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Hybrid passivity and finite gain stability theorem: stability and control of systems possessing passivity violations

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Abstract: The stability and control of systems possessing passivity violations is considered. The authors seek to exploit the finite gain characteristics of a plant over a range in which a passive mapping no longer exists while implementing a similar hybrid passive and finite gain controller. Using the dissipative systems framework the authors define a hybrid system: one which possesses a passive map, and finite gain characteristics when the passive map is destroyed. The definition of a hybrid system utilises a switching parameter to break the system into passive and finite gain regions. It is shown that this switching parameter is equivalent to an ideal low-pass filter and can be approximated by a Butterworth filter. The stability of two hybrid systems within a negative feedback interconnection is also considered. A hybrid passivity and finite gain stability theorem is developed using both Lyapunov and input–output techniques, which yield equivalent results. Sufficient conditions for the closed-loop system to be stable are presented, which resemble an amalgamation of the traditional passivity and small-gain theorems.

1 Introduction

The passivity theorem and the small gain theorem are two of the most influential stability results in feedback control theory [1, 2]. Briefly, the passivity theorem states that the negative feedback interconnection of two systems is stable if one system is passive while the other is very strictly passive. The small gain theorem states that the negative feedback interconnection of two systems is stable if the product of the system gains is strictly less than one. These theorems are extremely general in that the systems within the feedback loop may be time-varying and non-linear.

When closed-loop performance objectives (e.g. tracking, disturbance rejection etc.) are of great importance high-gain feedback compensation usually outperforms gain limited feedback compensation. The very nature of the small gain theorem limits the controller gain; when the plant gain is large the controller gain must be small, which can lead to poor closed-loop performance. The passivity theorem is much less restrictive compared to the small gain theorem insofar as the plant is permitted to have infinite gain (i.e. no damping or natural energy dissipation), while the control must have finite gain. Although the control gain must be finite it may be extremely large, usually leading to an excellent closed-loop performance. It is not surprising then that the passivity theorem has been successfully implemented in a multitude of linear and non-linear control applications such as network analysis and synthesis (i.e. electric circuits and power systems), robotics and aerospace applications [3, 4].

There are many plants which nominally possess a passive input–output map, for example, resistor-inductor-capacitor (RLC) circuits, robotic manipulators (which are in general non-linear in the multi-link context) and flexible structures. Although these plants are nominally passive, their passive input–output maps can be destroyed by additional dynamics and nonlinearities which are usually ignored; for example actuator dynamics, sensor dynamics or computational delays associated with digital implementation of the control may destroy passivity. These neglected dynamics represent ‘passivity violations’. Without a passive map, the traditional passivity theorem can no longer be used. It is undesirable to resort to the small gain theorem to guarantee stability because, as previously mentioned, the plant usually has very large gain (which in most mechanical applications is a result of little structural damping) which in turn requires the control to have very small gain, potentially resulting in poor closed-loop performance.

Considering linear, single-input single-output systems for a moment (so that we may speak in terms of gain and phase in the frequency domain), the vast majority of passivity violations, which are manifested as phase lead or lag outside of $\pm 90^\circ$, occur at very high frequencies. Most mechanical systems such as flexible space structures possess close to infinite gain at specific frequencies (i.e. the natural frequencies of the system) but have gain that rolls-off at high frequency. Similarly, controllers are almost always designed to roll-off in order to avoid the amplification of high-frequency signal noise. In the interest of recovering some of the high-gain compensation permitted via the traditional passivity theorem, it would be advantageous to be able to exploit the fact that the passivity violations experienced by a plant may only destroy the passive map intermittently, at high frequencies, where the gain of the plant and control has started to roll-off anyway. One might ask why not exploit the passive characteristics of a system at low frequency and the roll-off characteristics (that is the finite gain characteristics) of a system at high frequency where passivity is violated while designing a controller?

Blending both passive and finite gain characteristics to guarantee stability seems somewhat intuitive, yet few authors have investigated such an approach. In [5], stability of a linear time-invariant (LTI) plant and an LTI controller using both the passive (very strictly passive) and finite gain characteristics was considered. The plant must be strictly proper, positive real and the controller must be stable, positive real in specific frequency bands for the closed-loop system to be stable.

More recently, Griggs *et al.* [6, 7] have introduced a mixed small gain and passivity theorem for linear and non-linear systems. The mixed systems are divided into three operating ranges: when the plant and controller both possess gain strictly less than one independently, a mixture of gain strictly less than one and a form of passivity, and just a form of passivity. Although these works are a great step forward in an effort to exploit passive and finite gain characteristics, the separation of the mixed systems into three regions is somewhat impractical, overly restrictive and, in light of this current paper unnecessary. Additionally, the gain of a plant may not be strictly less than one at the point which a passive map is destroyed, which is necessary for the mixed theorem to be applied. For these reasons, we are inspired to extend, generalise and improve upon [6, 7] so that we may guarantee closed-loop stability of a larger class of systems.

In this paper, our motivation is the stability of systems which possess passivity violations. The finite gain characteristics of the plant will be exploited when passivity is violated. We will extend and generalise further the works of [6, 7]. We will present both Lyapunov and input–output stability proofs for general (linear or non-linear) systems within a negative feedback interconnection. Additionally, we will explore how to physically model the divide between passive input–output mappings and finite gain input–output mappings by using ideal low- and high-pass filters. Although we are inspired by [6, 7], we will differentiate our work by calling our systems ‘hybrid systems’ and our stability theorem a ‘hybrid passivity and finite gain stability theorem’.

2 Dissipative systems framework

The mathematical tool we will employ to prove stability is the dissipative systems framework presented in [8–10]. To start, the L_2-space is the space of square integrable functions defined by $L_2 = \{\mathbf{v}:\mathbb{R}^+ \to \mathbb{R}^m \mid \int_0^\infty \mathbf{v}^\mathsf{T}(t)\mathbf{v}(t)\,\mathrm{d}t < \infty\}$ where $\mathbf{v}$ is an arbitrary vector function of time and $\mathbf{v}^\mathsf{T}$ is its transpose. The L_2-space is a Hilbert space, where the inner product defines the norm

$$\langle \mathbf{w}, \mathbf{v}\rangle = \int_0^\infty \mathbf{w}^\top(t)\mathbf{v}(t)\,\mathrm{d}t \quad \|\mathbf{v}\|_2 = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

where $\mathbf{v} \in L_2$, $\mathbf{w} \in L_2$ and $\langle\cdot, \cdot\rangle$ is the inner product. The truncation of a vector function can be defined via

$$\mathbf{v}_T(t) = \begin{cases} \mathbf{v}(t), & 0 \le t \le T \\ \mathbf{0}, & t > T \end{cases}$$

The extended L_2-space is defined as $L_{2e} = \{\mathbf{v}:\mathbb{R}^+ \to \mathbb{R}^m \mid \mathbf{v}_T \in L_2, 0 \le T < \infty\}$ where $L_2 \subset L_{2e}$. The truncated L_2-norm is simply the L_{2T}-norm

$$\|\mathbf{v}\|_{2T} = \sqrt{\int_0^T \mathbf{v}^\mathsf{T}(t)\mathbf{v}(t)\,\mathrm{d}t}$$

The truncated inner product of two arbitrary signals is

$$\langle \mathbf{w}, \mathbf{v}\rangle_T = \int_0^T \mathbf{w}^\mathsf{T}(t)\mathbf{v}(t)\,\mathrm{d}t = \langle \mathbf{w}_T, \mathbf{v}_T\rangle$$

Given an operator $\boldsymbol{G}$, the operator adjoint can be defined using an inner product

$$\langle \mathbf{w}, \boldsymbol{G}\mathbf{v}\rangle = \langle \boldsymbol{G}^\sim \mathbf{w}, \mathbf{v}\rangle$$

where $\boldsymbol{G}^\sim$ is the operator adjoint.

If we assume that we have an input $\mathbf{w} \in L_2$ and an output $\mathbf{v} \in L_2$, the induced L_2-gain, or simply the system gain is $\gamma = \sup_{\mathbf{0} \neq \mathbf{w} \in L_2} (\|\mathbf{v}\|_2/\|\mathbf{w}\|_2)$. In the linear case the induced

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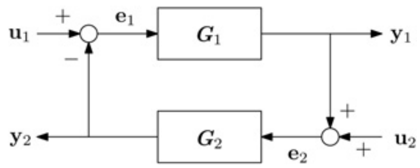

Figure 1 *General negative feedback interconnection of systems G_1 and G_2*

L_2-gain of a system is equivalent to the $\mathcal{H}_\infty$-norm of the system transfer matrix.

Consider the negative feedback interconnection of the linear or non-linear systems $G_1:L_{2e} \to L_{2e}$ and $G_2:L_{2e} \to L_{2e}$ presented in Fig. 1. We will assume each system has a state-space representation of the form

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \boldsymbol{f}_i(\mathbf{x}_i(t), \mathbf{e}_i(t)), \quad \mathbf{x}_i \in \mathbb{R}^n, \quad \mathbf{e}_i \in \mathbb{R}^m \\ \mathbf{y}_i(t) &= \boldsymbol{g}_i(\mathbf{x}_i(t), \mathbf{e}_i(t)), \quad \mathbf{y}_i \in \mathbb{R}^m \end{aligned} \tag{1}$$

where $\boldsymbol{f}_i:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\boldsymbol{g}_i:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. The admissible inputs $\mathbf{e}_1(t) = \mathbf{u}_1(t) - \mathbf{y}_2(t)$ and $\mathbf{e}_2(t) = \mathbf{u}_2(t) + \mathbf{y}_1(t)$ are assumed to be contained in L_{2e}. The system $\boldsymbol{G}_i$ is represented by (1) when $i = 1$ and the system $\boldsymbol{G}_2$ is represented by (1) when $i = 2$. Each system is:

1. 'zero-state-observable', that is $\mathbf{e}_i(t) = \mathbf{0}$ and $\mathbf{y}_i(t) = \mathbf{0}$ implies that $\mathbf{x}_i(t) = \mathbf{0}$;

2. completely reachable from the origin, that is for any t_1 and $\mathbf{x}_i(t_1)$, there exists a $t_0 \le t_1$ and an input $\mathbf{e}_i \in L_{2e}$ such that the states can move from $\mathbf{x}_i(t_0) = \mathbf{0}$ to $\mathbf{x}_i(t_1)$ in the finite time interval $t \in [t_0, t_1]$.

For linear systems, these two requirements are equivalent to the systems being observable and controllable.

Consider a supply rate $w_i(\mathbf{e}_i, \mathbf{y}_i):L_{2e} \times L_{2e} \to L_{2e}$ and an associated energy-like function called a storage function, $\phi_i:\mathbb{R}^n \to \mathbb{R}^+$. Each system described by (1) is dissipative with respect to a supply rate defined by

$$\begin{aligned}\int_0^T w_i(\mathbf{e}_i(t), \mathbf{y}_i(t))\,\mathrm{d}t &= \langle \mathbf{y}_i, \boldsymbol{Q}_i\mathbf{y}_i\rangle_T + 2\langle \mathbf{y}_i, \boldsymbol{S}_i\mathbf{e}_i\rangle_T + \langle \mathbf{e}_i, \boldsymbol{R}_i\mathbf{e}_i\rangle_T \\ &\ge \phi_i(\mathbf{x}_i(T)) - \phi_i(\mathbf{x}_i(0))\end{aligned} \tag{2}$$

$\forall T \ge 0$ where $w_i(\mathbf{e}_i(t), \mathbf{y}_i(t))$ is evaluated along the trajectories of (1) given $\mathbf{e}_i \in L_{2e}$, and $\boldsymbol{Q}_i$, $\boldsymbol{S}_i$ and $\boldsymbol{R}_i$ are bounded linear operators. We say that the system is $(\boldsymbol{Q}_i, \boldsymbol{S}_i, \boldsymbol{R}_i)$-dissipative. The supply rate can be thought of as the power input to the system. Storage functions represent the energy stored by the system at some time t and in some cases can be thought of as Lyapunov functions. The integral of a supply rate represents the energy supplied to the system during some time $t \in [0, T]$ from an external source. The initial energy of the system is $\phi_i(0) = \phi_i(\mathbf{x}_i(0))$.

In the traditional dissipative systems theory $\boldsymbol{Q}_i$, $\boldsymbol{S}_i$ and $\boldsymbol{R}_i$ are appropriately dimensioned constant matrices. Using the dissipative systems framework, passive, input strictly passive, output strictly passive, very strictly passive and finite gain systems can be defined.

Definition 1: A general square system with inputs $\mathbf{e} \in L_{2e}$ and outputs $\mathbf{y} \in L_{2e}$ mapped through the operator $\boldsymbol{G}:L_{2e} \to L_{2e}$ is 'very strictly passive' if there exists constants $\delta > 0$ and $\epsilon > 0$ such that

$$\begin{aligned}&\int_0^T \mathbf{e}^{\mathrm{T}}(t)\mathbf{y}(t)\,\mathrm{d}t \ge \delta\int_0^T \mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t)\,\mathrm{d}t + \epsilon\int_0^T \mathbf{y}^{\mathrm{T}}(t)\mathbf{y}(t)\,\mathrm{d}t, \\ &\quad \forall \mathbf{e} \in L_{2e}, \ \forall T \ge 0\end{aligned} \tag{3}$$

In addition to being very strictly passive, the system is said to be $(-\epsilon\mathbf{1}, (1/2)\mathbf{1}, -\delta\mathbf{1})$-dissipative.

An 'output strictly passive' system is one that satisfies (3) with $\delta = 0$, $\epsilon > 0$, and it is said to be $(-\epsilon\mathbf{1}, (1/2)\mathbf{1}, \mathbf{0})$-dissipative.

An 'input strictly passive' system is one that satisfies (3) with $\delta > 0$, $\epsilon = 0$, and it is said to be $(\mathbf{0}, (1/2)\mathbf{1}, -\delta\mathbf{1})$-dissipative.

A passive system is one that satisfies (3) with $\delta = 0$ and $\epsilon = 0$, and it is said to be $(\mathbf{0}, (1/2)\mathbf{1}, \mathbf{0})$-dissipative.

A system that is very strictly passive can also be called 'input–output strictly passive' or 'input strictly passive with finite gain'. The units of δ are gain and the units of ϵ are one over gain.

The passivity theorem [11–13] states that the negative feedback interconnection of the systems $\boldsymbol{G}_1:L_{2e} \to L_{2e}$ and $\boldsymbol{G}_2:L_{2e} \to L_{2e}$ is L_2-stable if $\epsilon_1 + \delta_2 > 0$ and $\epsilon_2 + \delta_1 > 0$ where δ_i and ϵ_i are the input and output strictly passive parameter of the system $\boldsymbol{G}_i$.

Definition 2: A general square system with inputs $\mathbf{e} \in L_{2e}$ and outputs $\mathbf{y} \in L_{2e}$ mapped though the operator $\boldsymbol{G}:L_{2e} \to L_{2e}$ possesses 'finite gain' if there exists $\gamma > 0$ such that

$$\begin{aligned}&\gamma^{-1}\int_0^T \mathbf{y}^{\mathrm{T}}(t)\mathbf{y}(t)\mathrm{d}t \le \gamma\int_0^T \mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t)\,\mathrm{d}t, \\ &\quad \forall \mathbf{e} \in L_{2e}, \quad \forall T \ge 0\end{aligned} \tag{4}$$

In addition to having finite gain, the system is said to be $(-\gamma^{-1}\mathbf{1}, \mathbf{0}, \gamma\mathbf{1})$-dissipative. The units of γ are gain. It is common for (4) to be written as $\|\mathbf{y}\|_{2T}^2 \le \gamma^2\|\mathbf{e}\|_{2T}^2$, but we are deliberately writing it in the above form.

The small gain theorem [11–13] states that the negative feedback interconnection of the systems $\boldsymbol{G}_1:L_{2e} \to L_{2e}$ and

$\boldsymbol{G}_2{:}L_{2e} \rightarrow L_{2e}$ is L_2-stable if the product of the system gains is less than one, $\gamma_1\gamma_2 < 1$, where γ_i is the gain of system $\boldsymbol{G}_i$.

3 Characteristics of hybrid passive and finite gain systems

3.1 Operators Q, S and R in the time and frequency domains

We will now proceed to define what a hybrid passive and finite gain system is. First, assume that we are dealing with a general non-linear system as presented in (1). We assume the system is dissipative with respect to a supply rate as in (2). If we express (2) in terms of integral expressions then we have

$$\int_0^\infty \mathbf{y}_T^\mathsf{T}(t)\boldsymbol{Q}\mathbf{y}_T(t)\,\mathrm{d}t + 2\int_0^\infty \mathbf{y}_T^\mathsf{T}(t)\boldsymbol{S}\mathbf{e}_T(t)\,\mathrm{d}t + \int_0^\infty \mathbf{e}_T^\mathsf{T}(t)\boldsymbol{R}\mathbf{e}_T(t)\,\mathrm{d}t \geq -\phi(0) \tag{5}$$

where we have dropped the subscript 'i' on $\mathbf{y}$, $\mathbf{e}$ and the system operators for convenience and moved the truncation from the integration limits to the signals themselves. Assuming that the operators $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$ are LTI, via Parseval's theorem (5) can be expressed in the frequency domain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^\mathsf{H}(j\omega)\mathbf{Q}(\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega + \frac{1}{\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} \mathbf{y}_T^\mathsf{H}(j\omega)\mathbf{S}(\omega)\mathbf{e}_T(j\omega)\mathrm{d}\omega\right\} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{e}_T^\mathsf{H}(j\omega)\mathbf{R}(\omega)\mathbf{e}_T(j\omega)\mathrm{d}\omega \geq -\phi(0) \tag{6}$$

where $\mathbf{e}_T^\mathsf{H}(j\omega) = \mathbf{e}_T^\mathsf{T}(-j\omega)$, $\mathbf{y}_T^\mathsf{H}(j\omega) = \mathbf{y}_T^\mathsf{T}(-j\omega)$ is the Hermitian or complex conjugate transpose of the Fourier transformed signals.

The Fourier transforms of the impulse responses of the operators $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$ are defined as

$$\mathbf{Q}(\omega) = -[\epsilon\alpha(\omega) + \gamma^{-1}(1-\alpha(\omega))]\mathbf{1} \tag{7a}$$

$$\mathbf{S}(\omega) = \frac{1}{2}\alpha(\omega)\mathbf{1} \tag{7b}$$

$$\mathbf{R}(\omega) = [\gamma(1-\alpha(\omega)) - \delta\alpha(\omega)]\mathbf{1} \tag{7c}$$

Notice the dimensional consistency of $\mathbf{Q}(\cdot)$ and $\mathbf{R}(\cdot)$: ϵ and γ^{-1} have the same units (one over gain), and δ and γ have the same units (gain). The variable $\alpha{:}\mathbb{R} \rightarrow \{0, 1\}$ dictates the switch between passive characteristics and finite gain characteristics. It can only take on two discrete values: zero or one. Mathematically, we will define α as

$$\begin{aligned}\alpha(\omega) &= \begin{cases} 1, & -\omega_\mathrm{c} < \omega < \omega_\mathrm{c} \\ 0, & |\omega| \geq \omega_\mathrm{c} \end{cases} \\ &= A(-j\omega)A(j\omega) \\ &= |A(j\omega)|^2 \end{aligned} \tag{8}$$

where ω_c is the critical frequency. When the system in question possesses passive characteristics $\alpha(\omega) = 1$, and when the system possess finite gain characteristics $\alpha(\omega) = 0$. The transfer function $A(s) \in \mathcal{H}_\infty$ and $A(s)A(-s)$ is the spectral factorisation of the Laplace transform of the inverse Fourier transform of $\alpha(\omega)$. The time domain equivalent of $A(s)$ is the causal convolution operator $\mathcal{A}$, where $\mathcal{A}{:}L_2 \rightarrow L_2$. We also define $\boldsymbol{\mathcal{A}} = \mathcal{A}\mathbf{1}$ and $\mathbf{A}(s) = A(s)\mathbf{1}$.

It is important to realise how α relates to an inner product when traversing from the frequency domain to the time domain via Parseval's theorem

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^\mathsf{H}(j\omega)(\alpha(\omega)\mathbf{1})\mathbf{y}_T(j\omega)\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^\mathsf{H}(j\omega)\mathbf{A}^\mathsf{T}(-j\omega)\,\mathbf{A}(j\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega \\ &= \int_0^\infty \mathbf{y}_T^\mathsf{T}(t)\boldsymbol{\mathcal{A}}^\sim\boldsymbol{\mathcal{A}}\mathbf{y}_T(t)\,\mathrm{d}t \\ &= \int_0^\infty (\boldsymbol{\mathcal{A}}\mathbf{y}_T(t))^\top(\boldsymbol{\mathcal{A}}\mathbf{y}_T(t))\,\mathrm{d}t \\ &= \langle \boldsymbol{\mathcal{A}}\mathbf{y}_T, \boldsymbol{\mathcal{A}}\mathbf{y}_T\rangle \end{aligned}$$

It will be advantageous to define the equality

$$\begin{aligned}\mathcal{B}^\sim\mathcal{B} + \mathcal{A}^\sim\mathcal{A} = 1 \Leftrightarrow B(-s)B(s) + A(-s)A(s) \\ = 1 \Leftrightarrow \beta(\omega) + \alpha(\omega) = 1\end{aligned} \tag{9}$$

where we have defined $\beta{:}\mathbb{R} \rightarrow \{0, 1\}$, $\beta(\omega) = 1 - \alpha(\omega)$, and $\mathcal{A}^\sim$ and $\mathcal{B}^\sim$ are the adjoint of $\mathcal{A}$ and $\mathcal{B}$, respectively. Using $\mathcal{A}$ and $\mathcal{B}$ the operators $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$ can be written as

$$\boldsymbol{Q} = -[\epsilon\mathcal{A}^\sim\mathcal{A} + \gamma^{-1}\mathcal{B}^\sim\mathcal{B}]\mathbf{1} \tag{10a}$$

$$\boldsymbol{S} = \frac{1}{2}\mathcal{A}^\sim\mathcal{A}\mathbf{1} \tag{10b}$$

$$\boldsymbol{R} = [\gamma\mathcal{B}^\sim\mathcal{B} - \delta\mathcal{A}^\sim\mathcal{A}]\mathbf{1} \tag{10c}$$

Returning to (5) and substituting the expressions for $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$ presented in (10) we have

$$\begin{aligned}&\langle \mathbf{y}_T, (-\epsilon\boldsymbol{\mathcal{A}}^\sim\boldsymbol{\mathcal{A}} - \gamma^{-1}\boldsymbol{\mathcal{B}}^\sim\boldsymbol{\mathcal{B}})\mathbf{y}_T\rangle \\ &\quad + \langle \mathbf{y}_T, \boldsymbol{\mathcal{A}}^\sim\boldsymbol{\mathcal{A}}\mathbf{e}_T\rangle + \langle \mathbf{e}_T, (\gamma\boldsymbol{\mathcal{B}}^\sim\boldsymbol{\mathcal{B}} - \delta\boldsymbol{\mathcal{A}}^\sim\boldsymbol{\mathcal{A}})\mathbf{e}_T\rangle \geq -\phi(0) \\ &\quad \Leftrightarrow \langle \boldsymbol{\mathcal{A}}\mathbf{y}_T, \boldsymbol{\mathcal{A}}\mathbf{e}_T\rangle - \epsilon\langle \boldsymbol{\mathcal{A}}\mathbf{y}_T, \boldsymbol{\mathcal{A}}\mathbf{y}_T\rangle - \delta\langle \boldsymbol{\mathcal{A}}\mathbf{e}_T, \boldsymbol{\mathcal{A}}\mathbf{e}_T\rangle \\ &\quad + \gamma\langle \boldsymbol{\mathcal{B}}\mathbf{e}_T, \boldsymbol{\mathcal{B}}\mathbf{e}_T\rangle - \gamma^{-1}\langle \boldsymbol{\mathcal{B}}\mathbf{y}_T, \boldsymbol{\mathcal{B}}\mathbf{y}_T\rangle \geq -\phi(0)\end{aligned} \tag{11}$$

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The inequality presented in (11) is representative of a hybrid system. The inner products containing $\mathcal{A}$ are associated with the passive part of the hybrid system, and the inner products containing $\mathcal{B}$ are associated with the finite gain part of the hybrid system.

It is important to realise that the parameters γ, ϵ and δ for a given system are defined by the dissipative inequality presented in (11). The stability proofs presented in Sections 4 and 5 combined with Corollary 1 presented in Section 6 will prescribe constraints on the parameters such that the negative feedback interconnection of two hybrid systems is stable.

3.2 Approximation of α

The parameter α divides system mapping characteristics into two regions: when a system possesses a passive input–output map (regardless of gain characteristics) and when a system does not possess a passive input–output map but possess finite gain. In reality, α is simply a mathematical abstraction but it is one that helps us distile and understand various relations between the system(s) in question. It will be advantageous to try and understand exactly what α represents, and if an approximation of α provides us with additional insight.

Previously in (8) we defined α to be one or zero as a function of frequency. This is essentially how an ideal low-pass filter is defined, as presented in Fig. 2. In reality an ideal low-pass filter can only be approximated, yet the hybrid system inequality (which is a dissipative inequality) presented in (11) is defined using an exact low-pass filter (and high-pass filter). However, as we will show, with the right approximation we can get arbitrarily close to the exact inequality representing a hybrid system.

Consider the construction of an Nth order low-pass Butterworth filter, which is an approximation of an ideal low-pass filter [14]

$$\alpha_N(\omega) = A_N(-j\omega)A_N(j\omega) = \frac{1}{1 + (\omega/\omega_c)^{2N}}$$

where

$$\lim_{N\to\infty} \alpha_N(\omega) = \alpha(\omega)$$

For example, the magnitude responses of $N = 10$ and $N = 30$ low-pass Butterworth filters are shown in Figs. 3*a* and *b*. Knowing that $\alpha_N(\omega) = A_N(-j\omega)A_N(j\omega)$, $A_N(-s)A_N(s)$ is

$$A_N(-s)A_N(s) = \frac{1}{1 + (-s^2/\omega_c^2)^N}$$

It follows that the causal transfer function representing an

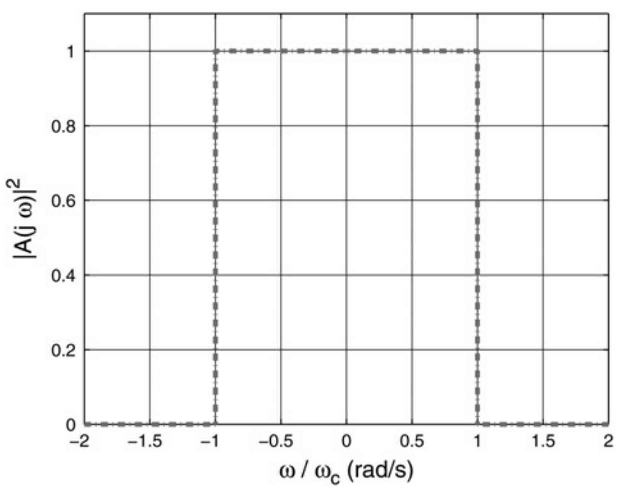

Figure 2 *Frequency response of an ideal low-pass filter*

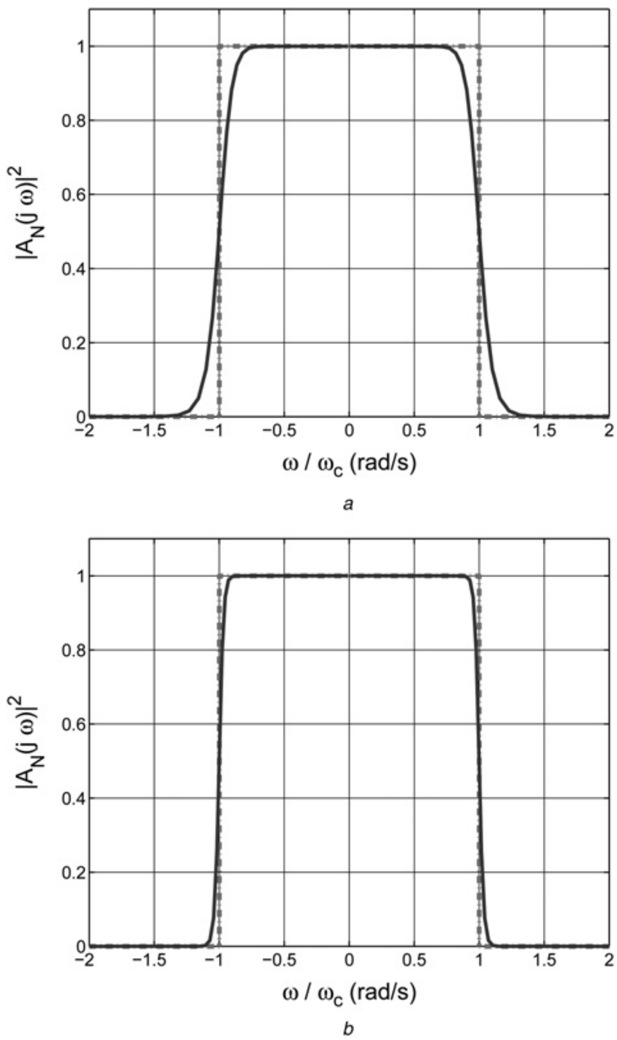

Figure 3 *Low-pass Butterworth filter responses*

a $N = 10$
b $N = 30$

Nth order low-pass Butterworth filter can be represented by

$$A_N(s) = \frac{1}{\prod_{k=1}^{N}((s - s_k)/\omega_c)}$$

where $s_k = \omega_c e^{(j(2k+N-1)\pi/2N)}$, $|s_k/\omega_c| = 1$, $\mathrm{Re}\{s_k\} < 0$ and $k = 1, \ldots, N$. Note that $A_N(s) \in \mathcal{H}_\infty$, and similarly $A_N(-s)$ is analytic in $\mathrm{Re}\{s\} < 0$ and therefore $A_N(-s) \in \mathcal{H}_\infty^\perp$. The operator $\mathcal{A}_N$ represents the action of an Nth order low-pass Butterworth filter in the time domain.

In a similar fashion, we can approximate β as $\beta_N(\omega) = B_N(-j\omega)B_N(j\omega)$. In order to construct $B_N(s)$, let $B_N(s/\omega_c) = A_N(\omega_c/s)$. We can now write

$$B_N\left(-\frac{s}{\omega_c}\right)B_N\left(\frac{s}{\omega_c}\right) = A_N\left(-\frac{\omega_c}{s}\right)A_N\left(\frac{\omega_c}{s}\right)$$

Letting $s = j\omega$ we have

$$\begin{aligned} B_N\left(-\frac{j\omega}{\omega_c}\right)B_N\left(\frac{j\omega}{\omega_c}\right) &= A_N\left(-\frac{\omega_c}{j\omega}\right)A_N\left(\frac{\omega_c}{j\omega}\right) \\ &= A_N\left(\frac{j\omega_c}{\omega}\right)A_N\left(-\frac{j\omega_c}{\omega}\right) \\ &= \frac{1}{1 + (\omega_c/\omega)^{2N}} = \frac{\omega^{2N}}{\omega^{2N} + \omega_c^{2N}} \end{aligned}$$

Therefore

$$B_N(-s)B_N(s) = \frac{1}{1 + (\omega_c^2/-s^2)^N}$$

Having defined both $A_N(-s)A_N(s)$ and $B_N(-s)B_N(s)$, we are able to show that the approximate form of the equality presented in (9) also holds

$$\begin{aligned} B_N(-s)B_N(s) + A_N(-s)A_N(s) = B_N(-j\omega)B_N(j\omega) \\ + A_N(-j\omega)A_N(j\omega) = 1 \end{aligned}$$

It follows that $\mathcal{B}_N^{\sim}\mathcal{B}_N + \mathcal{A}_N^{\sim}\mathcal{A}_N = 1$, and $\mathcal{B}_N$ is the time domain equivalent of an Nth order high-pass Butterworth filter.

Based on the construction of $\mathcal{A}_N$ and $\mathcal{B}_N$ we can approximate (11) as

$$\begin{aligned} &\langle \boldsymbol{\mathcal{A}}_N \mathbf{y}_T, \boldsymbol{\mathcal{A}}_N \mathbf{e}_T\rangle - \epsilon\langle \boldsymbol{\mathcal{A}}_N \mathbf{y}_T, \boldsymbol{\mathcal{A}}_N \mathbf{y}_T\rangle - \delta\langle \boldsymbol{\mathcal{A}}_N \mathbf{e}_T, \boldsymbol{\mathcal{A}}_N \mathbf{e}_T\rangle \\ &\quad + \gamma\langle \boldsymbol{\mathcal{B}}_N \mathbf{e}_T, \boldsymbol{\mathcal{B}}_N \mathbf{e}_T\rangle - \gamma^{-1}\langle \boldsymbol{\mathcal{B}}_N \mathbf{y}_T, \boldsymbol{\mathcal{B}}_N \mathbf{y}_T\rangle \geq -\phi(0) \end{aligned}$$

Any inner products containing the operator $\boldsymbol{\mathcal{A}}_N$ are associated with the passive nature of the system, and inner products containing the operator $\boldsymbol{\mathcal{B}}_N$ are associated with the finite gain nature of the system when passivity is violated. The LTI operators $\boldsymbol{\mathcal{A}}_N$ and $\boldsymbol{\mathcal{B}}_N$ filter the signals within the inner products. The inner products containing $\boldsymbol{\mathcal{A}}_N$ have their signals $\mathbf{e}_T$ and $\mathbf{y}_T$ filtered by an LTI low-pass filter, and the inner products containing $\boldsymbol{\mathcal{B}}_N$ have their signals filtered by an LTI high-pass filter. It is clear that as $N \to \infty$ we can exactly represent the hybrid system inequality presented in (11).

Remark 1: A hybrid system is distinctly different from a mixed system as presented in [6, 7] in two ways. First, three regions exist for which a mixed system: (i) has gain less than one, (ii) has gain less than one and a form of passivity (i.e. either passive, input strictly passive, output strictly passive or very strictly passive properties) and (iii) possesses only passive properties. We strive to break a system into two parts: when a system (a) possesses passive properties and (b) possesses finite gain properties, $\gamma < \infty$. As previously mentioned, our motivation to do so is a practical one: passivity violations. Given a system that is approaching a passivity violation, before the violation the system is passive in some sense and after the violation the system is not. Therefore region (ii) of a mixed system does not capture what is truly happening when a system loses its passive characteristics. It is quite possible that just before passivity is violated the gain of the system is less than one, but a stronger statement is the system is input strictly passive (i.e. passive with finite gain) before losing its passive characteristics.

Second, in some cases after passivity has been violated the gain may satisfy $\gamma < 1$, and the system would subscribe to the class of systems known as mixed systems as defined in [6, 7]. However, what if $1 \leq \gamma < \infty$ upon loss of passivity? Such a system would not be classified as mixed, although it would have finite attenuation when passivity was lost. Our systems' definition and the stability proofs presented in Sections 4 and 5 are much more general in that they permit $\gamma < \infty$ when passivity is violated; hence, we call such systems hybrid systems for they are truly hybrid in that they possess passive properties and finite gain properties.

As a final note, the frequency dependent matrices $\mathbf{Q}(\cdot)$, $\mathbf{S}(\cdot)$ and $\mathbf{R}(\cdot)$ defined in (7) are dimensionally consistent regardless of the value of α, unlike those defined in [6].

4 Lyapunov stability proof

Using a Lyapunov method we will determine criteria for the negative feedback interconnection of two hybrid systems to be stable. As is customary with Lyapunov proofs, we will assume no inputs, $\mathbf{u}_1(t) = \mathbf{u}_2(t) = \mathbf{0}$.

Before stating the main proof, we will review stability in the sense of Lyapunov [10, 15].

Lemma 1: A dynamic system described by (1) is globally asymptotically stable if a function $V:\mathbb{R}^n \to \mathbb{R}^+$ exists such that the following conditions are satisfied:

1. $V(\mathbf{0}) = 0$

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2. There exists a continuous function $a:\mathbb{R}^+ \to \mathbb{R}^+$ such that

(a) $a(0) = 0$

(b) $a(\|\mathbf{x}\|) \geq 0 \quad \forall \mathbf{x} \neq \mathbf{0}$ (we have dropped the subscript 'i' on $\mathbf{x}$ for simplicity)

(c) $a(\|\mathbf{x}\|) \to \infty$ as $\|\mathbf{x}\| \to \infty$ where $V(\mathbf{x}) \geq a(\|\mathbf{x}\|)$ (i.e. V is positive definite and radially unbounded).

3. When $\mathbf{u}(t) = \mathbf{0}$ and given a set of initial conditions, $\mathbf{x}(t_0) = \mathbf{x}_0$, the state trajectories of the system satisfy $V(\mathbf{x}(t_0)) \geq V(\mathbf{x}(t))$, $\forall t \geq t_0$ and $V(\mathbf{x}(t_0)) > V(\mathbf{x}(t_1))$, $\forall t_1 > t_0$ unless $\mathbf{x}_0 = \mathbf{0}$.

Theorem 1 (hybrid passivity and finite gain stability theorem – Lyapunov approach): Referring to Fig. 1, let $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$. The systems $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ could be linear or non-linear systems, but for generality we will assume that they are non-linear systems with a state-space form equivalent to that of (1). Each system is $(\boldsymbol{Q}_i, \boldsymbol{S}_i, \boldsymbol{R}_i)$-dissipative with respect to a supply rate w_i, and each has a corresponding storage function, ϕ_i, as described by (2). If (10) defines the operators $\boldsymbol{Q}_i$, $\boldsymbol{S}_i$ and $\boldsymbol{R}_i$ then a sufficient condition for the negative feedback interconnection to be globally asymptotically stable is

$$\left\langle \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \overbrace{\begin{bmatrix} -\boldsymbol{Q}_1 - \boldsymbol{R}_2 & \boldsymbol{S}_1 - \boldsymbol{S}_2^{\sim} \\ \boldsymbol{S}_1^{\sim} - \boldsymbol{S}_2 & -\boldsymbol{R}_1 - \boldsymbol{Q}_2 \end{bmatrix}}^{\mathbf{Q}} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \right\rangle_T > 0$$

Proof: Each system has a storage function, ϕ_i, described by (2). The storage functions can each be thought of as Lyapunov functions (i.e. $\phi_i \equiv V_i$) independently. We will choose our total system Laypunov function to be the sum of each storage function

$$V(\mathbf{x}(t)) = V_1(\mathbf{x}_1(t)) + V_2(\mathbf{x}_2(t))$$

where $\mathbf{x}^{\mathrm{T}}(t) = [\mathbf{x}_1^{\mathrm{T}}(t)\, \mathbf{x}_2^{\mathrm{T}}(t)]$. Adding together the storage function expressions for each system gives

$$\begin{aligned}
&\left(V_1(\mathbf{x}_1(0)) + \int_0^T \mathbf{y}_1^{\mathrm{T}}(t)\boldsymbol{Q}_1\mathbf{y}_1(t)\mathrm{d}t \right. \\
&\quad \left. + 2\int_0^T \mathbf{y}_1^{\mathrm{T}}(t)\boldsymbol{S}_1\mathbf{e}_1(t)\mathrm{d}t + \int_0^T \mathbf{e}_1^{\mathrm{T}}(t)\boldsymbol{R}_1\mathbf{e}_1(t)\mathrm{d}t \right) \\
&\quad + \left(V_2(\mathbf{x}_2(0)) + \int_0^T \mathbf{y}_2^{\mathrm{T}}(t)\boldsymbol{Q}_2\mathbf{y}_2(t)\mathrm{d}t \right. \\
&\quad \left. + 2\int_0^T \mathbf{y}_2^{\mathrm{T}}(t)\boldsymbol{S}_2\mathbf{e}_2(t)\mathrm{d}t + \int_0^T \mathbf{e}_2^{\mathrm{T}}(t)\boldsymbol{R}_2\mathbf{e}_2(t)\mathrm{d}t \right) \\
&\quad > V_1(\mathbf{x}_1(T)) + V_2(\mathbf{x}_2(T))
\end{aligned}$$

$\forall T > 0$. Note that the inequality is strict because the limits of integration are explicitly defined such that they are not equal to each other. Also, $\mathbf{u}_1(t) = \mathbf{u}_2(t) = \mathbf{0}$ which implies $\mathbf{e}_1(t) = -\mathbf{y}_2(t)$ and $\mathbf{e}_2(t) = \mathbf{y}_1(t)$. Making the appropriate substitutions for $\mathbf{e}_1$ and $\mathbf{e}_2$ it follows that

$$\begin{aligned}
&V_1(\mathbf{x}_1(0)) + (\langle \mathbf{y}_1, \boldsymbol{Q}_1\mathbf{y}_1\rangle_T + 2\langle \mathbf{y}_1, \boldsymbol{S}_1\mathbf{e}_1\rangle_T + \langle \mathbf{e}_1, \boldsymbol{R}_1\mathbf{e}_1\rangle_T) \\
&\quad + V_2(\mathbf{x}_2(0)) + (\langle \mathbf{y}_2, \boldsymbol{Q}_2\mathbf{y}_2\rangle_T + 2\langle \mathbf{y}_2, \boldsymbol{S}_2\mathbf{e}_2\rangle_T \\
&\quad + \langle \mathbf{e}_2, \boldsymbol{R}_2\mathbf{e}_2\rangle_T) > V_1(\mathbf{x}_1(T)) + V_2(\mathbf{x}_2(T)) \\
&\quad \Leftrightarrow \underbrace{V_1(\mathbf{x}_1(0)) + V_2(\mathbf{x}_2(0))}_{V(\mathbf{x}(0))} \\
&\quad + \left\langle \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \overbrace{\begin{bmatrix} \boldsymbol{Q}_1 + \boldsymbol{R}_2 & -\boldsymbol{S}_1 + \boldsymbol{S}_2^{\sim} \\ -\boldsymbol{S}_1^{\sim} + \boldsymbol{S}_2 & \boldsymbol{R}_1 + \boldsymbol{Q}_2 \end{bmatrix}}^{-\boldsymbol{Q}} \overbrace{\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}}^{\mathbf{y}(t)} \right\rangle_T \\
&\quad > \underbrace{V_1(\mathbf{x}_1(T)) + V_2(\mathbf{x}_2(T))}_{V(\mathbf{x}(T))} \\
&\quad \Leftrightarrow V(\mathbf{x}(0)) > V(\mathbf{x}(T)) + \langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T
\end{aligned}$$

To satisfy the third condition of Lemma 1, we need to show that $V(\mathbf{x}(t_0)) > V(\mathbf{x}(t_1))$ where $t_0 = 0$ and $t_1 = T$. The third condition of Lemma 1 will be satisfied if

$$\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0 \Rightarrow V(\mathbf{x}(0)) > V(\mathbf{x}(T))$$

Next, to show that condition 1 of Lemma 1 holds, consider the integral

$$V(\mathbf{x}(0)) - \int_0^t \mathbf{y}^{\mathrm{T}}(\tau)\boldsymbol{Q}\mathbf{y}(\tau)\mathrm{d}\tau \geq V(\mathbf{x}(t)) \tag{12}$$

By the zero-state observability criteria, if $\mathbf{x}(t) = \mathbf{0}$, then $\mathbf{y}(t) = \mathbf{0}$. This implies that $V(\mathbf{x}(0)) = V(\mathbf{x}(t)) = \mathbf{0}$ when $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{x}(0) = \mathbf{0}$.

We can express $\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0$ in the time domain or the frequency domain via Parseval's theorem

$$\begin{aligned}
&\int_0^\infty \mathbf{y}_T^{\mathrm{T}}(t)\boldsymbol{Q}\mathbf{y}_T(t)\mathrm{d}t > 0 \\
&\quad \Leftrightarrow \frac{1}{2\pi}\mathrm{Re}\left\{ \int_{-\infty}^{\infty} \mathbf{y}_T^{\mathrm{H}}(j\omega)\mathbf{Q}(\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega \right\} > 0 \\
&\quad \Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^{\mathrm{H}}(j\omega)\mathbf{Q}(\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega > 0
\end{aligned}$$

Traversing to the frequency domain can be justified if one notes that any signal such as an error signal, $\mathbf{e}_{i,T}(t)$, or output signal, $\mathbf{y}_{i,T}(t)$, can be expressed in terms of its Fourier transform, $\mathbf{e}_{i,T}(j\omega)$ and $\mathbf{y}_{i,T}(j\omega)$, regardless of the signals origin (such as the output of a non-linear plant in the case of $\mathbf{y}_i$). To show that condition two of Lemma 1 holds, consider the following

$$\int_0^{\infty} \mathbf{y}_T^{\mathsf{T}}(t)\mathbf{Q}\,\mathbf{y}_T(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^{\mathsf{H}}(j\omega)\mathbf{Q}(\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega$$

$$\geq \inf_{\omega\in\mathbb{R}} \underline{\lambda}(\mathbf{Q}(\omega))\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{y}_T^{\mathsf{H}}(j\omega)\mathbf{y}_T(j\omega)\mathrm{d}\omega$$

$$= \inf_{\omega\in\mathbb{R}} \underline{\lambda}(\mathbf{Q}(\omega))\int_0^{\infty} \mathbf{y}_T^{\mathsf{T}}(t)\mathbf{y}_T(t)\mathrm{d}t$$

where $\underline{\lambda}(\mathbf{Q}(\cdot))$ is the minimum eigenvalue of $\mathbf{Q}(\cdot)$. It follows that

$$V(\mathbf{x}(t)) \geq \inf_{\omega\in\mathbb{R}} \underline{\lambda}(\mathbf{Q}(\omega))\int_0^{\infty} \mathbf{y}_T^{\mathsf{T}}(t)\mathbf{y}_T(t)\mathrm{d}t$$

which satisfies condition two of Lemma 1.

Therefore the negative feedback interconnection of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ is asymptotically stable provided $\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0$. □

5 Input–output stability proof

Having determined a sufficient criteria for two hybrid systems within a negative feedback loop to be asymptotically stable using a Lyapunov method, we will now consider a system with inputs, $\mathbf{u}_1 \neq \mathbf{0}$ and $\mathbf{u}_2 \neq \mathbf{0}$, and prove asymptotic stability using an input–output method.

Theorem 2 (hybrid passivity and finite gain stability theorem – input–output approach): Referring to Fig. 1, the systems $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ could be linear or non-linear systems, but for generality we will assume they are non-linear systems with a state-space form equivalent to that of (1). Each system is $(\boldsymbol{Q}_i, \boldsymbol{S}_i, \boldsymbol{R}_i)$-dissipative with respect to a supply rate w_i, and each has a corresponding storage function, ϕ_i, as described by (2). If (10) defines the operators $\boldsymbol{Q}_i$, $\boldsymbol{S}_i$ and $\boldsymbol{R}_i$ then a sufficient condition for the negative feedback interconnection to be input–output stable (L_2-stable) is

$$\left\langle \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}, \overbrace{\begin{bmatrix} -\boldsymbol{Q}_1 - \boldsymbol{R}_2 & \boldsymbol{S}_1 - \boldsymbol{S}_2^{\sim} \\ \boldsymbol{S}_1^{\sim} - \boldsymbol{S}_2 & -\boldsymbol{R}_1 - \boldsymbol{Q}_2 \end{bmatrix}}^{\boldsymbol{Q}} \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}\right\rangle_T > 0$$

Proof: We wish to show that when $\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0$ bounded inputs lead to bounded outputs: $\mathbf{u}_1, \mathbf{u}_2 \in L_2 \Rightarrow \mathbf{y}_1, \mathbf{y}_2 \in L_2$. Consider the sum of the dissipation inequalities associated with each system over $t\in [0, T]$

$$\langle \mathbf{y}_1, \boldsymbol{Q}_1\mathbf{y}_1\rangle_T + 2\langle \mathbf{y}_1, \boldsymbol{S}_1\mathbf{e}_1\rangle_T + \langle \mathbf{e}_1, \boldsymbol{R}_1\mathbf{e}_1\rangle_T + \langle \mathbf{y}_2, \boldsymbol{Q}_2\mathbf{y}_2\rangle_T + 2\langle \mathbf{y}_2, \boldsymbol{S}_2\mathbf{e}_2\rangle_T + \langle \mathbf{e}_2, \boldsymbol{R}_2\mathbf{e}_2\rangle_T \geq 0$$

where $\phi_i(0) = 0$ corresponding to quiescent initial conditions. Referring to Fig. 1, $\mathbf{e}_1(t) = \mathbf{u}_1(t) - \mathbf{y}_2(t)$, $\mathbf{e}_2(t) = \mathbf{u}_2(t) + \mathbf{y}_1(t)$ and it follows that

$$\langle \mathbf{y}_1, \boldsymbol{Q}_1\mathbf{y}_1\rangle_T + 2\langle \mathbf{y}_1, \boldsymbol{S}_1(\mathbf{u}_1 - \mathbf{y}_2)\rangle_T + \langle (\mathbf{u}_1 - \mathbf{y}_2), \boldsymbol{R}_1(\mathbf{u}_1 - \mathbf{y}_2)\rangle_T + \langle \mathbf{y}_2, \boldsymbol{Q}_2\mathbf{y}_2\rangle_T + 2\langle \mathbf{y}_2, \boldsymbol{S}_2(\mathbf{u}_2 + \mathbf{y}_1)\rangle_T + \langle (\mathbf{u}_2 + \mathbf{y}_1), \boldsymbol{R}_2(\mathbf{u}_2 + \mathbf{y}_1)\rangle_T \geq 0$$

The above expression can be rearranged and put into a condensed truncated inner product form

$$\left\langle \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}, \begin{bmatrix} \boldsymbol{Q}_1 + \boldsymbol{R}_2 & -\boldsymbol{S}_1 + \boldsymbol{S}_2^{\sim} \\ -\boldsymbol{S}_1^{\sim} + \boldsymbol{S}_2 & \boldsymbol{R}_1 + \boldsymbol{Q}_2 \end{bmatrix} \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}\right\rangle_T + 2\left\langle \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}, \begin{bmatrix} \boldsymbol{S}_1 & \frac{1}{2}(\boldsymbol{R}_2^{\sim} + \boldsymbol{R}_2) \\ \frac{1}{2}(-\boldsymbol{R}_1^{\sim} - \boldsymbol{R}_1) & \boldsymbol{S}_2 \end{bmatrix} \begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}\right\rangle_T + \left\langle \begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}, \begin{bmatrix} \boldsymbol{R}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}_2 \end{bmatrix} \begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}\right\rangle_T \geq 0$$

By letting $\mathbf{y}^{\mathsf{T}}(t) = [\mathbf{y}_1^{\mathsf{T}}(t)\,\mathbf{y}_2^{\mathsf{T}}(t)]$, $\mathbf{u}^{\mathsf{T}}(t) = [\mathbf{u}_1^{\mathsf{T}}(t)\,\mathbf{u}_2^{\mathsf{T}}(t)]$ and using the self-adjoint properties of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ (i.e. $\boldsymbol{R}_1 \equiv \boldsymbol{R}_1^{\sim}$ and $\boldsymbol{R}_2 \equiv \boldsymbol{R}_2^{\sim}$) we have

$$\left\langle \mathbf{y}, \overbrace{\begin{bmatrix} \boldsymbol{Q}_1 + \boldsymbol{R}_2 & -\boldsymbol{S}_1 + \boldsymbol{S}_2^{\sim} \\ -\boldsymbol{S}_1^{\sim} + \boldsymbol{S}_2 & \boldsymbol{R}_1 + \boldsymbol{Q}_2 \end{bmatrix}}^{-\boldsymbol{Q}} \mathbf{y}\right\rangle_T + 2\left\langle \mathbf{y}, \overbrace{\begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{R}_2 \\ -\boldsymbol{R}_1 & \boldsymbol{S}_2 \end{bmatrix}}^{\boldsymbol{S}} \mathbf{u}\right\rangle_T + \left\langle \mathbf{u}, \overbrace{\begin{bmatrix} \boldsymbol{R}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}_2 \end{bmatrix}}^{\boldsymbol{R}} \mathbf{u}\right\rangle_T \geq 0$$

which can be written as simply

$$\langle \mathbf{u}, \boldsymbol{R}\mathbf{u}\rangle_T \geq \langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T - 2\langle \mathbf{y}, \boldsymbol{S}\mathbf{u}\rangle_T \qquad (13)$$

Noting that the operators $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$ are LTI, via Parseval's theorem we can transform the above expression into the frequency domain. For simplicity we will define a frequency domain inner product similar to the time domain inner product; for an arbitrary pair of truncated signals $\mathbf{w}_T(t)$ and $\mathbf{v}_T(t)$ which are zero before time $t = 0$ with corresponding Fourier transforms $\mathbf{w}_T(j\omega)$ and $\mathbf{v}_T(j\omega)$, through Parseval's theorem we will define the frequency domain inner product as

$$\int_0^{T} \mathbf{w}^{\mathsf{T}}(t)\mathbf{v}(t)\mathrm{d}t = \int_0^{\infty} \mathbf{w}_T^{\mathsf{T}}(t)\mathbf{v}_T(t)\mathrm{d}t = \langle \mathbf{w}_T, \mathbf{v}_T\rangle_{\omega} \triangleq \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} \mathbf{w}_T^{\mathsf{H}}(j\omega)\mathbf{v}_T(j\omega)\mathrm{d}\omega\right\}$$

Returning to (13), we must move the truncation from the inner product to the signals themselves, then we can utilise

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Parseval's theorem

$$\langle \mathbf{u}_T, \boldsymbol{R}\mathbf{u}_T\rangle \geq \langle \mathbf{y}_T, \boldsymbol{Q}\mathbf{y}_T\rangle - 2\langle \mathbf{y}_T, \boldsymbol{S}\mathbf{u}_T\rangle \Leftrightarrow \langle \mathbf{u}_T, \mathbf{R}\mathbf{u}_T\rangle_\omega \geq \langle \mathbf{y}_T, \mathbf{Q}\mathbf{y}_T\rangle_\omega - 2\langle \mathbf{y}_T, \mathbf{S}\mathbf{u}_T\rangle_\omega \quad (14)$$

where $\mathbf{Q}$, $\mathbf{S}$ and $\mathbf{R}$ are the Fourier transforms of the impulse responses of the operators $\boldsymbol{Q}$, $\boldsymbol{S}$ and $\boldsymbol{R}$. The matrices $\mathbf{Q}$, $\mathbf{S}$ and $\mathbf{R}$ are composed of $\mathbf{Q}_1(\cdot)$, $\mathbf{S}_1(\cdot)$, $\mathbf{R}_1(\cdot)$, $\mathbf{Q}_2(\cdot)$, $\mathbf{S}_2(\cdot)$ and $\mathbf{R}_2(\cdot)$ as defined in (7a)–(7c).

Let us look at the $\mathbf{S}$ and $\mathbf{R}$ matrices in more detail. The matrix $\mathbf{S}$ can be written as (see (15))

The matrix $\mathbf{R}$ equals (see (16))

In order to simplify (14) we must introduce various matrices; consider the matrix $\mathbf{R}$, but now modified by taking the absolute value of the elements within $\mathbf{R}$ (see equation at the bottom of the page)

Note that $\tilde{\mathbf{R}}$ is positive definite. Next consider the matrix

$$\bar{\mathbf{S}} = \mathbf{Q}^{-1/2}\mathbf{S}$$

Assuming that $\mathbf{Q}$ is positive definite (i.e. $\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0 \Leftrightarrow \langle \mathbf{y}_T, \mathbf{Q}\mathbf{y}_T\rangle_\omega > 0$), then $\bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}} = \mathbf{S}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{S}$ will also be positive definite. If $\mathbf{Q}$ is positive definite then $\mathbf{Q}^{1/2}$, $\mathbf{Q}^{-1}$ and $\mathbf{Q}^{-1/2}$ will be as well; these three matrices will be required in future simplifications.

Continuing on with the simplification of (14), we will utilise $\tilde{\mathbf{R}}$ within the inequality

$$\langle \mathbf{u}_T, \tilde{\mathbf{R}}\mathbf{u}_T\rangle_\omega \geq \langle \mathbf{u}_T, \mathbf{R}\mathbf{u}_T\rangle_\omega \geq \langle \mathbf{y}_T, \mathbf{Q}^{1/2}\mathbf{Q}^{1/2}\mathbf{y}_T\rangle_\omega - 2\langle \mathbf{y}_T, \mathbf{Q}^{1/2}\mathbf{Q}^{-1/2}\mathbf{S}\mathbf{u}_T\rangle_\omega$$

Adding $\langle \mathbf{u}_T, \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}}\mathbf{u}_T\rangle_\omega$ to both sides gives

$$\langle \mathbf{u}_T, (\tilde{\mathbf{R}} + \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}})\mathbf{u}_T\rangle_\omega \geq \langle \mathbf{y}_T, \mathbf{Q}^{1/2}\mathbf{Q}^{1/2}\mathbf{y}_T\rangle_\omega - 2\langle \mathbf{y}_T, \mathbf{Q}^{1/2}\mathbf{Q}^{-1/2}\mathbf{S}\mathbf{u}_T\rangle_\omega + \langle \mathbf{u}_T, \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}}\mathbf{u}_T\rangle_\omega$$

The structure of $\tilde{\mathbf{R}} + \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}}$ is one which will always be positive definite provided $\mathbf{Q}$ is positive definite, therefore, we can write

$$\sup_{\omega\in\mathbb{R}} \bar{\lambda}\langle \mathbf{u}_T, \mathbf{u}_T\rangle_\omega \geq \langle \mathbf{u}_T, (\tilde{\mathbf{R}} + \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}})\mathbf{u}_T\rangle_\omega$$

where $\bar{\lambda} > 0$ is the maximum eigenvalue of $\tilde{\mathbf{R}} + \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}}$. It follows that

$$\sup_{\omega\in\mathbb{R}} \bar{\lambda}\langle \mathbf{u}_T, \mathbf{u}_T\rangle_\omega \geq \langle \mathbf{y}_T, \mathbf{Q}^{1/2}\mathbf{Q}^{1/2}\mathbf{y}_T\rangle_\omega - 2\langle \mathbf{y}_T, \mathbf{Q}^{1/2}\bar{\mathbf{S}}\mathbf{u}_T\rangle_\omega + \langle \mathbf{u}_T, \bar{\mathbf{S}}^{\mathsf{H}}\bar{\mathbf{S}}\mathbf{u}_T\rangle_\omega$$

At this point we can use Parseval's theorem to transform the frequency domain inner products back into time domain inner products

$$\sup_{\omega\in\mathbb{R}} \bar{\lambda}\langle \mathbf{u}_T, \mathbf{u}_T\rangle \geq \langle \mathbf{y}_T, \boldsymbol{Q}^{1/2}\boldsymbol{Q}^{1/2}\mathbf{y}_T\rangle - 2\langle \mathbf{y}_T, \boldsymbol{Q}^{1/2}\bar{\boldsymbol{S}}\mathbf{u}_T\rangle + \langle \mathbf{u}_T, \bar{\boldsymbol{S}}^{\sim}\bar{\boldsymbol{S}}\mathbf{u}_T\rangle$$

Completing the square of the right-hand side of the above inequality and taking the square root of both sides yields

$$\sqrt{\sup_{\omega\in\mathbb{R}} \bar{\lambda}}\|\mathbf{u}_T\|_2 \geq \|\boldsymbol{Q}^{1/2}\mathbf{y}_T - \bar{\boldsymbol{S}}\mathbf{u}_T\|_2 \quad (17)$$

The inverse triangle inequality can be used to simplify the right-hand side of (17)

$$\|\boldsymbol{Q}^{1/2}\mathbf{y}_T - \bar{\boldsymbol{S}}\mathbf{u}_T\|_2 \geq \|\boldsymbol{Q}^{1/2}\mathbf{y}_T\|_2 - \|\bar{\boldsymbol{S}}\mathbf{u}_T\|_2$$

By using a matrix infinity norm, we can simplify $\|\boldsymbol{Q}^{1/2}\mathbf{y}_T\|_2$ in the above expression

$$\|\boldsymbol{Q}^{-1/2}\|_\infty\|\boldsymbol{Q}^{1/2}\mathbf{y}_T\|_2 \geq \|\boldsymbol{Q}^{-1/2}\boldsymbol{Q}^{1/2}\mathbf{y}_T\|_2 = \|\mathbf{y}_T\|_2 \Leftrightarrow \|\boldsymbol{Q}^{1/2}\mathbf{y}_T\|_2 \geq \|\boldsymbol{Q}^{-1/2}\|_\infty^{-1}\|\mathbf{y}_T\|_2$$

Similarly

$$\|\bar{\boldsymbol{S}}\|_\infty\|\mathbf{u}_T\|_2 \geq \|\bar{\boldsymbol{S}}\mathbf{u}_T\|_2$$

$$\mathbf{S} = \begin{bmatrix} \frac{1}{2}\alpha(\omega)\mathbf{1} & \mathbf{R}_2(\omega) \\ -\mathbf{R}_1(\omega) & \frac{1}{2}\alpha(\omega)\mathbf{1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\alpha(\omega)\mathbf{1} & (\gamma_2(1-\alpha(\omega)) - \delta_2\alpha(\omega))\mathbf{1} \\ -(\gamma_1(1-\alpha(\omega)) - \delta_1\alpha(\omega))\mathbf{1} & \frac{1}{2}\alpha(\omega)\mathbf{1} \end{bmatrix} \quad (15)$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2(\omega) \end{bmatrix} = \begin{bmatrix} (\gamma_1(1-\alpha(\omega)) - \delta_1\alpha(\omega))\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\gamma_2(1-\alpha(\omega)) - \delta_2\alpha(\omega))\mathbf{1} \end{bmatrix} \quad (16)$$

$$\tilde{\mathbf{R}} = \begin{bmatrix} (\gamma_1(1-\alpha(\omega)) + |\delta_1|\alpha(\omega))\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\gamma_2(1-\alpha(\omega)) + |\delta_2|\alpha(\omega))\mathbf{1} \end{bmatrix}$$

Then (17) becomes

$$\sqrt{\sup_{\omega\in\mathbb{R}}\bar{\lambda}}\|\mathbf{u}_T\|_2 + \|\bar{\mathbf{S}}\|_\infty\|\mathbf{u}_T\|_2 \geq \|\boldsymbol{Q}^{-1/2}\|_\infty^{-1}\|\mathbf{y}_T\|_2$$

$$\Leftrightarrow \|\mathbf{u}_T\|_2\underbrace{\left(\sqrt{\sup_{\omega\in\mathbb{R}}\bar{\lambda}} + \|\bar{\mathbf{S}}\|_\infty\right)\|\boldsymbol{Q}^{-1/2}\|_\infty}_{\nu} \geq \|\mathbf{y}_T\|_2$$

$$\Leftrightarrow \|\mathbf{u}_T\|_2\nu \geq \|\mathbf{y}_T\|_2$$

By letting $T\to\infty$ and assuming $\mathbf{u}\in L_2$ proves that $\mathbf{y}\in L_2$ if $\mathbf{u}\in L_2$, that is bounded inputs lead to bounded outputs. □

6 Structure and positive definiteness of Q

The sufficient condition for Theorems 1 and 2 to be satisfied is $\mathbf{Q}$ be positive definite, or in inner product form $\langle \mathbf{y}, \boldsymbol{Q}\mathbf{y}\rangle_T > 0 \Leftrightarrow \langle \mathbf{y}_T, \mathbf{Q}\mathbf{y}_T\rangle_\omega > 0$. Let us look at the matrix $\mathbf{Q}$ in more detail. The matrix $\mathbf{Q}$ can be written as (see (18))

$$\mathbf{Q} = -\begin{bmatrix} \mathbf{Q}_1(\omega) + \mathbf{R}_2(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_1(\omega) + \mathbf{Q}_2(\omega) \end{bmatrix}$$
$$= \begin{bmatrix} ((\epsilon_1+\delta_2)\alpha(\omega) + (\gamma_1^{-1}-\gamma_2)(1-\alpha(\omega)))\mathbf{1} & \mathbf{0} \\ \mathbf{0} & ((\epsilon_2+\delta_1)\alpha(\omega) + (\gamma_2^{-1}-\gamma_1)(1-\alpha(\omega)))\mathbf{1} \end{bmatrix} \quad (18)$$

and will be positive definite if $\mathbf{Q}_{11} > \mathbf{0}$ and $\det\{\mathbf{Q}\} > 0$. Recall that α is defined over $\omega\in(-\infty,\infty)$; when a system (a) has passive properties over the frequency range $(-\omega_c, \omega_c)$, $\alpha(\omega) = 1$. When a system (b) has finite gain properties over the frequency range $(-\infty, -\omega_c]\cup[\omega_c,\infty)$, $\alpha(\omega) = 0$.

Corollary 1: The matrix $\mathbf{Q}$ is positive definite if $\epsilon_1+\delta_2 > 0$, $\epsilon_2+\delta_1 > 0$, and $\gamma_1\gamma_2 < 1$.

Proof: When $\alpha(\omega) = 1$ the matrix $\mathbf{Q}$ reduces to

$$\mathbf{Q}|_{\alpha(\omega)=1} = \begin{bmatrix} (\epsilon_1+\delta_2)\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\epsilon_2+\delta_1)\mathbf{1} \end{bmatrix}$$

Clearly when $\epsilon_1+\delta_2 > 0$ and $\epsilon_2+\delta_1 > 0$, $\mathbf{Q}|_{\alpha(\omega)=1}$ is positive definite. When $\alpha(\omega) = 0$, $\mathbf{Q}$ becomes

$$\mathbf{Q}|_{\alpha(\omega)=0} = \begin{bmatrix} (\gamma_1^{-1}-\gamma_2)\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\gamma_2^{-1}-\gamma_1)\mathbf{1} \end{bmatrix}$$
$$= \begin{bmatrix} \gamma_1^{-1}(1-\gamma_1\gamma_2)\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \gamma_2^{-1}(1-\gamma_1\gamma_2)\mathbf{1} \end{bmatrix}$$

Clearly when $1 > \gamma_1\gamma_2$, $\mathbf{Q}|_{\alpha(\omega)=0}$ is positive definite. □

It is very interesting that $\epsilon_1+\delta_2 > 0$, $\epsilon_2+\delta_1 > 0$ and $\gamma_1\gamma_2 < 1$ are exactly the same as the inequalities stipulating stability via the traditional passivity and small gain theorems, yet in the traditional sense the parameters are defined globally, unlike in the hybrid sense.

Although the gain inequalities $\gamma_1\gamma_2 < 1$ can only be satisfied one way (in the region where passivity is violated), the passivity inequalities can be satisfied one of many ways; when the plant is passive and the control is very strictly passive (i.e. $\epsilon_1 = \delta_1 = 0$ and $\epsilon_2 > 0$, $\delta_2 > 0$), when the plant in input strictly passive and the control is input strictly passive (i.e. $\epsilon_1 = 0$, $\delta_1 > 0$ and $\epsilon_2 = 0$, $\delta_2 > 0$), when the plant is output strictly passive and the control is output strictly passive (i.e. $\epsilon_1 > 0$, $\delta_1 = 0$ and $\epsilon_2 > 0$, $\delta_2 = 0$), and when $\epsilon_1+\delta_2 > 0$ and $\epsilon_2+\delta_1 > 0$.

Notice the symmetry that is similar to that of the traditional passivity and small gain theorems. By symmetry we refer to the fact that the plant and control may switch roles. Should the plant be passive with gain γ_1 when passivity is violated and the control be very strictly passive with γ_2 when passivity is violated such that $\gamma_1\gamma_2 < 1$, then the feedback interconnection is L_2-stable. The converse may also be true, if the control is passive with gain γ_2 when passivity is violated and the plant is very strictly passive with gain γ_1 when passivity is violated such that $\gamma_1\gamma_2 < 1$, the feedback interconnection is L_2-stable. Other symmetrical switches are permitted as well.

Given the γ_1, ϵ_1 and δ_1 parameters for a hybrid plant (as defined by (11)), Corollary 1 stipulates that the parameters γ_2, ϵ_2 and δ_2 associated with the hybrid system representing the controller must satisfy specific inequalities for the negative feedback interconnection of the plant and controller to be stable via Theorems 1 and 2. Stated concisely, (11) defines the parameters associated with the plant and controller, Corollary 1 constrains them, and Theorems 1 and 2 guarantee the stability of the negative feedback interconnection.

7 Passivity and gain parameter definitions

The traditional passivity and small gain theorems (and equivalently in the traditional dissipative systems framework) specify the parameters γ_1, ϵ_1, δ_1, γ_2, ϵ_2 and δ_2 in a global sense. These global parameter definitions for linear systems are well known; for example, the relationship between positive real transfer functions and passive systems, strictly positive real transfer function and very strictly passive systems

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and the $\mathcal{H}_\infty$-norm of a transfer function and the maximum gain of a system are standard. Hybrid parameters are not defined globally; the passivity parameters are defined when the system in question possess passive properties, and the finite gain parameters are defined when passivity is violated.

Defining the γ_1, ϵ_1, δ_1, γ_2, ϵ_2 and δ_2 parameters for two hybrid systems is, at first, non-trivial. First, we must clarify that one of the systems, usually the plant $\boldsymbol{G}_1$, will govern the so-called switch between passivity and finite gain. The controller $\boldsymbol{G}_2$ is at the mercy of the plant in that when the plant loses its passive mapping but can then be described by a finite attenuation mapping, the controller must possess the appropriate passive mapping before the switch and a finite attenuation mapping after the switch. Therefore the ϵ_1, δ_1 and γ_1 parameters are defined first, and then the control system engineer must strategically design the controller to have appropriate ϵ_2, δ_2 and γ_2 values such that Corollary 1 is satisfied. Naturally, the control system designer may also want to optimally design the control parameters for superior system performance.

In the LTI case, the frequency response of a hybrid system can aid in defining the passivity and finite gain parameters. For example, a Nyquist plot clearly lays out the passive and finite gain characteristics of a single-input single-output system; the passive (i.e. positive real) portion of the system has its Nyquist plot lay in the right half portion of the Nyquist plane. The finite gain portion of the system has its Nyquist plot lay in the left half portion of the Nyquist plane. Given a plant with a corresponding Nyquist plot, ω_c (i.e. the critical frequency) is defined at the point when the Nyquist plot crosses the vertical axis of the Nyquist plane.

Defining the parameters for a non-linear system may be possible, but how to do so is still currently under investigation.

8 Conclusions

Our main motivation in this paper is guaranteeing the stability of originally passive systems which lose their passive characteristics. First, using the dissipative systems framework we defined a hybrid system. Additionally, we showed that approximating the parameters α and β using low- and high-pass Butterworth filters in the limit yields the exact definition of a hybrid system. Stability of two hybrid systems within a negative feedback loop was considered. Via Lyapunov and input–output techniques, we showed that stability of the closed-loop hinges on the positive definiteness of the matrix $\mathbf{Q}$. This condition was explored further, yielding the strict inequalities $\epsilon_1 + \delta_2 > 0$, $\epsilon_2 + \delta_1 > 0$ and $\gamma_1\gamma_2 < 1$, which when satisfied guarantee stability of the closed-loop system. The inequalities are equivalent to the inequalities delivered from the traditional passivity and small gain theorems, yet in the traditional sense the parameters are defined globally, unlike in the hybrid sense where they are defined when passivity is and is not violated.

The switching parameter α plays an important role in the hybrid systems framework. As previously mentioned, it is usually the plant which governs the switch between passive properties and finite gain properties, and hence defines ω_c, the critical frequency. However, if robustness of the control is of great concern, the control system designer may change the critical frequency artificially. This is not unreasonable in that the passivity violation, which dictates the switch, is usually only estimated and not accurately modelled or understood. Therefore a more robust control may be realised by simply enforcing the control to satisfy finite gain conditions prematurely.

Although hybrid systems are similar to mixed systems as defined by [6, 7], they are distinctly different as discussed in Remark 1. The mixed framework cannot accommodate plants which have $\gamma < \infty$ when no longer passive, only those with $\gamma < 1$. The hybrid passivity and finite gain stability theorem, however, can accommodate both $\gamma < 1$ and $\gamma < \infty$ situations. Although the mixed framework may be justly applied in specific situations, the hybrid framework handles a broader class of systems. Additionally, we define the $\mathbf{Q}(\cdot)$, $\mathbf{S}(\cdot)$, and $\mathbf{R}(\cdot)$ matricies (7) so that $\mathbf{Q}(\cdot)$ and $\mathbf{R}(\cdot)$ are dimensionally consistent; in the mixed framework this is not so.

In future work we plan to use the hybrid passivity and finite gain theorem to control practical, yet sufficiently complicated plants. For example, in the single-input single-output LTI context, the control of a single-link flexible robotic arm possessing actuator dynamics, sensor dynamics and other passivity destroying characteristics is quite difficult. In the multi-input multi-output LTI context, the vibration control of a two-dimensional flexible beam (representing a large communications antenna on a spacecraft, solar array main boom or solar sail structural member) using a tip-mounted control moment gyro as the actuator is a complicated problem in its own right, but is more complicated with the addition of passivity violating motor dynamics, sensor dynamics etc. Both these problems have been explored in great detail, and control via the hybrid passivity and finite gain stability theorem has worked well. Publication of these results is expected in the future. Additionally, we hope to investigate the formulation of optimal hybrid controllers. We wish to use a constrained $\mathcal{H}_2$ controller formulation, or an $\mathcal{H}_\infty$ control formulation, to be investigated in the near future.

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