Gain Scheduled SPR Controllers for Nonlinear Flexible Systems

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The problem of scheduling strictly positive real (SPR) dynamic compensation for control of nonlinear flexible systems which exhibit collocated inputs and outputs is explored. The major application is the robust motion control of structurally flexible systems whose dynamics possess significant configuration dependence. Included in this class are flexible robot manipulators. The issue of designing a linear time-invariant SPR compensator for control of a nonlinear system is examined. Controller performance is enhanced by scheduling a series of such designs and a scheduling algorithm is developed which preserves robust stability with respect to the nonlinear plant model. Global asymptotic stability of equilibrium setpoints is proven when the scheduled SPR compensator is used in conjunction with a proportional feedback gain. A numerical example employing a two-link flexible manipulator is used to illustrate the approach and compare the efficacy of different scheduling algorithms.

1 Introduction

It is well known that flexible structures with collocated force (torque) inputs and velocity (angular velocity) outputs exhibit the property known as passivity. This property is independent of the details of the mass and stiffness distributions and provides a mechanism for robust stabilization via the passivity theorem. This important result in input-output theory (Desoer and Vidyasagar, 1975) states that any strictly passive operator connected in negative feedback with a passive system yields input-output stability. Further work by Hill and Moylan (1977) extended the result to global asymptotic stability under suitable reachability and detectability hypotheses governing the state representations of each system.

Linear time-invariant (LTI) passive systems are characterized by positive real transfer functions. In the context of flexible structures, the positive real property is independent of the number of vibration modes in the model as well as the details of the mode shapes and natural frequencies. Hence, any strictly passive feedback yields robust stability since spillover instabilities, which can result from controller designs based on a reduced subset of modes, are avoided. LTI controllers which are strictly passive are closely related to the strictly positive real (SPR) property (Wen, 1988). In particular, an SPR feedback controller always stabilizes a passive system.

The use of dynamic SPR compensation for stabilization of large space structures was suggested by Benhabib et al. (1981). Since then, several authors have presented systematic methodologies for designing SPR control. We mention McLaren and Slater (1987) and Lozano and Joshi (1988), the latter of whom examined LQG weight selection such that the Kalman-Yakubovich Lemma was satisfied, i.e., the LQG controller is SPR. More recently, Haddad et al. (1994) have looked at the corresponding situation for H_∞ design.

The use of strictly passive compensation for nonlinear mechanical systems was considered by Takegaki and Arimoto (1981) who rigorously demonstrated setpoint regulation for a rigid robot using proportional-derivative control. Exploitation of the passivity property for rigid robots to prove stability for both setpoint regulation and tracking was explored by Wen and Bayard (1988). Paden and Riedle (1988) examined the extension of a constant derivative gain to dynamic SPR feedback which they called the "PR modification." Extensions to flexible systems were contemplated by Paden et al. (1990, 1993) and Lanari and Wen (1992). The latter pointed out that the approach could be applied to a very wide class of systems which includes flexible-link and flexible-joint robots. A common feature of the above approaches is the use of an LTI controller for stabilization of a nonlinear system. Although such an approach can guarantee stability over all configurations, it is unlikely that good performance can be achieved over the same.

In this paper, we address some issues related to the design of LTI controllers for nonlinear systems. Setpoint linearizations of the plant, after transformation to modal coordinates, are used as the basis for designing an SPR rate feedback which is added to a proportional term. We then examine the problem of scheduling a series of SPR designs and a scheduling algorithm is developed which preserves the basic stability of a single SPR controller. A series of LTI designs based on local linear models is advanced as a way of dealing with the performance issue. Other related work is that of Shamma and Athans (1990) where the local stability of gain scheduling in general was treated and Meressi and Paden (1994) where LTI H_∞ controllers were scheduled for controlling a two-link flexible manipulator. In this latter work, stability was obtained with respect to intermediate "frozen values" of the plant.

2 System Model and Problem Statement

We confine ourselves to a study of those flexible mechanical systems which are described by a model of the form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{k}(\mathbf{q}) = \mathbf{B}(\mathbf{q})\mathbf{u} + \mathbf{f}_{non}(\mathbf{q}, \dot{\mathbf{q}}),$$
$$\boldsymbol{\theta} = \mathbf{h}(\mathbf{q}), \quad \mathbf{y} = \dot{\boldsymbol{\theta}} = \mathbf{B}^T(\mathbf{q})\dot{\mathbf{q}}, \tag{1}$$

where $\mathbf{q} \in R^n$ are the generalized coordinates, $\mathbf{u} \in R^m$ are the control inputs, and $\boldsymbol{\theta} \in R^m$ are the measured configuration variables. Note that $\mathbf{B} = \partial \mathbf{h}^T(\mathbf{q})/\partial \mathbf{q}$ which expresses the collocation of actuation $\mathbf{u}$ and rate sensing $\mathbf{y}$. The mass matrix $\mathbf{M}$ is positive definite whereas the damping matrix $\mathbf{D}$ is nonnegative definite. The stiffness forces $\mathbf{k}(\mathbf{q}) = \partial V/\partial \mathbf{q}$ emanate from a nonnegative potential energy function $V(\mathbf{q})$. Given the kinetic energy $T = (1/2)\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$, the nonlinear inertial forces have the form $\mathbf{f}_{non} = \partial T/\partial \mathbf{q} - \dot{\mathbf{M}}\dot{\mathbf{q}}$. Defining the Lagrangian by $L = T - V$, the Hamiltonian satisfies

$$E = \dot{\mathbf{q}}^T \frac{\partial L}{\partial \dot{\mathbf{q}}} - L = T + V, \quad \dot{E} = \dot{\boldsymbol{\theta}}^T\mathbf{u} - \dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}}, \tag{2}$$

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where Lagrange's equations have been used. We desire globally stable tracking of constant setpoints $\boldsymbol{\theta} = \boldsymbol{\theta}_d$. It is assumed that the 'augmented potential energy' $V_a(\mathbf{q}) \triangleq V(\mathbf{q}) + \frac{1}{2}(\mathbf{h}(\mathbf{q}) - \boldsymbol{\theta}_d)^T\mathbf{K}_p(\mathbf{h}(\mathbf{q}) - \boldsymbol{\theta}_d)$ with $\mathbf{K}_p = \mathbf{K}_p^T > \mathbf{O}$ is a positive definite function of $\mathbf{q}$, $\mathbf{q} \neq \mathbf{q}_d$, where $\mathbf{q}_d$ uniquely satisfies $\boldsymbol{\theta}_d = \mathbf{h}(\mathbf{q}_d)$ and $V_a(\mathbf{q}_d)$ is a global minimum.

With an appropriate interpretation of the symbols, the above model applies to a wide variety of important systems: large space structures (Benhabib et al., 1981), spacecraft attitude dynamics (Hughes, 1986), six-DOF vehicle dynamics, and rigid or flexible robot manipulators (Sincarsin and Hughes, 1990; Hughes and Sincarsin, 1990; Lanari and Wen, 1992; Wang and Vidyasagar, 1992; Damaren, 1996). Although the general case embodied by (1) is studied here, it is helpful to consider the case of flexible-link manipulators. To this end, the coordinates can be partitioned as $\mathbf{q} = \text{col}\{\boldsymbol{\theta}, \mathbf{q}_e\}$, where $\boldsymbol{\theta}$ are the joint variables, $\mathbf{q}_e$ are the elastic coordinates, and $\mathbf{u}$ are the joint torques. If cantilevered shape functions are used to discretize the link deflections, $\mathbf{k}(\mathbf{q}) = \mathbf{K}\mathbf{q}$ and the matrices can be partitioned consistent with $\mathbf{q}$ as: $\mathbf{B}^T = [\mathbf{1}\ \mathbf{O}]$, $\mathbf{K} = \text{diag}\{\mathbf{O}, \mathbf{K}_{ee}\}$, $\mathbf{D} = \text{diag}\{\mathbf{O}, \mathbf{D}_{ee}\}$ with $\mathbf{K}_{ee} = \mathbf{K}_{ee}^T > \mathbf{O}$ and $\mathbf{D}_{ee} = \mathbf{D}_{ee}^T > \mathbf{O}$.

The key notions required for an input-output study of (1) (Vidyasagar, 1993) are now presented. L_∞ is the set of essentially bounded functions and L_2 is the set of square-integrable functions defined on $t \in [0, \infty]$. The truncation of a function $\mathbf{u}(t)$ is defined as $\mathbf{u}_T(t) = \mathbf{u}(t)$, $t \leq T$, $\mathbf{u}_T(t) = \mathbf{0}$, $t > T$, and the extended space $L_{2e} = \{\mathbf{u} | \mathbf{u}_T \in L_2, \forall T > 0\}$. We also define $\|\mathbf{u}\|_T \triangleq [\int_0^T \mathbf{u}^T\mathbf{u}dt]^{1/2}$. The motion equation (1) can be interpreted as an operator G mapping $\mathbf{u} \in L_{2e}$ into $\mathbf{y} = G\mathbf{u} \in L_{2e}$. Such a system is strictly passive if there exists $\epsilon > 0$ such that

$$\int_0^T \mathbf{u}^T G\mathbf{u}dt \geq \epsilon\|\mathbf{u}\|_T^2, \quad \forall\mathbf{u} \in L_{2e}, \forall T > 0. \tag{3}$$

If (3) is satisfied with $\epsilon = 0$, then the system is *passive*. It is readily verified that the system given in (1) is passive:

$$\int_0^T \mathbf{y}^T\mathbf{u}dt = E(T) + \int_0^T \dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}}dt \geq 0 \quad (E(0) \equiv 0)$$

which follows from integration of the second expression in (2).

Now consider the feedback system shown in Fig. 1. Given that G is passive we wish to select the controller H so that bounded disturbance (or feedforward) signals $\mathbf{u}_d \in L_2$ imply that $\mathbf{y} = \dot{\boldsymbol{\theta}} \in L_2$, i.e., the system is L_2-stable. The key result which will be exploited is one form of the passivity theorem (Desoer and Vidyasagar, 1975) which states that the system is L_2-stable if G is passive and H is strictly passive. We also desire that $\lim_{t\to\infty}\mathbf{q}(t) = \mathbf{q}_d$ when $\mathbf{u}_d = \mathbf{y}_d = \mathbf{0}$. Although input-output ideas better motivate our results, this latter objective will require Lyapunov-style arguments. Tracking of time-varying trajectories $\mathbf{y}_d(t) = \dot{\boldsymbol{\theta}}_d \neq \mathbf{0}$ can be accomplished by selecting a feedforward $\mathbf{u}_d$ such that the map from $(\mathbf{u} - \mathbf{u}_d)$ to $(\mathbf{y} - \mathbf{y}_d)$ is passive. Then choosing $\mathbf{u} - \mathbf{u}_d = -H(\mathbf{y} - \mathbf{y}_d)$ with H strictly passive, places the tracking rate errors in L_2. Again, additional arguments are needed to show that $\boldsymbol{\theta}(t) \to \boldsymbol{\theta}_d(t)$. It bears noting that passive systems are also minimum phase; hence, stabilization and tracking can be accomplished via feedback linearization approaches (Vidyasagar, 1993).

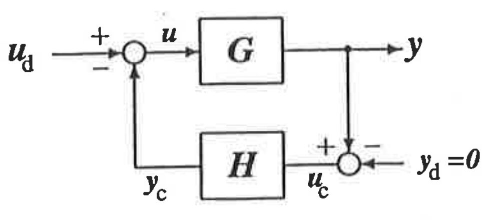

Fig. 1 Mechanical system with feedback

The simplest choice for H is a constant derivative feedback gain, $H\dot{\boldsymbol{\theta}} = \mathbf{K}_d\dot{\boldsymbol{\theta}}$ with $\mathbf{K}_d = \mathbf{K}_d^T > \mathbf{O}$. Position control also demands the introduction of an integral term (a position feedback with respect to $\boldsymbol{\theta}$). In terms of Laplace transforms, $\mathbf{y}_c(s) = \mathbf{H}(s)\mathbf{u}_c(s)$ where $\mathbf{H}(s) = s^{-1}\mathbf{K}_p + \mathbf{K}_d$ which represents PD control (in general, $\mathbf{H}(s)$ will denote the transfer function of an LTI operator H). Since we wish to concentrate on the design of the rate feedback, the proportional term can be transfered to the plant using the technique of loop transformation (Desoer and Vidyasagar, 1975). The input-output properties, including stability, of the system in Figure 1 are equivalent to those of the system in Fig. 2 when L is a linear operator. Selecting $\mathbf{L}(s) = s^{-1}\mathbf{K}_p$, G' is equivalent to the structure with the proportional loop closed and H' is just the rate feedback part of the controller. Since G' itself consists of the negative feedback interconnection of passive systems (the integral operator is passive), it is also passive. Hence, L_2-stability can be established by showing that H' is strictly passive. In the following section, we drop the primes and H refers to the rate feedback and G to G'.

3 SPR Controller Design

For rigid structures, it has been widely established that PD control provides excellent regulation. However, for flexible systems, performance improvements can be expected using a dynamic compensator. The proposed controller structure consists of a proportional feedback loop coupled with a dynamic SPR design for velocity feedback. In the frequency domain, assuming $\boldsymbol{\theta}_d$ is constant and $\mathbf{u}_d = \mathbf{0}$,

$$\mathbf{u}(s) = -\mathbf{K}_p[\boldsymbol{\theta}(s) - \boldsymbol{\theta}_d] - \mathbf{H}(s)\dot{\boldsymbol{\theta}}(s),$$
$$\mathbf{H}(s) = \mathbf{K}_c(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_c + \epsilon\mathbf{1} \tag{4}$$

where $\epsilon > 0$. If $\epsilon = 0$, $\mathbf{H}(s)$ is strictly proper and, according to Tao and Ioanou (1988), is SPR if and only if

(i) $\mathbf{H}(s)$ is real for real s and $\mathbf{H}(s)$ is analytic for $\mathcal{Re}\{s\} \geq 0$;
(ii) $\mathbf{H}(j\omega) + \mathbf{H}^H(j\omega) > \mathbf{O}$, $\quad -\infty < \omega < \infty$;
(iii) $\lim_{\omega\to\infty} \omega^2[\mathbf{H}(j\omega) + \mathbf{H}^H(j\omega)] > \mathbf{O}$.

A more useful characterization uses the triplet $(\mathbf{A}_c, \mathbf{K}_c, \mathbf{K}_c)$ (assumed minimal) and is known as the Kalman-Yakubovich Lemma (Wen, 1988; Tao and Ioannou, 1988): the system $\mathbf{H}(s)$ is SPR if and only if there exists positive definite matrices $\mathbf{P}_0$ and $\mathbf{Q}_0$ such that

$$\mathbf{P}_0\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P}_0 = -\mathbf{Q}_0, \quad \mathbf{P}_0\mathbf{K}_c = \mathbf{K}_c^T. \tag{5}$$

The controller structure is similar to that advocated by Paden and Riedle (1988) and Lanari and Wen (1992). Both considered (4) with a general derivative feedback $-\mathbf{K}_d\dot{\boldsymbol{\theta}}$ in lieu of $-\epsilon\dot{\boldsymbol{\theta}}$ which was interpreted as part of an SPR controller which is not strictly proper (the matrix $\mathbf{K}_d$ represents the high frequency gain matrix). Addition of the $\mathbf{K}_d$ term (or $\epsilon\mathbf{1}$ here) renders $\mathbf{H}(s)$ strictly passive and the passivity theorem can be directly employed to demonstrate L_2-stability. In its absence, $\mathbf{H}(s)$ is not strictly passive as can be seen by applying Parseval's theorem (Desoer and Vidyasagar, 1975, Examples VI.4.2 and VI.4.1). However, L_2-stability can still be established with $\mathbf{H}(s)$ merely SPR using loop transformations (Benhabib et al., 1981; Marquez and Damaren, 1995). An advantage of the strictly proper case is the first-order rolloff when viewed as a function of $\dot{\boldsymbol{\theta}}(s)$. This provides further robustness with respect to unmodeled high frequency effects including rate sensor noise. In spite of this, our proof of global asymptotic stability in the scheduled case will require strict passivity ($\epsilon > 0$ and arbitrarily small) but we set $\epsilon = 0$ in our numerical work.

The design of $\mathbf{H}(s)$ is treated separately from the selection of $\mathbf{K}_p$. The latter is typically chosen to meet steady-state error specifications in the presence of constant disturbances. Closing

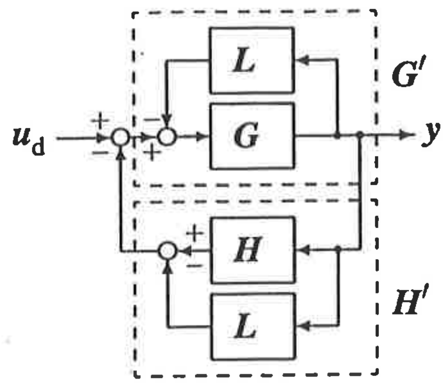

Fig. 2 Loop transformed system

the proportional loop in the model (1), effectively augments the stiffness term $\mathbf{k}(\mathbf{q})$ to produce

$$\frac{\partial V_a(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{k}(\mathbf{q}) + \mathbf{B}\mathbf{K}_p\mathbf{h}(\mathbf{q}) \quad (\boldsymbol{\theta}_d = \mathbf{0}) \tag{6}$$

where V_a was defined after Eq. (2). As noted by Lanari and Wen (1992), this produces observability of the rigid modes (i.e., those satisfying $\mathbf{k}(\mathbf{q}) = \mathbf{0}$) from $\dot{\boldsymbol{\theta}}$. The design of the LTI controller $\mathbf{H}(s)$ for the nonlinear system (1) is most easily treated using an LTI setpoint linearization. Assuming that the structural damping is small and poorly known, $\mathbf{D}$ can be taken as $\mathbf{O}$ for control system design. Linearizing (1) in the vicinity of a constant configuration $\bar{\mathbf{q}}$ and letting $\delta\mathbf{q} = \mathbf{q} - \bar{\mathbf{q}}$ yields

$$\mathbf{M}(\bar{\mathbf{q}})\delta\ddot{\mathbf{q}} + \mathbf{K}_a(\bar{\mathbf{q}})\delta\mathbf{q} = \mathbf{B}(\bar{\mathbf{q}})\mathbf{u}, \quad \mathbf{y} = \dot{\boldsymbol{\theta}} = \mathbf{B}^T(\bar{\mathbf{q}})\delta\dot{\mathbf{q}}, \tag{7}$$

where $\mathbf{K}_a(\mathbf{q}) \triangleq \partial^2 V_a/(\partial\mathbf{q}\partial\mathbf{q}^T) = \partial\mathbf{k}/\partial\mathbf{q}^T + \mathbf{B}(\mathbf{q})\mathbf{K}_p\mathbf{B}^T(\mathbf{q})$ is the effective stiffness matrix. In many applications $\mathbf{K}_a > \mathbf{O}$ which is assumed for the duration of this section.

For controller design, consider modal transformation of (7). Let $\Omega_\alpha > 0$ and $\mathbf{e}_\alpha$, $\alpha = 1, \ldots, n$, denote the vibration frequencies and eigenvectors of the corresponding second order eigenproblem ($\mathbf{u} \equiv \mathbf{0}$) and assume that $\mathbf{e}_\alpha^T\mathbf{M}\mathbf{e}_\beta = \delta_{\alpha\beta}$. If the eigenmatrix $\mathbf{E} \triangleq \text{row}\,\{\mathbf{e}_\alpha\}$ is used to transform the system according to $\delta\mathbf{q}(t) = \mathbf{E}\boldsymbol{\eta}(t)$, then (7) can be represented by the first-order state space model

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u}, \quad \mathbf{y} = \bar{\mathbf{C}}\mathbf{x} \tag{8}$$

with

$$\mathbf{x} = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \boldsymbol{\Omega}\boldsymbol{\eta} \end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{O} & -\boldsymbol{\Omega} \\ \boldsymbol{\Omega} & \mathbf{O} \end{bmatrix},$$

$$\bar{\mathbf{B}} = \bar{\mathbf{C}}^T = \begin{bmatrix} \mathbf{E}^T\mathbf{B} \\ \mathbf{O} \end{bmatrix}, \quad \boldsymbol{\Omega} = \text{diag}\,\{\Omega_\alpha\}. \tag{9}$$

It is a simple matter to show that the linear system is passive since $\mathbf{G}(s) = \bar{\mathbf{C}}(s\mathbf{1} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}$ is positive real (Anderson, 1967). A major advantage of the modal space representation is the ability to systematically reduce the order of the model and hence the controller which will typically be of the same order as the plant model used for its design. This modal truncation procedure does not affect the passivity (positive realness) of the system given by (8) and (9).

Given the LTI plant model $\mathbf{G}(s)$, the development of an optimal controller $\mathbf{H}(s)$ which is SPR represents an unsolved problem. However, as noted in the introduction, a number of systematic approaches for developing SPR controllers have been suggested, most of which rely on the Kalman-Yakubovich Lemma. We use the method suggested by Benhabib et al. (1981). Given the plant (8), choose a matrix $\mathbf{K}_c$ such that $\mathbf{A}_c = \bar{\mathbf{A}} - \bar{\mathbf{B}}\mathbf{K}_c$ is stable. The classical LQR synthesis is used here which determines $\mathbf{K}_c$ in terms of the solution of an algebraic Riccati solution associated with the minimization of

$$J = \frac{1}{2}\int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{10}$$

with $\mathbf{Q} = \mathbf{Q}^T > \mathbf{O}$ and $\mathbf{R} = \mathbf{R}^T > \mathbf{O}$. Assuming there are no trivial rows in $\mathbf{E}^T\mathbf{B}$, all modes of (8) are controllable and observable which guarantees that $\mathbf{A}_c$ has eigenvalues with negative real parts. Then, given another positive definite matrix $\mathbf{Q}_0$, one can solve the Lyapunov equation in (5) for its unique positive definite solution $\mathbf{P}_0$ and take $\mathbf{K}_e = \mathbf{K}_c^T\mathbf{P}_0$. This ensures that $\mathbf{H}(s) = \mathbf{K}_c(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$ is SPR. The positive definite matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{Q}_0$ are free design parameters.

The SPR controller is guaranteed to stabilize not only the LTI system $\mathbf{G}(s)$ regardless of the number of retained modes, but also the nonlinear plant G (which includes the proportional loop) on the basis of the passivity theorem (Marquez and Damaren, 1995). Although $\mathbf{H}(s)$ should perform well in the vicinity of the setpoint used for its design, it may not work well outside the linear range. If large excursions of the configuration variables $\mathbf{q}$ are to be handled, the use of several SPR controllers is suggested, each designed for a different configuration encountered en route. In the next section we address the design of a scheduling algorithm which preserves stability.

4 Gain Scheduled SPR Controllers

Consider the basic feedback structure of Fig. 1 and recall that G (formerly G') consists of the nominal system (1) with the proportional feedback loop closed. Let $\mathbf{x} \triangleq \text{col}\,\{\dot{\mathbf{q}}, \mathbf{q}\}$ denote the state of G, which is passive, and assume that $\mathbf{y}_c = H\mathbf{u}_c$ is implemented using N parallel strictly passive controllers:

$$\mathbf{y}_i = H_i\mathbf{u}_i, \quad \int_0^T \mathbf{y}_i^T\mathbf{u}_i\,dt \geq \epsilon_i\|\mathbf{u}_i\|_T^2,$$

$$\epsilon_i > 0, \quad i = 1, \ldots, N, \tag{11}$$

where the relationship between the $\mathbf{u}_i$ and $\mathbf{u}_c$ is to be determined. The net controller output is taken to be

$$\mathbf{y}_c(t) = \sum_{i=1}^{N} s_i(\mathbf{x}, t)\mathbf{y}_i(t) \tag{12}$$

where $s_i(\mathbf{x}, t)$ are the scheduling signals. For example, with two controllers we might use linear interpolation with $s_1 = (1 - \lambda)$, $s_2 = \lambda$, where $\lambda \in [0, 1]$ parametrizes the controller design. Such a scheme was considered by Meressi and Paden (1994) in the control of a two-link flexible manipulator. The scheduled controllers were not SPR but were designed using an H_∞ methodology and stability was only guaranteed with respect to the frozen values of the corresponding interpolated LTI plants.

There are two obvious possibilities for the $s_i(\mathbf{x}, t)$. If one uses time, $s_i \equiv s_i(t)$ can be interpreted as scheduling based upon a prescribed reference trajectory $\boldsymbol{\theta}_d(t)$. Alternatively, one can schedule based on measurements of the controlled system, $s_i \equiv s_i(\mathbf{x})$. In general, it will be assumed that they satisfy

$$\sum_{i=1}^{N} s_i^2(\mathbf{x}, t) \geq \alpha > 0, \quad s_i(\mathbf{x}, t) \in L_{2e}\ \forall\mathbf{x}\forall t,$$

$$s_i(\mathbf{x}, t) \in L_\infty\ \forall\mathbf{x} \in \mathbf{L}_\infty, \tag{13}$$

which guarantees that at least one controller is in use at any time, each scheduling signal is square-integrable on any finite time interval, and the time dependence is bounded.

Theorem 1: If the controller $\mathbf{y}_c = H\mathbf{u}_c$ satisfies (11)–(13), then H is strictly passive if the individual controller inputs satisfy

$$\mathbf{u}_i(t) = s_i\mathbf{u}_c(t),\ i = 1, \ldots, N. \tag{14}$$

Proof: Consider the following:

$$\int_0^T \mathbf{y}_c^T\mathbf{u}_c dt = \int_0^T \mathbf{u}_c^T \sum_{i=1}^N s_i\mathbf{y}_i\, dt = \sum_{i=1}^N \int_0^T \mathbf{y}_i^T\mathbf{u}_i\, dt$$
$$\geq \sum_{i=1}^N \epsilon_i\|\mathbf{u}_i\|_T^2 = \sum_{i=1}^N \epsilon_i\|s_i\mathbf{u}_c\|_T^2 \geq \epsilon\|\mathbf{u}_c\|_T^2 \tag{15}$$

where $\epsilon = \alpha \min(\epsilon_i) > 0$. This establishes the result. □

We conclude that almost any scheduling signal preserves the strict passivity of the controller and will provide L_2-stability for a passive plant provided the scheduling signals are used to form the overall output *and* the individual controller inputs. It is beneficial if they further satisfy $s_i(\mathbf{x}(t), t) \equiv 0$, $i \neq j$, $i \neq k$, for some j and k at each t so that only two dynamic controllers are utilized at any time. The overall system is depicted in Figure 3.

Let us further specialize the above result to the case where each $\boldsymbol{H}_i$ is implemented as

$$\mathbf{y}_i = \mathbf{C}_i\mathbf{x}_i + \epsilon_i\mathbf{u}_i,\quad \dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i,\quad \mathbf{u}_i = s_i\dot{\boldsymbol{\theta}}. \tag{16}$$

If $\mathbf{H}_i(s) = \mathbf{C}_i(s\mathbf{1} - \mathbf{A}_i)^{-1}\mathbf{B}_i$ is taken to be SPR, then there exists $\mathbf{P}_i > \mathbf{O}$ and $\mathbf{Q}_i > \mathbf{O}$ such that

$$\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P}_i = -\mathbf{Q}_i,\quad \mathbf{P}_i\mathbf{B}_i = \mathbf{C}_i^T. \tag{17}$$

The control input to the structure (1) is taken to be

$$\mathbf{u}(t) = \mathbf{u}' = -\mathbf{K}_p(\boldsymbol{\theta}(t) - \boldsymbol{\theta}_d) - \sum_{i=1}^N s_i\mathbf{y}_i(t). \tag{18}$$

Although the proportional gain $\mathbf{K}_p$ could also be scheduled, for simplicity it will not be here. If $\mathbf{u} = \mathbf{u}' + \mathbf{u}_d$, then $\mathbf{u}_d \in L_2 \Rightarrow \mathbf{y} = \dot{\boldsymbol{\theta}} \in L_2$.

In addition to the assumptions of Section 2, we require an additional detectability condition to show position tracking when $\mathbf{u}_d = \mathbf{0}$. Consider (1) with the proportional loop closed:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{k}(\mathbf{q}) + \mathbf{B}\mathbf{K}_p[\mathbf{h}(\mathbf{q}) - \mathbf{h}(\mathbf{q}_d)] = \mathbf{f}_{non}(\mathbf{q}, \dot{\mathbf{q}}). \tag{19}$$

We assume that $\mathbf{D}\dot{\mathbf{q}}(t) = \mathbf{0}$ and $\dot{\boldsymbol{\theta}}(t) = \mathbf{B}^T\dot{\mathbf{q}}(t) = \mathbf{0}$ imply that $\dot{\mathbf{q}} = \mathbf{0}$. This *zero rate detectability* condition is similar to the zero state detectability hypothesis of Lanari and Wen (1992).

Theorem 2: The feedback system given by (1) and (16)–(18) yields global asymptotic stability of the equilibrium $\mathbf{q} = \mathbf{q}_d$ where $\boldsymbol{\theta}_d = \mathbf{h}(\mathbf{q}_d)$.

Proof: Using the Hamiltonian in (2), define the positive definite function

$$\mathcal{V} = E + \tfrac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_d)^T\mathbf{K}_p(\boldsymbol{\theta} - \boldsymbol{\theta}_d) + \tfrac{1}{2}\sum_{i=1}^N \mathbf{x}_i^T\mathbf{P}_i\mathbf{x}_i.$$

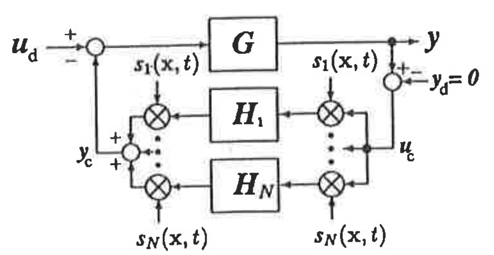

Fig. 3 Gain scheduled system

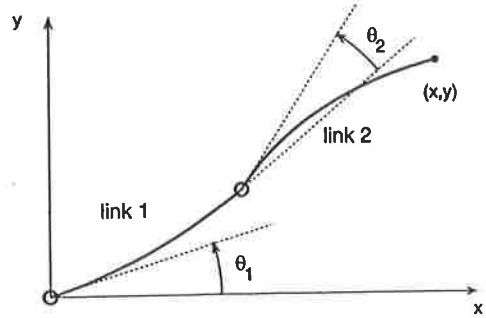

Fig. 4 Two-link manipulator

Differentiating and using (2) and (16)–(18) gives

$$\dot{\mathcal{V}} = \dot{\boldsymbol{\theta}}^T\mathbf{u} - \dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}} + \dot{\boldsymbol{\theta}}^T\mathbf{K}_p(\boldsymbol{\theta} - \boldsymbol{\theta}_d)$$
$$+ \sum_{i=1}^N [\tfrac{1}{2}\mathbf{x}_i^T(\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P}_i)\mathbf{x}_i + \mathbf{x}_i^T\mathbf{P}_i\mathbf{B}_i s_i\dot{\boldsymbol{\theta}}]$$
$$= -\sum_{i=1}^N \dot{\boldsymbol{\theta}}^T(s_i\mathbf{y}_i) - \tfrac{1}{2}\sum_{i=1}^N \mathbf{x}_i^T\mathbf{Q}_i\mathbf{x}_i + \sum_{i=1}^N s_i(\mathbf{C}_i\mathbf{x}_i)^T\dot{\boldsymbol{\theta}} - \dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}}$$
$$\leq -\tfrac{1}{2}\sum_{i=1}^N \mathbf{x}_i^T\mathbf{Q}_i\mathbf{x}_i - \dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}} - \epsilon\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}}$$

where ϵ is defined by (13) and (15) et seq. Since $\dot{\mathcal{V}} \leq 0$, we have shown Lyapunov stability when $\mathbf{u}_d = \mathbf{0}$. In addition, $\{\mathbf{x}_i, \mathbf{D}\dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}\} \in L_2 \cap L_\infty$ and $\{\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\theta}\} \in L_\infty$. From (18), $\mathbf{u} \in L_\infty$ since the s_i are bounded using (13). The motion equations (1) and (16) yield boundedness of $\{\dot{\mathbf{x}}_i, \ddot{\mathbf{q}}, \ddot{\boldsymbol{\theta}}\}$ which makes $\{\mathbf{x}_i, \mathbf{D}\dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}\}$ uniformly continuous and therefore $\mathbf{x}_i(t) \to \mathbf{0}$, $\mathbf{D}\dot{\mathbf{q}}(t) \to \mathbf{0}$, and $\dot{\boldsymbol{\theta}}(t) \to \mathbf{0}$ as $t \to \infty$. Hence, $\mathbf{u}(t) \to -\mathbf{K}_p(\boldsymbol{\theta}(t) - \boldsymbol{\theta}_d)$ and invoking the zero rate detectability assumption we have $\dot{\mathbf{q}}(t) \to \mathbf{0}$. Using (19), $\mathbf{k}(\mathbf{q}) + \mathbf{B}\mathbf{K}_p[\mathbf{h}(\mathbf{q}) - \mathbf{h}(\mathbf{q}_d)] = \partial V_a(\mathbf{q})/\partial\mathbf{q} \to \mathbf{0}$ as $t \to \infty$. Since $V_a(\mathbf{q}_d)$ is a global minimum, we must have $\mathbf{q}(t) \to \mathbf{q}_d$. □

Note that if $\mathbf{D} \equiv \mathbf{O}$, one can still prove the above result if $\dot{\boldsymbol{\theta}} = \mathbf{0} \Rightarrow \dot{\mathbf{q}} = \mathbf{0}$ in (19), i.e., the rate coordinates are detectable via $\dot{\boldsymbol{\theta}}$.

5 Numerical Example

In the previous sections, a methodology for designing SPR controllers and a scheduling scheme which preserves the basic stability of a single SPR compensator has been established. The controllers will now be implemented in the simulation of a planar two-link flexible manipulator (Fig. 4). In-plane bending of each link is modeled using the analytical clamped-free mode shapes for spatial discretization. A complete description of the simulation methodology can be found elsewhere (Damaren and Sharf, 1995); we use the I(E) model detailed there. Four bending modes per link are used. The links are identical with length 1 m, cross-sectional area $A = 6 \times 10^{-5}$ m^2, bending stiffness EI = 1.4 N·m^2, and mass per unit length $\rho = 0.15$ kg/m^2. This is based on the model used by Meressi and Paden (1994).

We wish to track the joint-space trajectories

$$\theta_j(t) = (\theta_{jT} - \theta_{j0})\left(\frac{t}{T} - \frac{1}{2\pi}\sin\frac{2\pi t}{T}\right) + \theta_{j0},\ j = 1, 2, \tag{20}$$

where $\theta_{1,0} = 0$, $\theta_{2,0} = -\pi/2$, $\theta_{1,T} = \theta_{2,T} = \pi/2$, and $T = 10$ sec. Tracking of the corresponding (but noncollocated) end-effector trajectories is seen as a secondary goal. The N configurations for controller design are selected according to $\overline{\boldsymbol{\theta}}_{di} = \boldsymbol{\theta}_d(t_i)$, $t_i = (i - 1)T/(N - 1)$, $i = 1, \ldots, N$ with $t_1 = T$ if $N = 1$. The linearized system given by (8) and (9) is formed at each configuration where the proportional feedback gain is cho-

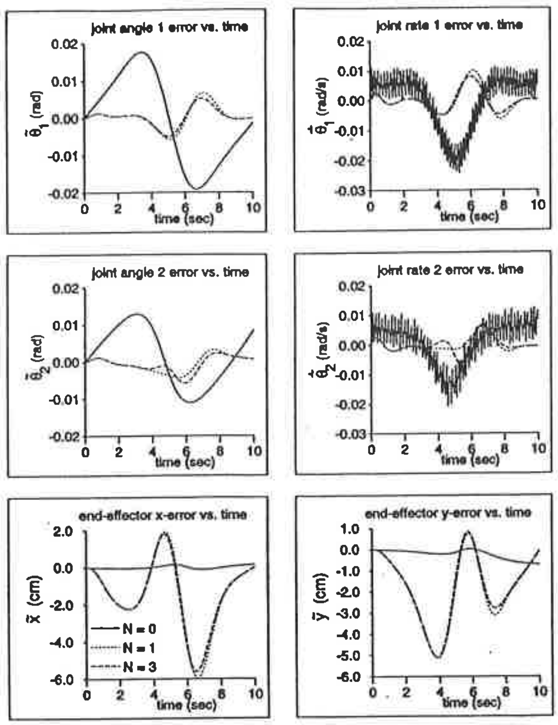

Fig. 5 Tracking errors

sen as $\mathbf{K}_p = \mathbf{B}^T\mathbf{M}(\bar{\theta}_{dN})\mathbf{B}\cdot\Omega_c^2$, $\Omega_c = 1$ rad/s (i.e., proportional to the joint-space rigid mass matrix at the terminal configuration). After modal transformation, only the first 4 of the 10 unconstrained vibration modes are retained for controller design. The SPR controllers at each setpoint are designed using the procedures of Section 3 with $\mathbf{Q} = \mathbf{1}$, $\mathbf{R} = (2.44 \times 10^{-4})\mathbf{1}$, $\mathbf{Q}_0 = (2048)\mathbf{1}$, and are of 8th order. With this choice of $\mathbf{Q}$, the LQR cost functional in (10) penalizes the total energy in the modes retained for controller design. The matrix $\mathbf{Q}_0$ was selected so that the eigenvalues of the composite system matrix,

$$\begin{bmatrix} \bar{\mathbf{A}} & -\bar{\mathbf{B}}\mathbf{K}_c \\ \mathbf{K}_e\bar{\mathbf{C}} & \mathbf{A}_c \end{bmatrix}$$

were at least as fast as those of $\mathbf{A}_c = \bar{\mathbf{A}} - \bar{\mathbf{B}}\mathbf{K}_c$ emanating from the LQR synthesis.

Two choices for the scheduling signal are used. The first is based on the second joint angle and satisfies

$$s_i(\theta_2(t)) = \begin{cases} \sqrt{\dfrac{\theta_2(t) - \theta_2(t_{i-1})}{\theta_2(t_i) - \theta_2(t_{i-1})}}, & \theta_2(t_{i-1}) \le \theta_2(t) \le \theta_2(t_i) \\ \sqrt{\dfrac{\theta_2(t_{i+1}) - \theta_2(t)}{\theta_2(t_{i+1}) - \theta_2(t_i)}}, & \theta_2(t_i) \le \theta_2(t) \le \theta_2(t_{i+1}) \\ 0, & \text{otherwise.} \end{cases}$$

If $\theta_2(t) \le \theta_2(0)$, then $s_1 = 1$, $s_i \equiv 0$, $i \ne 1$. If $\theta_2(t) \ge \theta_2(T)$, then $s_N = 1$, $s_i \equiv 0$, $i \ne N$. This is designated Type I. Type II corresponds to scheduling based on the reference trajectory (i.e., time):

$$s_i(t) = \begin{cases} \sqrt{\dfrac{t - t_{i-1}}{t_i - t_{i-1}}}, & t_{i-1} \le t \le t_i \\ \sqrt{\dfrac{t_{i+1} - t}{t_{i+1} - t_i}}, & t_i \le t \le t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

with $s_N(t) = 1$, $t \ge T$.

A feedforward signal ($\mathbf{u}_d$) is also included and is determined from the inverse dynamics of the corresponding rigid arm. As well, $\mathbf{y}_d(t)$ in Fig. 3 is set to $\dot{\boldsymbol{\theta}}_d(t)$ and $\epsilon_i = 0$, $i = 1 \dots N$. In Figure 5, the joint angle, joint rate, and end-effector position tracking errors are shown for no feedback controller ($N = 0$), $N = 1$, and $N = 3$ (using Type I scheduling). The notation $(\tilde{\cdot})$ denotes tracking error so that $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_d$. We have also collected the relative tracking errors for each variable in Table 1. These are defined according to $E_{rel}(\cdot) = \|(\tilde{\cdot})\|_T/\|(\cdot)_d\|_T$. The end-effector variables are collected into $\boldsymbol{\rho} \triangleq [x\, y]^T$. In all cases, stability is maintained as expected.

Clearly, the error performance of the joint angle and joint rate tracking improves as the number of scheduled controllers is increased. However the improvement is not great. This can be attributed to the lack of configuration dependence of the vibration modes of a two-link manipulator. The vibration frequencies and mode shapes of this system do not vary greatly with the value of θ_2. One expects that more complicated systems would derive greater benefit from the scheduling strategy. From the table, we see that Type I scheduling offers a modest improvement in the joint space tracking over Type II for $N = 3$. This is in keeping with the heuristic that "one should schedule based on the true configuration rather than the desired one." Interested readers may compare our results with those of De Luca and Siciliano (1993) where inversion-based nonlinear controllers were used to accomplish similar goals with a similar arm.

While the end-effector tracking errors largely improve with increasing N for $N \ge 1$, the best performance is achieved for no feedback control at all ($N = 0$) which also yields the poorest joint-space tracking. We conclude that the rigid inverse dynamics torques do not invert the torque to joint rate map very well for the flexible arm. In some cases they can provide a good approximation to the inverse of the torque to end-effector rate map (Damaren, 1995; Damaren, 1996). Damaren (1996) has shown that flexible manipulators will exhibit this property when carrying large payloads. Based on the current results, one is led to speculate that there is a wider class of flexible-link robots which also possess this property.

6 Concluding Remarks

A scheme has been presented whereby one can schedule N SPR controller designs and preserve the basic stability afforded by a single SPR controller in feedback with a passive plant. Both input-output stability and the global asymptotic stability of a constant setpoint follow from rather mild assumptions in addition to the passivity property. The latter is independent of the details of the mass and stiffness distribution and only relies on the collocation of force actuation and rate sensing.

The design of each controller was based on the philosophy that an LTI controller is most easily designed using an LTI plant. By transforming the linearized system into modal form, the order of the system could be easily reduced while preserving the passive nature of the model. It was suggested that this was a simple way to develop lower order SPR controllers. Our numerical results illustrated the stability results and showed that scheduling could improve the performance achievable using a single controller. While this improvement was not significant

Table 1 Relative tracking errors

N	Type	$E_{rel}(\theta)$	$E_{rel}(\dot{\theta})$	$E_{rel}(\rho)$	$E_{rel}(\dot{\rho})$
0	—	9.38×10^{-3}	2.71×10^{-2}	2.00×10^{-3}	2.52×10^{-3}
1	—	2.36×10^{-3}	1.06×10^{-2}	2.09×10^{-2}	4.69×10^{-2}
2	I	2.36×10^{-3}	1.06×10^{-2}	2.09×10^{-2}	4.69×10^{-2}
3	I	2.07×10^{-2}	9.90×10^{-3}	2.01×10^{-2}	4.50×10^{-2}
4	I	2.02×10^{-2}	9.57×10^{-3}	2.01×10^{-2}	4.50×10^{-2}
3	II	2.14×10^{-2}	1.05×10^{-2}	2.00×10^{-2}	4.46×10^{-2}

for the system under study, it is expected that more complex systems could greatly benefit from the scheduling result. However, the performance of the scheduled controller can only be as good as that of its constituents in the vicinity of the design point. More research is required on the development of optimal controllers subject to a strictly positive real constraint.

References

Anderson, B. D. O., 1967. "A System Theory Criterion for Positive Real Matrices," *SIAM J. Control,* Vol. 5, No. 2, pp. 171–182.

Benhabib, R. J., Iwens, R. P., and Jackson, R. L., 1981. "Stability of Large Space Structure Control Systems Using Positivity Concepts," *AIAA J. Guidance and Control,* Vol. 4, No. 5, pp. 487–494.

Damaren, C. J., 1995. "Passivity Analysis for Flexible Multilink Space Manipulators," *AIAA J. Guidance, Control, and Dynamics,* Vol. 18, No. 2, pp. 272–279.

Damaren, C. J., 1996. "Approximate Inverse Dynamics and Passive Feedback for Flexible Manipulators Carrying Large Payloads," *IEEE Trans. Robotics and Automation,* Vol. 12, No. 1, pp 131–138.

Damaren, C. J., and Sharf, I., 1995. "Simulation of Flexible-Link Manipulators with Inertial and Geometric Nonlinearities," ASME Journal Dynamic Systems, Measurement, and Control, Vol. 117, No. 1, pp. 74–87.

De Luca, A., and Siciliano, B., 1993. "Inversion-Based Nonlinear Control of Robot Arms with Flexible Links," *AIAA J. Guidance, Control, and Dynamics,* Vol. 16, No. 6, pp. 1169–1176.

Desoer, C. A., and Vidyasagar, M., 1975. *Feedback Systems: Input-Output Properties,* Academic Press, New York.

Haddad, W. M., Bernstein, D. S., and Wang, Y. W., 1994. "Dissipative H_2/H_∞ Controller Synthesis," *IEEE Trans. Automatic Control,* Vol. 39, No. 4, pp. 827–831.

Hill, D. J. & Moylan, P. J., 1977. "Stability Results for Nonlinear Feedback Systems," *Automatica,* Vol. 13, pp. 377–382.

Hughes, P. C., 1986. *Spacecraft Attitude Dynamics,* John Wiley, Toronto.

Hughes, P. C., and Sincarsin, G. B., 1990. "Dynamics of Elastic Multibody Chains: Part B—Global Dynamics," *Dynamics and Stability of Systems,* Vol. 4, Nos. 3 and 4, pp. 227–243.

Lanari, L., and Wen, J. T., 1992. "Asymptotically Stable Set Point Control Laws for Flexible Robots," *Systems & Control Letters,* Vol. 19, pp. 119–129.

Lozano-Leal, R., and Joshi, S. M., 1988. "On the Design of Dissipative LQG-Type Controllers," *Proc. 27th IEEE Decision and Control Conference,* Vol. 2, pp. 1645–1646.

McLaren, M. D., and Slater, G. L., 1987. "Robust Multivariable Control of Large Space Structures Using Positivity," *AIAA J. Guidance, Control, and Dynamics,* Vol. 10, No. 4, pp. 393–400.

Marquez, H. J., and Damaren, C. J., 1995. "Comments on 'Strictly Positive Real Transfer Functions Revisited,' " *IEEE Trans. on Automatic Control,* Vol. 40, No. 3, pp. 478–479.

Meressi, T., and Paden, B., 1994. "Gain Scheduled H_∞ Controllers for a Two Link Flexible Manipulator," *AIAA J. Guidance, Control, and Dynamics,* Vol. 17, No. 3, pp. 537–543.

Paden, B., Chen., D., Ledesma, R., and Bayo, E., 1993. "Exponentially Stable Tracking Control for Multijoint Flexible-Link Manipulators," ASME Journal Dynamic Systems, Measurement, and Control, Vol. 115, No. 1, pp. 53–59.

Paden, B., and Riedle, B., 1988. "A Positive-Real Modification of a Class of Nonlinear Controllers for Robot Manipulators," *Proc. 1988 American Control Conference,* Atlanta, GA, pp. 1782–1785.

Paden, B., Riedle, B., and Bayo, E., 1990. "Exponentially Stable Tracking Control for Multi-Joint Flexible-Link Manipulators," *Proc. 1990 American Control Conference,* San Diego, CA, pp. 680–684.

Shamma, J. S., and Athans, M., 1990. "Analysis of Gain Scheduled Control for Nonlinear Plants," *IEEE Trans. Automatic Control,* Vol. 35, No. 8, pp. 898–907.

Sincarsin, G. B., and Hughes, P. C., 1990. "Dynamics of Elastic Multibody Chains: Part A—Body Motion Equations," *Dynamics and Stability of Systems,* Vol. 4, Nos. 3 and 4, pp. 537–543.

Takegaki, M., and Arimoto, S., 1981. "A New Feedback Method for Dynamic Control of Manipulators," ASME Journal Dynamic Systems, Measurement, and Control, Vol. 103, pp. 119–125.

Tao, G., and Ioannou, P. A., 1988. "Strictly Positive Real Matrices and the Lefschetz-Kalman-Yakubovich Lemma," *IEEE Trans. Automatic Control,* Vol. 33, No. 12, pp. 1183–1185.

Vidyasagar, M., 1993. *Nonlinear Systems Analysis,* 2nd Ed., Prentice-Hall, Englewood Cliffs, NJ.

Wang, D., and Vidyasagar, M., 1992. "Passive Control of a Stiff Flexible Link," *Int. J. of Robotics Research,* Vol. 11, No. 6, pp. 572–578.

Wen, J. T., 1988. "Time Domain and Frequency Domain Conditions for Strict Positive Realness," *IEEE Trans. Automatic Control,* Vol. 33, No. 10, pp. 988–992.

Wen, J. T., and Bayard, D. S., 1988. "New Class of Control Laws for Robotic Manipulators Part 1. Non-adaptive Case," *Int. J. Control,* Vol. 47, No. 5, pp. 1361–1385.