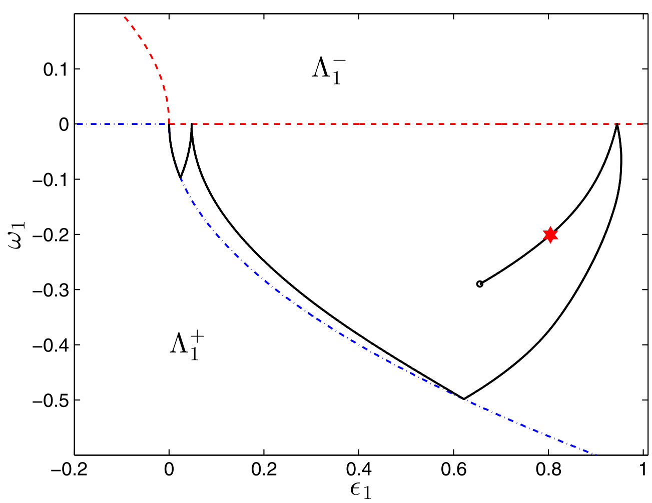


Fig. 3 Trajectory of $(\epsilon_1(t), \omega_1(t))$. The star identifies the transition $-1 \to +1$ in automaton $\mathcal{H}(\delta)$.

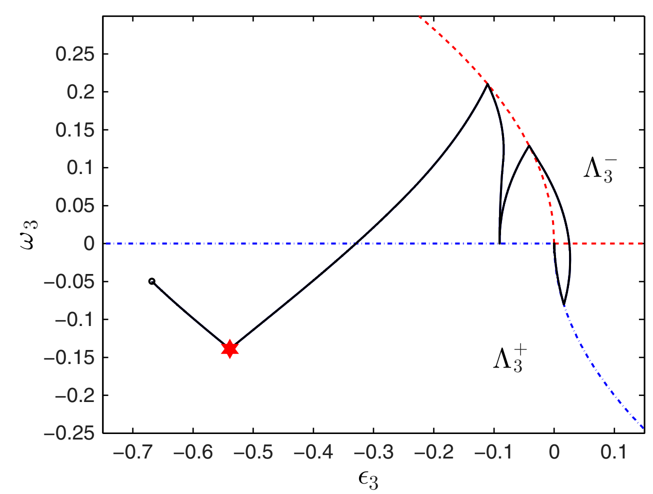


Fig. 5 Trajectory of $(\epsilon_3(t), \omega_3(t))$. The star identifies the transition $-1 \to +1$ in automaton $\mathcal{H}(\delta)$. In this case a switch in the control input value is triggered as $\mathcal{H}(\delta)$ switches value of parameter h.

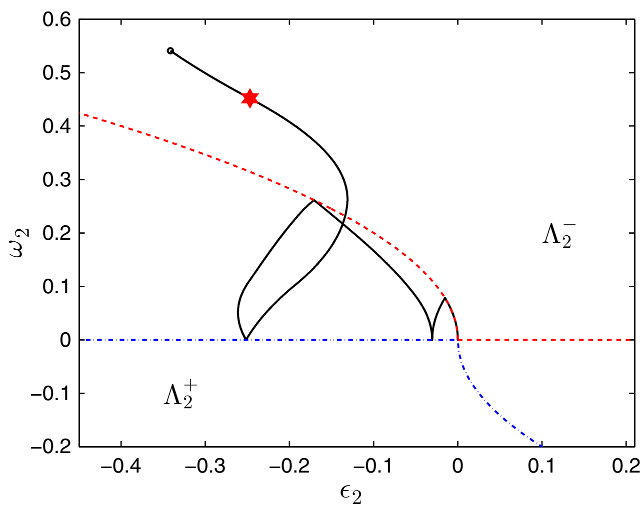


Fig. 4 Trajectory of $(\epsilon_2(t), \omega_2(t))$. The star identifies the transition $-1 \to +1$ in automaton $\mathcal{H}(\delta)$.

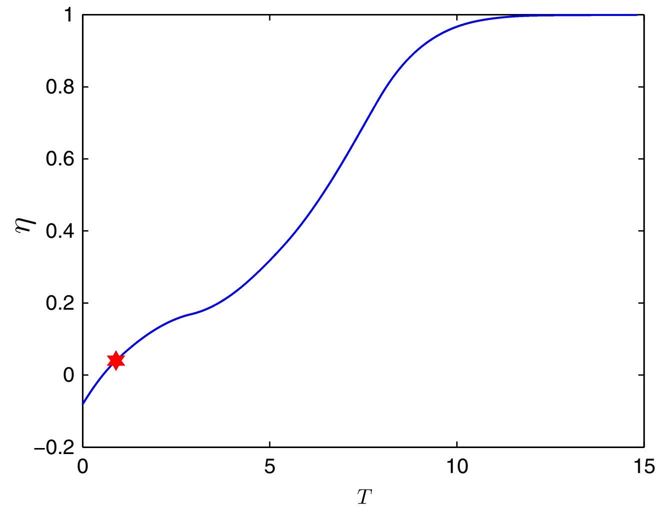


Fig. 6 Time history of state η. The red star identifies the transition $-1 \to +1$ in automaton $\mathcal{H}(\delta)$.

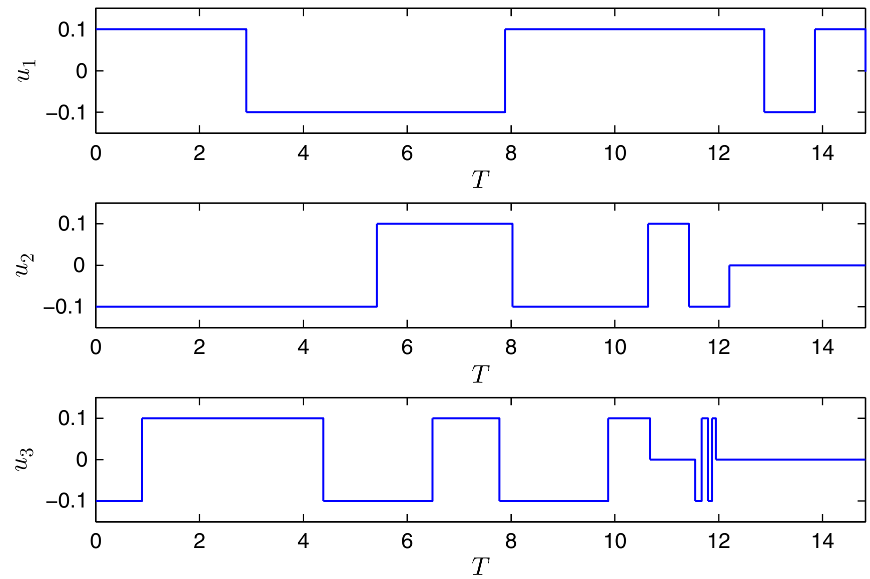


Fig. 7 Controller switching history.

In the following, the output of a simulation is presented so as to provide a better understanding of the controller performance. The initial conditions were randomly chosen at $q(0) = (0.6556, -0.3414, -0.6686, -0.0809)$ and $\omega(0) = (-0.2901, 0.5410, -0.0496)$ (ω is nondimensional). Since $\eta(0) < -\delta$, automaton $\mathcal{H}(\delta)$ is initialized at state $h = -1$. Automata $\mathcal{A}_i(p, -1)$ are then initialized according to the rules presented in Eq. (3), with $\xi_i = (-\epsilon_i, \omega_i)$. In this case, the set of initials states of the automata $\mathcal{A}_i(p, -1)$ is $\{q_2^1, q_1^2, q_1^3\}$, which implies that $u^\star = (+\bar{u}, -\bar{u}, -\bar{u})$.

The trajectories of states (ϵ_i, ω_i), with $i = 1, 2, 3$, are shown in Figs. 3–5. It is immediate to notice that the proposed controller steers each (ϵ_i, ω_i) to a neighborhood of $(\epsilon_i, \omega_i) = (0, 0)$. Figure 6 clearly shows that the controller steers the spacecraft state (q, ω) to a neighborhood of $(\mathbb{1}, 0)$, in that $\eta(t)$ is steered toward $+1$. Automaton $\mathcal{H}(\delta)$ undergoes a single state transition $-1 \to +1$ to prevent the insurgence of unwinding. As shown in Fig. 6, η overshoots past the threshold $\eta \geq \delta$, triggering the state transition $-1 \to +1$ in automaton $\mathcal{H}(\delta)$. After the transition, the low-level controller stabilizes what is now the “closest” equilibrium in X, that is, $(\mathbb{1}, 0)$, instead of forcing the state back to $(-\mathbb{1}, 0)$. This clearly shows how the action of supervisor $\mathcal{H}(\delta)$ prevents the insurgence of unwinding in the closed-loop dynamics. The state transition of $\mathcal{H}(\delta)$ is indicated in the figures by a red star.

Figure 7 shows the switching history of the three control inputs on–off state u_i. Clearly, the proposed controller successfully avoids the generation of high-frequency switching behaviors.

Table 3 Mean, Max, and Min of e_ω, e_ϕ, T across all the simulations performed

	e_ω	e_ϕ, rad	T
Mean	0.318	1.2682	12.458
Max	0.4688	2.213	20.2
Min	0.0528	0.0206	1.67

Table 4 Mean, Max, and Min of n_i, f_i with $i = 1, 2, 3$ across all the simulations performed

	n_1	n_2	n_3	f_1	f_2	f_3
Mean	5.2492	5.2264	5.3106	0.4102	0.4076	0.4137
Max	31	30	29	2.7184	2.0492	2.3374
Min	2	2	2	0.1286	0.1192	0.125

V. Robustness with Respect to Measurement Noise

This section presents a Monte Carlo numerical investigation of the robustness properties of the proposed controller against measurement noise. Let $(\hat{q}, \hat{\omega})$ denote the measured state. When measurement noise is considered, automata $\mathcal{H}(\delta)$ and $\mathcal{A}_i(p_i, h)$, with $i = 1, 2, 3$, undergo state transitions when the measured states $\hat{\eta}$ and $\hat{\xi}_i = (h\hat{\epsilon}_i, \hat{\omega}_i)$ satisfy the state transition conditions, instead of η and $\xi_i = (h\epsilon_i, \omega_i)$. The quaternion measurements $\hat{q}$ are generated as follows (see [25,26]) $\hat{q} = ([\tilde{q}\oplus]q)/\|[\tilde{q}\oplus]q\|$, where $[\tilde{q}\oplus]$ is defined as

$$[\tilde{q}\oplus] = \begin{bmatrix} \tilde{\eta} & \tilde{\epsilon}_3 & -\tilde{\epsilon}_2 & \tilde{\epsilon}_1 \\ -\tilde{\epsilon}_3 & \tilde{\eta} & \tilde{\epsilon}_1 & \tilde{\epsilon}_2 \\ \tilde{\epsilon}_2 & -\tilde{\epsilon}_1 & \tilde{\eta} & \tilde{\epsilon}_3 \\ -\tilde{\epsilon}_1 & -\tilde{\epsilon}_2 & -\tilde{\epsilon}_3 & \tilde{\eta} \end{bmatrix}$$

Vector $\tilde{q}$ is generated as $\tilde{q} = (1/2\delta q, +1)$, where $\delta q \in \mathbb{R}^3$ is sampled from a zero-mean Gaussian distribution with covariance $\sigma_q^2 = (0.001)I\,\mathrm{deg}^2$.

The angular velocity measurements are generated as $\hat{\omega} = \omega + \delta\omega$, where $\delta\omega \in \mathbb{R}^3$ is sampled from a zero-mean Gaussian distribution with covariance $\sigma_\omega^2 = (9 \cdot 10^{-8})I$ (ω is nondimensional). Parameters δ_1 and δ_2 were taken with values $\delta_1 = 1 \cdot 10^{-3}$ and $\delta_2 = 3 \cdot 10^{-3}$. In this case, particular care must be adopted in choosing parameters δ_1, δ_2 so to avoid the insurgence of high-frequency switching at the origin. An in-depth analysis of this issue can be found in [20]. A total of 6000 simulations were performed, obtaining a success rate of SR = 1.0. The results of the numerical study are summarized in Tables 3 and 4.

It can be seen from Table 3 that the performances indices characterizing the closed-loop behavior of the system during the transient are comparable to the nominal performances presented in Table 1. This suggests that the performance degradation induced by the presence of measurement noise is minimal. The number of switches and the switching frequency (see Table 4) also remain comparable to the nominal case (see Table 2). The decrease in the mean number of required switches to meet the control specifications is easily explained by the fact that the values for parameters δ_1 and δ_2 were chosen larger than in the nominal case.

VI. Conclusions

The paper presents a novel hybrid bang-bang controller that solves the bang-bang attitude control problem. The proposed controller successfully stabilizes any desired spacecraft attitude without inducing high-frequency switching of the actuators. Extensive simulation analysis suggests that the proposed controller may in fact yield global practical stability of the target spacecraft attitude. It was also verified that the proposed controller meets the control specifications in the presence of measurement noise.

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