Controllability and Observability of Gyroelastic Vehicles

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Stored angular momentum devices such as reaction wheels and control moment gyros have been used extensively for space vehicle attitude control. They represent a potential source of actuation for vibration and shape control of large space structures where they can potentially be distributed in large numbers. The vibration characteristics of these gyroelastic vehicles are affected by the presence of stored angular momentum and, hence, so are the conditions for controllability and observability. In this paper, these conditions are derived for systems modeled as gyroelastic continua, i.e., vehicles with continuous distributions of mass, stiffness, and gyricity (stored angular momentum). The conditions are expressed in terms of the gyroelastic modes and cover the case of pointwise actuators and those modeled in a continuum fashion. A numerical example is used to show that the degree of controllability in the continuum case can be interpreted as that corresponding to the limit of a sequence of pointwise control problems. The observability conditions are developed for a general class of measurements. The concept of a gyroelastic node is introduced and related to the problem of locating sensors.

I. Introduction

THE prospect of very large spacecraft in orbit has received a great deal of consideration. Such structures will be very flexible and require active control to achieve desired pointing accuracy and acceptable vibration levels. Judicious location of actuators and sensors is required to achieve these objectives. A potential source of actuation is stored angular momentum, hereafter termed *gyricity,* in the form of reaction wheels and control moment gyros (CMGs). These are linear devices capable of high-bandwidth, large-output torques. Other researchers have considered the use of gyric devices for vibration control. Aubrun and Margulies[1] present a detailed study of a device referred to as the "gyrodamper," which consists of a CMG collocated with an angular rate sensing gyroscope. One of their findings was that the use of many small units was preferable to the use of one large one from the point of view of achievable damping factors.

An important feature of stored angular momentum devices is the introduction of gyric torques into the motion equation as well as active control terms. This alters the modal characteristics of the structure. For very large spacecraft, the changes can be significant, as many devices are required for adequate control. The notion of a gyroelastic continuum has been introduced recently[2-5] as a model for structures with many lumped sources of gyricity. Constrained gyroelastic structures are handled in Ref. 2, and the dynamics of gyroelastic vehicles are treated in Refs. 3-5. Modal analyses and some key numerical examples are presented in the latter references. Although a continuous distribution of stored angular momentum is an important contribution of the model, pointwise descriptions are not exempt. An optimal control theory has been advanced that utilizes a continuum description of the gyricity distribution.[6] In Ref. 6, it was pointed out that the mere presence of gyricity, i.e., an open-loop configuration, could be beneficial in suppressing unwanted vibrations.

In this paper, we present the modal form of the equations of motion for a gyroelastic structure subjected to a control force distribution that can be either pointwise or distributed. Controllability and observability conditions are then derived in terms of the system modal parameters. These conditions can be used to determine the minimum number of sensors and pointwise actuators required to control the structure effectively. However, the relative information present in the controllability norms provides information that can be used to optimize the gyricity distribution, which can be interpreted as an actuator location function. They can also be used to show how the number of actuators contributes to overall controllability of the structure. We shall explore this aspect in a numerical example where controllability stemming from a continuous gyricity distribution as well as an equivalent pointwise distribution is considered. In elastic structures (flexible structures with no gyricity), it is well known[8] that one can render all of the elastic modes observable with one sensor, provided it is not located at a node (zero crossing) of any of the mode shapes. We shall generalize the notion of a node to the gyroelastic case and provide guidelines for locating sensors.

We should point out that controllability and observability conditions for large, spinning spacecraft, which have the same (mathematical) form as gyroelastic structures, were determined by Juang and Balas.[7] However, they considered only modes with nonzero frequencies since the stiffness matrix was assumed to be positive definite. A complete analysis of discrete parameter gyroelastic systems has been performed by Hughes and Skelton,[8] who derived controllability and observability conditions in terms of the system modes. The approach taken here yields identical results for pointwise actuators, but the derivation is somewhat different because continuum modeling of the control forces is encompassed by the techniques.

II. Dynamics of Gyroelastic Structures

A gyroelastic vehicle V is taken to consist of a number of flexible appendages, collectively denoted by E, attached to a rigid body R (see Fig. 1). An origin O is affixed to the rigid body, which can be made arbitrarily small. More general topologies are also possible.[4] The vehicle contains a distribution of gyricity $\mathbf{h}_s(\mathbf{r})$ which, for now, we restrict to be constant with respect to a local reference frame at $\mathbf{r}$. This function represents the stored angular momentum/unit volume.[2]

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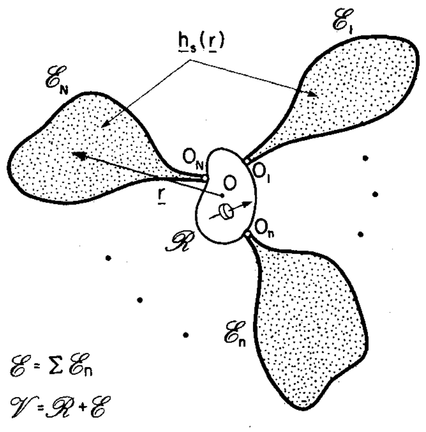

Fig. 1 Gyroelastic vehicle.

The total displacement field can be written as

$$\mathbf{w}(\mathbf{r},t) = \mathbf{w}_0(t) - \mathbf{r}^\times \boldsymbol{\theta}(t) + \begin{cases} \mathbf{u}_e(\mathbf{r},t), & \mathbf{r}\in E \\ \mathbf{0}, & \mathbf{r}\in R \end{cases} \tag{1}$$

where $\mathbf{w}_0$ is the translation of O, $\boldsymbol{\theta}$ is the small rotation of R relative to inertial space, and $\mathbf{u}_e(\mathbf{r},t)$ the small elastic deformation of E relative to R. The cross operator, $(\cdot)^\times$, is used to define the vector cross product; i.e., if $\mathbf{r} = [x\ y\ z]^T$, then

$$\mathbf{r}^\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

The rigid portion of the displacement in Eq. (1) is simply

$$\mathbf{u}_r(\mathbf{r},t) = \mathbf{U}_r(\mathbf{r})\mathbf{q}_r(t) \tag{2}$$

where $\mathbf{q}_r = \mathrm{col}\{\mathbf{w}_0, \boldsymbol{\theta}\}$ and $\mathbf{U}_r$ are the rigid degrees of freedom:

$$\mathbf{U}_r = [\mathbf{1} \quad -\mathbf{r}^\times] \tag{3}$$

It has been shown[2-5] that the equation of motion for V is

$$\mathfrak{M}\ddot{\mathbf{w}} + \mathcal{G}\dot{\mathbf{w}} + \mathcal{K}\mathbf{u}_e = \mathbf{f}_c(\mathbf{r},t) \tag{4}$$

where $\mathbf{f}_c$ is the external control force/volume distribution. Disturbances will be neglected in this analysis. The *stiffness operator* $\mathcal{K}$ is symmetric,

$$\int_V \mathbf{w}_1^T \mathcal{K}\mathbf{w}_2 \,\mathrm{d}V = \int_V \mathbf{w}_2^T \mathcal{K}\mathbf{w}_1 \,\mathrm{d}V$$

but is only positive semidefinite with respect to $\mathbf{w}$ since the rigid motions in Eq. (2) lie in the null space of $\mathcal{K}$. The *mass operator* $\mathfrak{M}$ is $\sigma(\mathbf{r})\mathcal{I}$, where $\sigma(\mathbf{r})$ is the mass density and $\mathcal{I}$ represents the identity operator. This operator is symmetric and positive definite with respect to $\mathbf{w}$. Finally, the gyricity operator[2] is

$$\mathcal{G} \triangleq -\tfrac{1}{4}\nabla^\times \mathbf{h}_s^\times \nabla^\times \tag{5}$$

This operator is rendered skew symmetric if we assume that $\mathbf{h}_s$ vanishes on ∂V, the boundary of V. We do not necessarily require $\mathbf{h}_s(\mathbf{r})$ to be continuous; the Dirac delta function can be used to model pointwise gyros:

$$\mathbf{h}_s(\mathbf{r}) = \sum_{i=1}^{n} \mathbf{h}_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{6}$$

Both forms of gyricity will be considered here.

Let us now consider the use of CMGs for active control. Associated with the nominal gyricity distribution $\mathbf{h}_s(\mathbf{r})$ is a distribution of gimbal angles,

$$\boldsymbol{\beta}(\mathbf{r},t) \triangleq \mathrm{col}\{\beta_x, \beta_y, \beta_z\} \tag{7}$$

which is assumed small, i.e.,

$$\|\boldsymbol{\beta}\| \ll 1 \tag{8}$$

The column of angles $\boldsymbol{\beta}(\mathbf{r},t)$ represents the angular displacement of the gyricity element $\mathbf{h}_s(\mathbf{r})\mathrm{d}V$ with respect to the local reference frame at $\mathbf{r}$. It can be shown[6] that the resulting force on the vehicle is

$$\mathbf{f}_h(\mathbf{r},t) = -\mathcal{G}\dot{\mathbf{w}} + \mathcal{H}\boldsymbol{v}$$

where the first term is present on the left-hand side of Eq. (4) and

$$\mathcal{H} \triangleq \tfrac{1}{2}\nabla^\times \mathbf{h}_s^\times, \qquad \boldsymbol{v} \triangleq \dot{\boldsymbol{\beta}} \tag{9}$$

The (distributed) control force in Eq. (4) is therefore

$$\mathbf{f}_c(\mathbf{r},t) = \mathcal{H}\boldsymbol{v} \tag{10}$$

where $\boldsymbol{v}(\mathbf{r},t)$ is the (distributed) control variable. Substituting Eq. (10) into Eq. (4) gives the desired second-order form of the motion equation:

$$\mathfrak{M}\ddot{\mathbf{w}} + \mathcal{G}\dot{\mathbf{w}} + \mathcal{K}\mathbf{u}_e = \mathcal{H}\boldsymbol{v} \tag{11}$$

The preceding expression for the control force is valid in the pointwise case if we make the following definitions:

$$\mathcal{H}\boldsymbol{v} = \sum_{i=1}^{n} \mathcal{H}_i \boldsymbol{v}_i(t), \qquad \mathcal{H} = \mathrm{row}\{\mathcal{H}_i\}, \qquad \boldsymbol{v}(t) = \mathrm{col}\{\boldsymbol{v}_i\} \tag{12}$$

and, more specifically, for pointwise CMGs,

$$\mathcal{H}_i = \tfrac{1}{2}\nabla^\times \delta(\mathbf{r} - \mathbf{r}_i)\mathbf{h}_i^\times, \qquad \boldsymbol{v}_i = \dot{\boldsymbol{\beta}}_i \tag{13}$$

where $\boldsymbol{\beta}_i = \mathrm{col}\{\beta_{xi}, \beta_{yi}, \beta_{zi}\}$ are the gimbal angles associated with the ith gyro. In this case, $\boldsymbol{v}(t)$ is finite-dimensional. Although we will be concerned primarily with the CMG cases embodied in Eqs. (9) and (13), many of our results extend to more general controls of the form of Eqs. (10) and (12).

We shall now introduce a generalized notation for handling both descriptions of the control force distribution. First, we shall indicate by $\mathcal{U}$ the space of all possible controls $\boldsymbol{v}$. For continuum controls, inner products involving the control variable will be written as

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_{\mathcal{U}} = \int_V \boldsymbol{v}_1^T(\mathbf{r},t)\boldsymbol{v}_2(\mathbf{r},t)\,\mathrm{d}V \tag{14}$$

and, in the pointwise case, as

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_{\mathcal{U}} = \boldsymbol{v}_1^T(t)\boldsymbol{v}_2(t) \tag{15}$$

Let us define a set of admissible controls as

$$\mathcal{U}_{\mathrm{ad}} \triangleq \left\{\boldsymbol{v} \in \mathcal{U} \,\middle|\, \int_0^T \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{U}}\,\mathrm{d}t < \infty\right\} \tag{16}$$

where T is the terminal time of interest. The adjoint of the input operator $\mathcal{H}$ is defined by

$$\int_V \mathbf{w}^T \mathcal{H}\boldsymbol{v}\,\mathrm{d}V = \langle \mathcal{H}^*\mathbf{w}, \boldsymbol{v} \rangle_{\mathcal{U}} \tag{17}$$

For continuum CMGs, the adjoint operator produces a function of $\mathbf{r}$,

$$\mathcal{H}^*\mathbf{w} = -\tfrac{1}{2}\mathbf{h}_s^\times \nabla^\times \mathbf{w} \tag{18}$$

and, for pointwise devices,

$$\mathcal{H}^*\mathbf{w} = \mathrm{col}\{-\tfrac{1}{2}\mathbf{h}_i^\times(\nabla^\times\mathbf{w})_{\mathbf{r}=\mathbf{r}_i}\} \tag{19}$$

is simply a column matrix. In deriving these relations, we have made use of the following form of integration by parts[2]:

$$\int_V \boldsymbol{\phi}^T\nabla^\times\boldsymbol{\psi}\,\mathrm{d}V = \int_V (\nabla^\times\boldsymbol{\phi})^T\boldsymbol{\psi}\,\mathrm{d}V$$

if $\boldsymbol{\phi}$ or $\boldsymbol{\psi}$ vanishes on the boundary of V.

III. Modal Analysis

The goal of this section is to write down the modal equations governing the gyroelastic dynamics as given by Eq. (11). We begin with a brief eigenanalysis of the first-order form of the operator equation and present the orthonormality conditions that exist among the eigenfunctions. A modal expansion that expresses the solution for the motion as a linear combination of the eigenfunctions is used, and the equations governing the modal coordinates are derived. They are then placed in a standard first-order matrix form from which the controllability and observability conditions can be extracted. Some care is taken in describing the input operator/matrix since distributed and pointwise controls will be encompassed by the treatment.

The motion equation (4) in first-order form is

$$\mathcal{E}\dot{\chi} + \mathcal{S}\chi = \mathcal{B}v \tag{20}$$

where

$$\mathcal{E} \triangleq \begin{bmatrix} \mathcal{M} & \mathbf{O} \\ \mathbf{O} & \mathcal{K} \end{bmatrix}, \quad \mathcal{S} \triangleq \begin{bmatrix} \mathcal{G} & \mathcal{K} \\ -\mathcal{K} & \mathbf{O} \end{bmatrix}, \quad \chi \triangleq \begin{bmatrix} \dot{\mathbf{w}} \\ \mathbf{u}_e \end{bmatrix}, \quad \mathcal{B} \triangleq \begin{bmatrix} \mathcal{H} \\ \mathbf{O} \end{bmatrix}$$

At this juncture, let us introduce the following inner product:

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle \triangleq \int_V \boldsymbol{\phi}^T\boldsymbol{\psi}\,\mathrm{d}V \tag{21}$$

The operator $\mathcal{E}$ is symmetric which, in the notation of this inner product, means

$$\langle \chi_1, \mathcal{E}\chi_2 \rangle = \langle \mathcal{E}\chi_1, \chi_2 \rangle \tag{22}$$

for χ_1 and χ_2 satisfying the boundary conditions. In addition, owing to our choice of the state description χ, $\mathcal{E}$ is positive definite, i.e.,

$$\langle \chi, \mathcal{E}\chi \rangle > 0 \qquad (\chi \neq \mathbf{0}) \tag{23}$$

The operator $\mathcal{S}$ is skew symmetric:

$$\langle \chi_1, \mathcal{S}\chi_2 \rangle = -\langle \mathcal{S}\chi_1, \chi_2 \rangle$$

The eigenproblem, which has been discussed in detail elsewhere,[3,4] is

$$\lambda_\alpha\mathcal{E}\chi_\alpha + \mathcal{S}\chi_\alpha = \mathbf{0} \tag{24}$$

The properties of $\mathcal{E}$ and $\mathcal{S}$ dictate that λ_α and χ_α appear in complex-conjugate pairs and that λ_α is purely imaginary. Therefore, we can write

$$\lambda_\alpha = j\omega_\alpha, \qquad \chi_\alpha = \boldsymbol{\phi}_\alpha + j\boldsymbol{\psi}_\alpha \quad (\alpha = -\infty\cdots\infty) \tag{25}$$

where $\omega_{-\alpha} = -\omega_\alpha$, $\omega_\alpha > 0$ for $\alpha > 0$, and $\boldsymbol{\phi}_{-\alpha} = \boldsymbol{\psi}_\alpha$. (Notice that 0 is excluded from the range of α.) The real eigenfunctions $\boldsymbol{\phi}_\alpha$ and $\boldsymbol{\psi}_\alpha$ have the form

$$\boldsymbol{\phi}_\alpha = \begin{bmatrix} -\omega_\alpha\mathbf{v}_\alpha \\ \mathbf{u}_{e\alpha} \end{bmatrix}, \qquad \boldsymbol{\psi}_\alpha = \begin{bmatrix} \omega_\alpha\mathbf{u}_\alpha \\ \mathbf{v}_{e\alpha} \end{bmatrix} \tag{26}$$

where $\mathbf{u}_\alpha = \mathbf{u}_{r\alpha} + \mathbf{u}_{e\alpha}$, $\mathbf{u}_{-\alpha} = \mathbf{v}_\alpha$, $\mathbf{u}_{r,-\alpha} = \mathbf{v}_{r\alpha}$, and $\mathbf{u}_{e,-\alpha} = \mathbf{v}_{e\alpha}$. The rigid-body portion of the mode shape, $\mathbf{u}_{r\alpha} \triangleq \mathbf{w}_{0\alpha} - \mathbf{r}^\times\boldsymbol{\theta}_\alpha$, can be written compactly as

$$\mathbf{u}_{r\alpha} = \mathbf{U}_r\mathbf{t}_\alpha \tag{27}$$

where $\mathbf{t}_\alpha = \mathrm{col}\{\mathbf{w}_{0\alpha}, \boldsymbol{\theta}_\alpha\}$.

Since the system is unconstrained, there are also zero-frequency eigenfunctions:

$$\lambda = 0, \qquad \mathcal{S}\mathbf{X}_r = \mathbf{O} \tag{28}$$

The matrix function $\mathbf{X}_r$, whose columns span the null space of $\mathcal{S}$, can be partitioned as

$$\mathbf{X}_r = \begin{bmatrix} \mathbf{U}_{\dot{r}} \\ \mathbf{U}_e \end{bmatrix} = \begin{bmatrix} \mathbf{1} & -\mathbf{r}^\times\mathbf{a} \\ \mathbf{O} & \mathbf{u}_a \end{bmatrix} \tag{29}$$

The first "column" represents the translational modes, which are unaffected by gyricity, while the second consists of the *pseudorigid modes*, which may be described as uniform rotations about an axis (axes) $\mathbf{a}$ with the elastic appendages of the vehicle in a constant deformed state $\mathbf{u}_a(\mathbf{r})$.[5] We shall refer to $\mathbf{X}_r$ as the *rigid rate modes*. Using Eqs. (29) and (3), the top portion of the rate modes can be expressed as a linear combination of the rigid degrees of freedom:

$$\mathbf{U}_{\dot{r}} = \mathbf{U}_r\mathbf{T}_r, \qquad \mathbf{T}_r = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{O} & \mathbf{a} \end{bmatrix} \tag{30}$$

The orthonormality relationships among the eigenfunctions can be summarized as follows:

$$\langle \boldsymbol{\phi}_\alpha, \mathcal{E}\boldsymbol{\phi}_\beta \rangle = 2\omega_\alpha^2\delta_{\alpha\beta}, \qquad \langle \boldsymbol{\phi}_\alpha, \mathcal{E}\mathbf{X}_r \rangle = \mathbf{O}$$

$$\langle \mathbf{X}_r, \mathcal{E}\mathbf{X}_r \rangle \triangleq \mathbf{M}_r > O \tag{31a}$$

$$\langle \boldsymbol{\psi}_\alpha, \mathcal{S}\boldsymbol{\phi}_\beta \rangle = 2\omega_\alpha^3\delta_{\alpha\beta} \tag{31b}$$

By virtue of Eq. (28), $\boldsymbol{\phi}_\alpha$ are also orthogonal to $\mathbf{X}_r$ with respect to $\mathcal{S}$.

The general solution for the motion of our system can be expressed in terms of the rate modes $\mathbf{X}_r$ and the eigenfunctions $\boldsymbol{\phi}_\alpha$:

$$\chi(\mathbf{r},t) = \mathbf{X}_r(\mathbf{r})\eta_r(t) + \sum_{\beta=-\infty}^{\infty}\boldsymbol{\phi}_\beta(\mathbf{r})\eta_\beta(t) \tag{32}$$

Substituting into Eq. (20) and operating with $\langle \mathbf{X}_r, \cdot \rangle$, $\langle \boldsymbol{\phi}_\alpha, \cdot \rangle$, and $\langle \boldsymbol{\phi}_{-\alpha}, \cdot \rangle$ while observing the orthogonality conditions, we arrive at the modal equations of motion,

$$\mathbf{M}_r\dot{\eta}_r = f_r \tag{33}$$

$$\dot{\eta}_\alpha - \omega_\alpha\eta_{-\alpha} = -\frac{1}{2\omega_\alpha}f_{-\alpha} \qquad \alpha = 1\cdots\infty$$

$$\dot{\eta}_{-\alpha} + \omega_\alpha\eta_\alpha = \frac{1}{2\omega_\alpha}f_\alpha$$

where

$$f_r \triangleq \int_V \mathbf{U}_{\dot{r}}^T\mathcal{H}v(t)\,\mathrm{d}V, \qquad f_\alpha \triangleq \int_V \mathbf{u}_\alpha^T\mathcal{H}v(t)\,\mathrm{d}V$$

The equations for the elastic modal coordinates can be written conveniently as

$$\dot{\eta}_e + \boldsymbol{\Omega}_e\eta_e = \tfrac{1}{2}\boldsymbol{\Omega}_e^{-T}f_e \tag{34}$$

where

$$\eta_e \triangleq \mathrm{col}\{\eta_\alpha, \eta_{-\alpha}\},$$

$$\Omega_e \triangleq \mathrm{diag}\left\{\omega_\alpha \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}, \qquad f_e \triangleq \mathrm{col}\{f_\alpha, f_{-\alpha}\}$$

They form a system of *bicoupled* equations.

Given the expression Eq. (17) for the adjoint operator $\mathcal{K}^*$, we can further write the modal forces as

$$f_r = B_r v, \qquad f_e = B_e v \tag{35}$$

where B_r and B_e are linear operators:

$$B_r(\cdot) \triangleq \langle \mathcal{K}^* \mathbf{U}_r, (\cdot)\rangle_{\mathcal{U}}, \qquad B_e(\cdot) \triangleq \mathrm{col}\{\langle \mathcal{K}^* \mathbf{u}_\alpha, (\cdot)\rangle_{\mathcal{U}}\} \tag{36}$$

Let us also define

$$\mathbf{H}_r \triangleq (\mathcal{K}^* \mathbf{U}_r)^T, \qquad \mathbf{H}_e \triangleq \mathrm{col}\{(\mathcal{K}^* \mathbf{u}_\alpha)^T\} \tag{37}$$

Note that when the space of controls is finite-dimensional, the operator B_r is simply the matrix $\mathbf{H}_r$ and B_e is the matrix $\mathbf{H}_e$. The adjoints of the operators B_r and B_e, defined such that

$$\eta_r^T B_r v = \langle B_r^* \eta_r, v\rangle_{\mathcal{U}}, \qquad \eta_e^T B_e v = \langle B_e^* \eta_e, v\rangle_{\mathcal{U}} \tag{38}$$

are, in general, given by

$$B_r^* = \mathbf{H}_r^T, \qquad B_e^* = \mathbf{H}_e^T \tag{39}$$

Equations (33) and (34) do not completely describe the rigid-body position and attitude. To this end, note that the rigid portion of the top of the expansion in Eq. (32) is

$$\dot{\mathbf{u}}_r = U_r \eta_r + \sum_{\alpha=-\infty}^{\infty} \omega_\alpha \mathbf{u}_{r\alpha} \eta_{-\alpha} \tag{40}$$

Upon examining Eqs. (2), (27), and (30), we can extract

$$\dot{\mathbf{q}}_r = \mathbf{T}_r \eta_r + \sum_{\alpha=-\infty}^{\infty} \omega_\alpha \mathbf{t}_\alpha \eta_{-\alpha} \tag{41}$$

or, in matrix form,

$$\dot{\mathbf{q}}_r = \mathbf{T}_r \eta_r - \mathbf{T}_e \Omega_e \eta_e \tag{42}$$

where $\mathbf{T}_e \triangleq \mathrm{row}\{\mathbf{t}_\alpha\}$.

Equations (33), (34), and (42), which describe the dynamics of the gyroelastic structure, can now be consolidated:

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + B v, \qquad \mathbf{x}(0) = \mathbf{0} \tag{43}$$

where

$$\mathbf{x} \triangleq \mathrm{col}\{\mathbf{q}_r, \eta_r, \eta_e\}, \qquad \mathcal{A} \triangleq \begin{bmatrix} \mathbf{O} & \mathbf{T}_r & -\mathbf{T}_e\Omega_e \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\Omega_e \end{bmatrix},$$

$$B \triangleq \begin{bmatrix} \mathbf{O} \\ \mathbf{M}_r^{-1} \mathbf{B}_r \\ \tfrac{1}{2}\Omega_e^{-T} B_e \end{bmatrix} \tag{44}$$

Although we have considered an infinite number of elastic modes to this point, it will now be assumed that the dimension of η_e is $2N$, where N is the number of gyroelastic mode pairs and N is arbitrarily large. The number of rate modes (the dimension of η_r) is denoted by n_r, and the dimension of $\mathbf{q}_r$ is, in general, $n_r \equiv 6$. Hence, the state vector $\mathbf{x} \in \mathfrak{X}$, where the state space $\mathfrak{X}$ is simply R^l, $l = n_r + n_r + 2N$, and $\mathcal{A}$ is a constant $l \times l$ matrix. The linear operator B maps $\mathcal{U}$ into $\mathfrak{X}$.

IV. Controllability Conditions

The reachable subspace of the state space is defined as

$$\mathfrak{R} \triangleq \{\mathbf{x}_d \in \mathfrak{X} \,|\, \exists T(0<T<\infty) \text{ and } v \in \mathcal{U}_{\mathrm{ad}} \ s.t.\ \mathbf{x}(T) = \mathbf{x}_d\} \tag{45}$$

where *s.t.* denotes "such that." The system of Eq. (44) is controllable if the reachable subspace is the entire state space. Using the result provided in the Appendix, this is true if, and only if, for all finite, positive T, the matrix

$$\mathbf{X}(T) \triangleq \int_0^T e^{\mathcal{A}(T-\tau)} B B^* e^{\mathcal{A}^T(T-\tau)}\, \mathrm{d}\tau \tag{46}$$

is positive definite, where

$$B^* = [\mathbf{O} \quad B_r^* \mathbf{M}_r^{-1} \quad \tfrac{1}{2} B_e^* \Omega_e^{-1}]$$

Controllability of $(\mathcal{A}, B)$ is equivalent to controllability of $(\mathcal{A}, \tilde{B})$, where

$$\tilde{B} = \begin{bmatrix} \mathbf{O} \\ B_r \\ B_e \end{bmatrix} \tag{47}$$

Henceforth, then, we will consider the pair $(\mathcal{A}, \tilde{B})$ and replace BB^* in Eq. (46) with $\mathbf{S} \triangleq \tilde{B}\tilde{B}^*$.

The quantity $\mathbf{S}$ is a positive-semidefinite matrix and can be partitioned as

$$\mathbf{S} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_{rr} & \mathbf{S}_{re} \\ \mathbf{O} & \mathbf{S}_{re}^T & \mathbf{S}_{ee} \end{bmatrix}$$

$$\mathbf{S}_{re} \triangleq \mathrm{row}\{\mathbf{S}_{r\alpha}\}, \qquad \mathbf{S}_{ee} \triangleq \mathrm{matrix}\{S_{\alpha\beta}\} \tag{48}$$

where, from Eqs. (36–39),

$$\mathbf{S}_{rr} = \langle \mathcal{K}^*\mathbf{U}_r, \mathcal{K}^*\mathbf{U}_r\rangle_{\mathcal{U}}, \qquad \mathbf{S}_{r\alpha} = \langle \mathcal{K}^*\mathbf{U}_r, \mathcal{K}^*\mathbf{u}_\alpha\rangle_{\mathcal{U}},$$

$$S_{\alpha\beta} = \langle \mathcal{K}^*\mathbf{u}_\alpha, \mathcal{K}^*\mathbf{u}_\beta\rangle_{\mathcal{U}} \tag{49}$$

If the rank of $\mathbf{S}$ is, say, k, then it can be factored as

$$\mathbf{S} = \hat{\mathbf{B}}\hat{\mathbf{B}}^T \tag{50}$$

where the column dimension of $\hat{\mathbf{B}}$ is k. The "square root" factorization in Eq. (50) can be realized by

$$\hat{\mathbf{B}} = \mathrm{col}\{\mathbf{O}, \hat{\mathbf{B}}_r, \hat{\mathbf{B}}_e\}, \qquad \hat{\mathbf{B}}_e = \mathrm{col}\{\hat{\mathbf{b}}_\alpha^T, \hat{\mathbf{b}}_{-\alpha}^T\} \tag{51}$$

where $\hat{\mathbf{B}}_r$ and $\hat{\mathbf{b}}_\alpha$ are such that

$$\hat{\mathbf{B}}_r\hat{\mathbf{B}}_r^T = \mathbf{S}_{rr}, \qquad \hat{\mathbf{B}}_r\hat{\mathbf{b}}_\alpha = \mathbf{S}_{r\alpha}, \qquad \hat{\mathbf{b}}_\alpha^T\hat{\mathbf{b}}_\beta = S_{\alpha\beta} \tag{52}$$

After replacing BB^* in Eq. (46) by $\mathbf{S} = \tilde{B}\tilde{B}^* = \hat{\mathbf{B}}\hat{\mathbf{B}}^T$, we see that positive-definiteness of $\mathbf{X}$ for all T is equivalent to controllability of the pair $(\mathcal{A}, \hat{\mathbf{B}})$. The matrix $\mathbf{X}$ becomes the controllability Grammian matrix of the pair $(\mathcal{A}, \hat{\mathbf{B}})$.

In the subsequent analysis, there is no need to identify $\hat{\mathbf{B}}_r$ and $\hat{\mathbf{B}}_e$ explicitly, since from the point of view of the derivations, they are purely artificial. However, in the case of pointwise actuators, the factorization is achieved by identifying $\hat{\mathbf{B}}_r = \mathbf{H}_r$ and $\hat{\mathbf{B}}_e = \mathbf{H}_e$, and the rank k of $\mathbf{S}$ is the number of independent controls. In the distributed case, the factorization can be interpreted as identifying a pointwise space of controls (the domain of $\hat{\mathbf{B}}$, $\mathbf{R}^k$) that behaves identically to the distributed controls from the point of view of controllability.

Upon examining the partitions for $\mathcal{A}$ [Eq. (44)] and $\hat{\mathbf{B}}$ [Eq. (51)], we notice that the last two sets of equations corresponding to the elastic and rigid rate modes are uncoupled. Hence, we can examine their controllability separately. Let us begin with the elastic modes or, to be specific, the matrix pair $(\mathbf{\Omega}_e, \hat{\mathbf{B}}_e)$.

Theorem 1. The pair $(\mathbf{\Omega}_e, \hat{\mathbf{B}}_e)$ is controllable if, and only if,

$$\hat{\mathbf{b}}_\alpha^T \hat{\mathbf{b}}_\alpha = S_{\alpha\alpha} \neq 0 \text{ or } \hat{\mathbf{b}}_{-\alpha}^T \hat{\mathbf{b}}_{-\alpha} = S_{-\alpha,-\alpha} \neq 0, \quad \alpha = 1, \ldots, N$$

where $S_{\alpha\alpha}$ is defined in Eq. (49).

Proof. The controllability matrix is given by

$$\mathbf{\Gamma}_e \triangleq [\hat{\mathbf{B}}_e \ \mathbf{\Omega}_e\hat{\mathbf{B}}_e \ \mathbf{\Omega}_e^2\hat{\mathbf{B}}_e \cdots \mathbf{\Omega}_e^{2N-1}\hat{\mathbf{B}}_e]$$

$$= \operatorname{col}\left\{\begin{bmatrix} \hat{\mathbf{b}}_\alpha^T & -\omega_\alpha\hat{\mathbf{b}}_{-\alpha}^T & -\omega_\alpha^2\hat{\mathbf{b}}_\alpha^T & \cdots & (-1)^N\omega_\alpha^{2N-1}\hat{\mathbf{b}}_{-\alpha}^T \\ \hat{\mathbf{b}}_{-\alpha}^T & \omega_\alpha\hat{\mathbf{b}}_\alpha^T & -\omega_\alpha^2\hat{\mathbf{b}}_{-\alpha}^T & \cdots & (-1)^{N+1}\omega_\alpha^{2N-1}\hat{\mathbf{b}}_\alpha^T \end{bmatrix}\right\}$$

If we define

$$\hat{\mathbf{\Gamma}}_\alpha \triangleq \begin{bmatrix} \hat{\mathbf{b}}_\alpha^T & -\omega_\alpha\hat{\mathbf{b}}_{-\alpha}^T \\ \hat{\mathbf{b}}_{-\alpha}^T & \omega_\alpha\hat{\mathbf{b}}_\alpha^T \end{bmatrix}, \qquad \lambda_\alpha \triangleq -\omega_\alpha^2$$

then the controllability matrix can be factored to read

$$\mathbf{\Gamma}_e = \operatorname{diag}\{\hat{\mathbf{\Gamma}}_1, \hat{\mathbf{\Gamma}}_2, \ldots, \hat{\mathbf{\Gamma}}_N\} \begin{bmatrix} \mathbf{1} & \lambda_1\mathbf{1} & \cdots & \lambda_1^{N-1}\mathbf{1} \\ \mathbf{1} & \lambda_2\mathbf{1} & \cdots & \lambda_2^{N-1}\mathbf{1} \\ \vdots & \vdots & & \vdots \\ \mathbf{1} & \lambda_N\mathbf{1} & \cdots & \lambda_N^{N-1}\mathbf{1} \end{bmatrix}$$

where $\mathbf{1}$ is the $2k \times 2k$ unit matrix. The $2kN \times 2kN$ matrix on the right yields the Vandermonde determinant, which is nonzero for distinct λ_α (ω_α). Therefore, the rank of $\mathbf{\Gamma}_e$ is given by

$$\operatorname{rank} \mathbf{\Gamma}_e = \sum_{\alpha=1}^{N} \operatorname{rank} \hat{\mathbf{\Gamma}}_\alpha$$

and hence for controllability we require

$$\operatorname{rank} \hat{\mathbf{\Gamma}}_\alpha = 2, \qquad \alpha = 1, \ldots, N$$

However, the rank of $\hat{\mathbf{\Gamma}}_\alpha$ is equal to the rank of

$$\mathbf{\Gamma}_\alpha = \begin{bmatrix} \hat{\mathbf{b}}_\alpha^T & \hat{\mathbf{b}}_{-\alpha}^T \\ \hat{\mathbf{b}}_{-\alpha}^T & -\hat{\mathbf{b}}_\alpha^T \end{bmatrix}$$

and requiring $\mathbf{\Gamma}_\alpha$ to have full rank (i.e., 2) is the same as requiring that $\mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\alpha^T$ be positive definite. This 2×2 matrix is simply

$$\mathbf{\Gamma}_\alpha\mathbf{\Gamma}_\alpha^T = \begin{bmatrix} \hat{\mathbf{b}}_\alpha^T\hat{\mathbf{b}}_\alpha + \hat{\mathbf{b}}_{-\alpha}^T\hat{\mathbf{b}}_{-\alpha} & 0 \\ 0 & \hat{\mathbf{b}}_\alpha^T\hat{\mathbf{b}}_\alpha + \hat{\mathbf{b}}_{-\alpha}^T\hat{\mathbf{b}}_{-\alpha} \end{bmatrix}$$

from which the result follows immediately. □

Using the definition of $S_{\alpha\alpha}$ and $S_{-\alpha,-\alpha}$ in Eq. (49), the controllability conditions can be written as

$$\langle \mathcal{H}^*\mathbf{u}_\alpha, \mathcal{H}^*\mathbf{u}_\alpha \rangle_{\mathcal{U}} \neq 0 \quad \text{or} \quad \langle \mathcal{H}^*\mathbf{u}_{-\alpha}, \mathcal{H}^*\mathbf{u}_{-\alpha} \rangle_{\mathcal{U}} \neq 0$$

$$\alpha = 1 \cdots N \tag{53}$$

We emphasize that these conditions apply only to the case of distinct frequencies. Examining the controllability matrix corresponding to the rigid rate modes, we arrive at Theorem 2.

Theorem 2. The pair $(\mathbf{O}, \hat{\mathbf{B}}_r)$ is controllable if, and only if,

$$\det \hat{\mathbf{B}}_r\hat{\mathbf{B}}_r^T = \det \mathbf{S}_{rr} = \det\{\langle \mathcal{H}^*\mathbf{U}_r, \mathcal{H}^*\mathbf{U}_r \rangle_{\mathcal{U}}\} \neq 0 \tag{54}$$

the proof of which is obvious when one realizes that the preceding is equivalent to rank $\hat{\mathbf{B}}_r = n_r$.

The preceding conditions say nothing about the ability to control the spacecraft's attitude and position since $\mathbf{q}_r$ cannot be expressed as a linear combination of $\boldsymbol{\eta}_e$ and $\boldsymbol{\eta}_r$. To this end, let us follow a path that is reminiscent of Hughes and Skelton.[8]

Theorem 3. The conditions for controllability of the gyroelastic system of Eq. (43) are Eqs. (53), (54), and

$$\det[\mathbf{T}_e(\mathbf{S}_{ee} - \mathbf{S}_{re}^T\mathbf{S}_{rr}^{-1}\mathbf{S}_{re})\mathbf{T}_e^T + \mathbf{T}_r\mathbf{S}_{rr}\mathbf{T}_r^T] \neq 0 \tag{55}$$

which can be interpreted as the condition for the attitude and position variables, $\mathbf{q}_r$.

Proof. Controllability of $(\mathcal{A}, \hat{\mathbf{B}})$ is equivalent to controllability of $(\mathbf{E}, \mathbf{F})$, where $\mathbf{E} = \mathcal{A}^2$ and $\mathbf{F} = [\hat{\mathbf{B}} \ \mathcal{A}\hat{\mathbf{B}}]$. Furthermore, controllability of $(\mathbf{E}, \mathbf{F})$ is equivalent to controllability of $(\mathbf{\Upsilon}^{-1}\mathbf{E}\mathbf{\Upsilon}, \mathbf{\Upsilon}^{-1}\mathbf{F})$, where

$$\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{T}_e \\ \mathbf{O} & \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{1} \end{bmatrix}, \qquad \mathbf{\Upsilon}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} & -\mathbf{T}_e \\ \mathbf{O} & \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{1} \end{bmatrix} \tag{56}$$

This new pair is

$$\left\{\begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{\Omega}_e^2 \end{bmatrix}, \begin{bmatrix} -\mathbf{T}_e\hat{\mathbf{B}}_e & \mathbf{T}_r\hat{\mathbf{B}}_r \\ \hat{\mathbf{B}}_r & \mathbf{O} \\ \hat{\mathbf{B}}_e & -\mathbf{\Omega}_e\hat{\mathbf{B}}_e \end{bmatrix}\right\}$$

The third set of equations corresponds to the elastic modes discussed previously.

The first and second partitions lead to the following controllability condition:

$$\operatorname{rank} \mathbf{P} = n_r + n_r, \qquad \mathbf{P} \triangleq \begin{bmatrix} -\mathbf{T}_e\hat{\mathbf{B}}_e & \mathbf{T}_r\hat{\mathbf{B}}_r \\ \hat{\mathbf{B}}_r & \mathbf{O} \end{bmatrix}$$

Clearly, the preceding requirement necessitates condition (54). It can also be expressed in terms of the determinant:

$$\det \mathbf{P}\mathbf{P}^T = \det\begin{bmatrix} \mathbf{T}_e\mathbf{S}_{ee}\mathbf{T}_e^T + \mathbf{T}_r\mathbf{S}_{rr}\mathbf{T}_r^T & -\mathbf{T}_e\mathbf{S}_{re}^T \\ -\mathbf{S}_{re}\mathbf{T}_e^T & \mathbf{S}_{rr} \end{bmatrix} \neq 0$$

Using an elementary expansion for the determinant, this condition can be rewritten as

$$\det \mathbf{S}_{rr} \cdot \det[\mathbf{T}_e(\mathbf{S}_{ee} - \mathbf{S}_{re}^T\mathbf{S}_{rr}^{-1}\mathbf{S}_{re})\mathbf{T}_e^T + \mathbf{T}_r\mathbf{S}_{rr}\mathbf{T}_r^T] \neq 0$$

where controllability of the rate modes (54) has been enforced in assuming $\mathbf{S}_{rr}^{-1}$ exists. Thus, we arrive at the final condition (55). □

V. Observability Conditions

In the interest of generality, measurements of gyroelastic behavior will be expressed as

$$y_i(t) = \int_V \mathcal{Y}_i(\mathbf{r})\dot{\mathbf{w}}(\mathbf{r}, t)\, \mathrm{d}V + \int_V \mathcal{Z}_i(\mathbf{r})\mathbf{w}(\mathbf{r}, t)\, \mathrm{d}V$$

$$i = 1, \ldots, M \tag{57}$$

where the kernels $\mathcal{Y}_i$, $\mathcal{Z}_i$ are assumed to be a function of $\mathbf{r}$ and may contain differential operators. For example, if $y_i = \mathbf{n}^T\mathbf{w}(\mathbf{r}, t)$, i.e., a pointwise deflection measurement in the direction $\mathbf{n}$, then

$$\mathcal{Y}_i = \mathbf{O}, \qquad \mathcal{Z}_i = \delta(\mathbf{r}_i - \mathbf{r})\mathbf{n}^T \tag{58}$$

If one senses the structure's rotational rate about an axis $\mathbf{a}$ at $\mathbf{r} = \mathbf{r}_i$, then

$$\mathcal{Y}_i = \tfrac{1}{2}\mathbf{a}^T\delta(\mathbf{r} - \mathbf{r}_i)\nabla^\times, \qquad \mathcal{Z}_i = \mathbf{O} \tag{59}$$

Averaging-type sensors can be accommodated by replacing the delta function with a Heavisidelike function defined to be 1 on the averaged region and 0 elsewhere. From the expansions (2) and (32), we have

$$\dot{\mathbf{w}} = \mathbf{U}_{\dot{r}}\boldsymbol{\eta}_r + \sum_{\alpha=-\infty}^{\infty} \mathbf{u}_\alpha \omega_\alpha \eta_{-\alpha}$$

$$\mathbf{w} = \mathbf{u}_r + \mathbf{u}_e = \mathbf{U}_r\mathbf{q}_r + \mathbf{U}_e\boldsymbol{\eta}_r + \sum_{\alpha=-\infty}^{\infty} \mathbf{u}_{e\alpha}\eta_\alpha$$

Substituting the preceding expressions into Eq. (57), we arrive at

$$y_i(t) = \mathbf{Z}_{ir}\mathbf{q}_r(t) + (\mathbf{Y}_{i\dot{r}} + \mathbf{Z}_{ie})\boldsymbol{\eta}_r(t) + \sum_{\alpha=-\infty}^{\infty}(-\omega_\alpha \mathbf{y}_{i,-\alpha} + \mathbf{z}_{ie,\alpha})\eta_\alpha(t)$$

where

$$\mathbf{Z}_{ir} = \int_V \mathcal{Z}_i \mathbf{U}_r \,\mathrm{d}V, \qquad \mathbf{Z}_{ie} = \int_V \mathcal{Z}_i \mathbf{U}_e \,\mathrm{d}V$$

$$\mathbf{z}_{ie,\alpha} = \int_V \mathcal{Z}_i \mathbf{u}_{e\alpha}\,\mathrm{d}V$$

$$\mathbf{Y}_{i\dot{r}} = \int_V \mathcal{Y}_i \mathbf{U}_{\dot{r}}\,\mathrm{d}V, \qquad \mathbf{y}_{i\alpha} = \int_V \mathcal{Y}_i \mathbf{u}_\alpha\,\mathrm{d}V \tag{60}$$

Assembling the measurements into a vector $\mathbf{y} \triangleq \mathrm{col}\{y_i\}$ and truncating the infinite series at N elastic mode pairs yields

$$\mathbf{y}(t) = \mathbf{Z}_r\mathbf{q}_r + \mathbf{C}_r\boldsymbol{\eta}_r + \mathbf{C}_e\boldsymbol{\eta}_e$$

where

$$\mathbf{Z}_r = \mathrm{col}_i\{\mathbf{Z}_{ir}\}, \quad \mathbf{C}_r = \mathrm{col}_i\{\mathbf{Y}_{i\dot{r}} + \mathbf{Z}_{ie}\}$$

$$\mathbf{C}_e = \mathrm{col}_i\{\mathrm{row}_\alpha\{-\omega_\alpha \mathbf{y}_{i,-\alpha} + \mathbf{z}_{ie,\alpha}\}\} \tag{61}$$

If we suppress the control variable ($\boldsymbol{\upsilon} \equiv \mathbf{0}$), the gyroelastic system can be described by

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x}, \qquad \mathbf{C} \triangleq [\mathbf{Z}_r\ \mathbf{C}_r\ \mathbf{C}_e] \tag{62}$$

where $\mathbf{x}$ and $\mathcal{A}$ are defined in Eq. (44).

Observability of a system with this form was discussed by Hughes and Skelton.[8] Here, we derive similar results using a different approach. Observability of ($\mathbf{C}$, $\mathcal{A}$) is equivalent to controllability of ($\mathcal{A}^T$, $\mathbf{C}^T$), where

$$\mathcal{A}^T = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{T}_r^T & \mathbf{O} & \mathbf{O} \\ \boldsymbol{\Omega}_e\mathbf{T}_e^T & \mathbf{O} & \boldsymbol{\Omega}_e \end{bmatrix}, \qquad \mathbf{C}^T = \begin{bmatrix} \mathbf{Z}_r^T \\ \mathbf{C}_r^T \\ \mathbf{C}_e^T \end{bmatrix}$$

The first set of equations implies that we must have rank $\mathbf{Z}_r^T = n_r$ or

$$\det[\mathbf{Z}_r^T\,\mathbf{Z}_r] = \det\left[\sum_{i=1}^{M} \mathbf{Z}_{ir}^T\mathbf{Z}_{ir}\right] \neq 0 \tag{63}$$

which mathematically expresses the need for n_r independent measurements in observing the attitude and position.

To uncover the remaining conditions, note that controllability conditions for ($\mathcal{A}^T$, $\mathbf{C}^T$) are equivalent to those of ($\mathbf{E}$, $\mathbf{F}$), where $\mathbf{E} \triangleq \mathcal{A}^{2T}$ and $\mathbf{F} \triangleq [\mathbf{C}^T\quad \mathcal{A}^T\mathbf{C}^T]$. Furthermore, controllability for this pair implies and is implied by controllability of ($\boldsymbol{\Upsilon}^T\mathbf{E}\boldsymbol{\Upsilon}^{-T}$, $\boldsymbol{\Upsilon}^T\mathbf{F}$), where $\boldsymbol{\Upsilon}$ was defined in Eq. (56). This new pair is

$$\left\{\begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \boldsymbol{\Omega}_e^2 \end{bmatrix}, \begin{bmatrix} \mathbf{Z}_r^T & \mathbf{O} \\ \mathbf{C}_r^T & \mathbf{T}_r^T\mathbf{Z}_r^T \\ \mathbf{T}_e^T\mathbf{Z}_r^T + \mathbf{C}_e^T & -\boldsymbol{\Omega}_e(\mathbf{T}_e^T\mathbf{Z}_r^T + \mathbf{C}_e^T) \end{bmatrix}\right\}$$

From the first and second partitions, we get the condition

$$\mathrm{rank}\begin{bmatrix} \mathbf{Z}_r^T & \mathbf{O} \\ \mathbf{C}_r^T & \mathbf{T}_r^T\mathbf{Z}_r^T \end{bmatrix} = n_r + n_{\dot{r}}$$

If we enforce condition (63), the preceding requirement is equivalent to

$$\det[\mathbf{C}_r^T\mathbf{C}_r + \mathbf{T}_r^T\mathbf{Z}_r^T\mathbf{Z}_r\mathbf{T}_r - \mathbf{C}_r^T\mathbf{Z}_r(\mathbf{Z}_r^T\mathbf{Z}_r)^{-1}\mathbf{Z}_r^T\mathbf{C}_r] \neq 0 \tag{64}$$

From the third partition, we glean that ($\boldsymbol{\Omega}_e$, $\mathbf{T}_e^T\mathbf{Z}_r^T + \mathbf{C}_e^T$) must be controllable. Given the definitions of $\mathbf{T}_e$ [Eq. (42)], $\mathbf{Z}_r$ [Eqs. (61) and (60)], and $\mathbf{C}_e$ [Eq. (61)], let us define

$$\mathbf{T}_e^T\mathbf{Z}_r^T + \mathbf{C}_e^T = \mathrm{col}\{\hat{\mathbf{c}}_\alpha^T, \hat{\mathbf{c}}_{-\alpha}^T\}$$

$$\hat{\mathbf{c}}_\alpha \triangleq \mathrm{col}_i\{-\omega_\alpha \mathbf{y}_{i,-\alpha} + \mathbf{z}_{ie,\alpha} + \mathbf{Z}_{ir}\mathbf{t}_\alpha\} \tag{65}$$

Using the definitions of $\mathbf{z}_{ie,\alpha}$, $\mathbf{Z}_{ir}$ [Eq. (60)], and $\mathbf{u}_{r\alpha}$ [Eq. (27)], we have

$$\mathbf{z}_{ie,\alpha} + \mathbf{Z}_{ir}\mathbf{t}_\alpha \equiv \mathbf{z}_{i\alpha}, \qquad \mathbf{z}_{i\alpha} \triangleq \int_V \mathcal{Z}_i\mathbf{u}_\alpha\,\mathrm{d}V \tag{66}$$

and, therefore, the definition of $\hat{\mathbf{c}}_\alpha$ becomes $\mathrm{col}_i\{-\omega_\alpha\mathbf{y}_{i,-\alpha} + \mathbf{z}_{i\alpha}\}$. We are now in a position to apply Theorem 1 directly. The elastic modes are observable if, and only if,

$$\hat{\mathbf{c}}_\alpha^T\hat{\mathbf{c}}_\alpha \neq 0 \quad \text{or} \quad \hat{\mathbf{c}}_{-\alpha}^T\hat{\mathbf{c}}_{-\alpha} \neq 0, \qquad \alpha = 1, \ldots, N \tag{67}$$

The conditions (63), (64), and (67) constitute the complete observability conditions.

The following definitions are natural in light of the preceding conditions.

Definition 1. A mode pair α exhibits a gyroelastic node in the direction $\mathbf{n}$ at $\mathbf{r} = \mathbf{r}_i$ if

$$\mathbf{n}^T\mathbf{u}_\alpha(\mathbf{r}_i) = \mathbf{n}^T\mathbf{v}_\alpha(\mathbf{r}_i) = 0$$

Definition 2. A mode pair α exhibits a gyroelastic node with respect to the axis $\mathbf{a}$ at $\mathbf{r}_i$ if

$$\mathbf{a}^T(\nabla^\times\mathbf{u}_\alpha)_{\mathbf{r}=\mathbf{r}_i} = \mathbf{a}^T(\nabla^\times\mathbf{v}_\alpha)_{\mathbf{r}=\mathbf{r}_i} = 0$$

Consider a single deflection measurement of the form of Eq. (58). We then have $\hat{\mathbf{c}}_\alpha = \mathbf{n}^T\mathbf{u}_\alpha(\mathbf{r}_1)$. Given the observability conditions in Eq. (67), we can say that one measurement of this form is sufficient for observability of a mode pair provided the pair does not exhibit a node at $\mathbf{r} = \mathbf{r}_1$ in the direction $\mathbf{n}$. Similar comments apply to an angular rate measurement of the form of Eq. (59).

VI. Numerical Example

We shall now employ a numerical example to illustrate our findings. Let us consider the bending (in two dimensions) of the uniform, free-free rod in Fig. 2. Such a model can be thought of as an equivalent continuum model of many space structure components of interest.[9] The rod's mass density (per unit length) is ρ and the stiffness operator for this case is

$$\mathcal{K} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\frac{\mathrm{d}^4}{\mathrm{d}x^4}$$

where B_1 and B_2 are the bending stiffnesses and $\beta^2 = B_1/B_2 = 1.5$. The quantity $B \triangleq (B_1B_2)^{1/2}$ is the geometric bending stiffness. This model has been used[3,4] previously to illustrate the properties of gyroelastic structures.

The deflection function is

$$\mathbf{w}(\mathbf{r}, t) = \begin{bmatrix} w_1(x, t) \\ w_2(x, t) \end{bmatrix} = \begin{bmatrix} w_{1,0}(t) - x\theta_2(t) + u_{1e}(x, t) \\ w_{2,0}(t) - x\theta_1(t) + u_{2e}(x, t) \end{bmatrix} \tag{68}$$

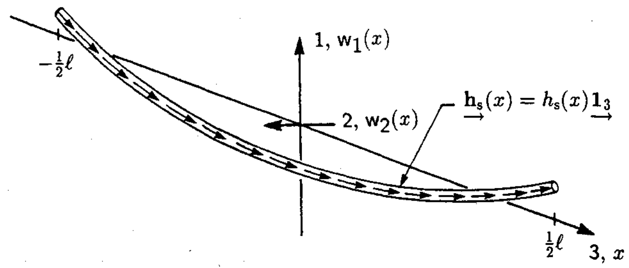

Fig. 2 Gyroelastic rod for numerical example.

where $\mathbf{q}_r = \text{col}\{w_{1,0}, w_{2,0}, \theta_1, \theta_2\}$ are the position and attitude of the rod. The gyricity distribution $h_s(x)$ is directed along the longitudinal axis of the rod and vanishes at both ends, i.e., $h_s(\ell/2) = h_s(-\ell/2) = 0$. The input operator and its adjoint, for double-gimbal CMGs, have the forms

$$\mathcal{H}\dot{\beta} = \begin{bmatrix} -\dfrac{\partial(h_s\dot{\beta}_1)}{\partial x} \\ -\dfrac{\partial(h_s\dot{\beta}_2)}{\partial x} \end{bmatrix} \tag{69}$$

and

$$\mathcal{H}^*\mathbf{w} = \begin{bmatrix} h_s(x)\dfrac{\partial w_1}{\partial x} \\ h_s(x)\dfrac{\partial w_2}{\partial x} \end{bmatrix} \tag{70}$$

in accordance with Eqs. (9) and (18). The gimbal angles have been denoted by $\beta = \text{col}\{\beta_1, \beta_2\}$. Note the absence of the factor ½, which stems from the one-dimensional nature of the rod.

Let us begin our study of controllability with the rate modes described by Eq. (29). From Ref. 5, the axes of rotation for the pseudorigid modes are defined by

$$\tilde{\mathbf{h}}_T\mathbf{a} = \mathbf{0}, \quad \tilde{\mathbf{h}}_T \triangleq \begin{bmatrix} 0 & -h_T \\ h_T & 0 \end{bmatrix}, \quad h_T \triangleq \int_{-\ell/2}^{\ell/2} h_s(x)\,\mathrm{d}x \tag{71}$$

where $(\tilde{\cdot})$ defines the cross product in two dimensions and h_T is the total gyricity. If $h_T \neq 0$, then there are no pseudorigid modes (the rotation about the 3 axis is not modeled in this two-dimensional example). In this case, the beam is incapable of rotating uniformly about the 1 or 2 axis and the rigid rotations that exist in the absence of gyricity become a precessional mode pair that exhibits a nonzero vibration frequency.[4,5] We shall return to this situation in a moment.

Zero Total Gyricity ($h_T = 0$)

If $h_T = 0$, which will occur if the gyricity distribution is skew symmetric, then there are two pseudorigid modes corresponding to rotations about the 1 and 2 axes:

$$\mathbf{U}_r = [\mathbf{1}_1 \ \mathbf{1}_2 \ -\tilde{\mathbf{x}}\mathbf{a}_1 \ -\tilde{\mathbf{x}}\mathbf{a}_2]$$

$$\mathbf{a}_1 = \mathbf{1}_1 \triangleq [1 \ 0]^T, \quad \mathbf{a}_2 = \mathbf{1}_2 \triangleq [0 \ 1]^T \tag{72}$$

The first two columns of $\mathbf{U}_r$ represent the two uniform translations of the rod. From Eqs. (70) and (72),

$$\mathcal{H}^*\mathbf{U}_r = [\mathbf{O} \ -\tilde{\mathbf{h}}_s(x)\mathbf{A}], \qquad \mathbf{A} \triangleq [\mathbf{a}_1 \ \mathbf{a}_2] = \mathbf{1} \tag{73}$$

where the zero partition reveals that the translational rate modes are not controllable. This is not surprising given that the gyricity distribution cannot supply a net force. The matrix $\mathbf{S}_{rr}$ defined in Eq. (49) has the form

$$\mathbf{S}_{rr} = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_{\theta\theta} \end{bmatrix}, \qquad \mathbf{S}_{\theta\theta} = \mathbf{A}^T\int_{-\ell/2}^{\ell/2} h_s^2(x)\,\mathrm{d}x\,\mathbf{A} \tag{74}$$

which establishes the controllability of the pseudorigid modes since $\det \mathbf{S}_{\theta\theta} \neq 0$ if $h_s(x) \not\equiv 0$. This is the reduced rotational form of the condition (54).

The vibration modes can be partitioned as

$$\mathbf{u}_\alpha = \begin{bmatrix} u_{1,\alpha}(x) \\ u_{2,\alpha}(x) \end{bmatrix} = \begin{bmatrix} w_{1,0\alpha} - x\theta_{2,\alpha} + u_{1e,\alpha} \\ w_{2,0\alpha} + x\theta_{1,\alpha} + u_{2e,\alpha} \end{bmatrix} \tag{75}$$

with similar notation for $\mathbf{v}_\alpha = \mathbf{u}_{-\alpha}$. The definition of $S_{\alpha\beta}$ [Eq. (49)] and the preceding descriptions for $\mathbf{u}_\alpha$ and $\mathcal{H}^*$ give

$$S_{\alpha\beta} = \int_{-\ell/2}^{\ell/2} h_s^2(x)\left[\frac{\partial u_{1,\alpha}}{\partial x}\frac{\partial u_{1,\beta}}{\partial x} + \frac{\partial u_{2,\alpha}}{\partial x}\frac{\partial u_{2,\beta}}{\partial x}\right]\mathrm{d}x \tag{76}$$

For each mode pair, we shall define the controllability norm:

$$C_\alpha^2 \triangleq \frac{S_{\alpha\alpha} + S_{-\alpha,-\alpha}}{\omega_\alpha^2}, \qquad \alpha = 1,2,3,\ldots \tag{77}$$

and the controllability condition in Eq. (53) for each mode pair is equivalent to $C_\alpha > 0$. We include the factor ω_α^2 in this definition because it would have resulted if we had used $\boldsymbol{B}$, given by Eq. (44), instead of $\tilde{\boldsymbol{B}}$ [Eq. (47)], in the derivation of the controllability conditions. For vibration control using CMGs, the condition in Eq. (53) is equivalent to requiring that there be at least one gyro located where the modal slope $\nabla^\times\mathbf{u}_\alpha$ or $\nabla^\times\mathbf{v}_\alpha$ is nonzero. It must also be such that the output torque $\mathbf{h}_s^\times\dot{\beta}$ is not perpendicular to both $\nabla^\times\mathbf{u}_\alpha$ and $\nabla^\times\mathbf{v}_\alpha$.

Lastly, let us deal with the attitude variables $\theta = [\theta_1 \ \theta_2]^T$. The quantity $\mathbf{S}_{r\alpha}$ [Eq. (49)] takes the form

$$\mathbf{S}_{r\alpha} = \begin{bmatrix} \mathbf{0} \\ \mathbf{S}_{\theta\alpha} \end{bmatrix}, \qquad \mathbf{S}_{\theta\alpha} = \mathbf{A}^T\int_{-\ell/2}^{\ell/2} h_s^2(x)\begin{bmatrix} -\dfrac{\partial u_{2,\alpha}}{\partial x} \\ \dfrac{\partial u_{1,\alpha}}{\partial x} \end{bmatrix}\mathrm{d}x \tag{78}$$

when $h_T = 0$. The zero partition is, once again, evidence of the translational modes. Clearly, we cannot have controllability of the position variables $\mathbf{w}_0 = [w_{1,0} \ w_{2,0}]^T$. However, the condition for attitude controllability can be salvaged from the collapsed form of Eq. (55). Taking only the rotational partitions, the criterion becomes

$$\det\left[\sum_{\alpha=-N}^{N}\sum_{\beta=-N}^{N}\theta_\alpha(S_{\alpha\beta} - \mathbf{S}_{\theta\alpha}^T\mathbf{S}_{\theta\theta}^{-1}\mathbf{S}_{\theta\beta})\theta_\beta^T + \mathbf{A}\mathbf{S}_{\theta\theta}\mathbf{A}^T\right] \neq 0 \tag{79}$$

where we have recalled the forms of $\mathbf{T}_r$ [Eq. (30)] and $\mathbf{T}_e$ [Eqs. (42) and (27)]. The two matrices that are summed here are nonnegative definite; that the first is at least positive semidefinite follows from the positive-definiteness of $\mathbf{S}_{\theta\theta}$ and the semidefiniteness of the matrix $\mathbf{S}$ in Eq. (48). Therefore, the condition is equivalent to asking that the sum be positive definite. When the total gyricity is zero ($h_T = 0$), we know from Eqs. (72) and (73) that $\mathbf{A} = \mathbf{1}$. In this case, controllability of the pseudorigid modes ($\det \mathbf{S}_{\theta\theta} \neq 0$) is enough to guarantee that the attitude will be controllable.

Nonzero Total Gyricity ($h_T \neq 0$)

The case where $h_T \neq 0$ will now be treated. As an example, let us choose the gyricity distribution to be

$$h_s(x) = -\frac{h_A\pi}{2\ell}\cos\frac{3\pi x}{\ell}, \qquad h_A \triangleq \int_{-\ell/2}^{\ell/2}|h_s|\,\mathrm{d}x \tag{80}$$

where h_A is the net gyricity. This distribution is symmetric about the origin, and the total gyricity is $h_T = h_A/3$. The determination of the gyroelastic mode shapes can be accomplished using a Ritz approximation. Here, we use the finite element method as described in Ref. 3. The first four mode pairs are depicted in Fig. 3 for the case where $\hat{h}_A = 1.0$ (we nondimensionalize according to $h_A^2 = \rho B \ell^2 \hat{h}_A^2$, $\hat{\omega}_\alpha^2 = \omega_\alpha^2 B/\rho\ell^4$). The mode shapes $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$ are plotted as well as the characteristic motion[2] $\{\mathbf{u}_\alpha \cos\omega_\alpha t - \mathbf{v}_\alpha \sin\omega_\alpha t\}$. The influence of gyricity on the modes is clearly in evidence, and the first pair is identified as the precessional mode. The controllability norms $\mathbf{C}_\alpha$ defined by Eq. (77) are tabulated in Table 1 for the first eight mode pairs ($\mathbf{C}_\alpha^2 = \hat{\mathbf{C}}_\alpha^2 h_A^2/B$). Given the way in which $\mathbf{C}_\alpha$ is defined, they fall off with α, although not monotonically.

Since this gyricity distribution does not exhibit pseudorigid modes, the partitions $\mathbf{S}_{\theta\theta}$ and $\mathbf{S}_{\theta\alpha}$ in Eqs. (74) and (78), respectively, do not exist. Therefore, the condition for attitude controllability [Eq. (79)], degenerates to

$$\det\left[\sum_{\alpha=-N}^{N}\sum_{\beta=-N}^{N}\boldsymbol{\theta}_\alpha S_{\alpha\beta}\boldsymbol{\theta}_\beta^T\right] \neq 0, \qquad \boldsymbol{\theta}_\alpha = \mathrm{col}\{\theta_{1,\alpha},\ \theta_{2,\alpha}\}$$

This expression would have resulted from Theorem 3 had the rate modes, which in this case consist of just the translations, been ignored. Note that $\theta_{1,\alpha}$ and $\theta_{2,\alpha}$ are the rotational components of the unconstrained vibration mode in Eq. (75). From Fig. 3, the second and fourth modes do not contribute to controllability of the attitude since $\boldsymbol{\theta}_\alpha = \mathbf{0}$ for both of them.

A pointwise distribution of CMGs will now be considered in accordance with Eq. (6):

$$\hat{h}_s(x) = \sum_{i=1}^{n} h_i \delta(x - x_i) \tag{81}$$

The quantities h_i and x_i are determined from the continuous distribution in Eq. (80) by partitioning the rod into n equal segments denoted by V_i. The point x_i is located at the center of V_i, and we define

$$h_i = \int_V h_s(x)\,\mathrm{d}x$$

The Cauchy-Schwartz inequality dictates that

$$\frac{1}{\ell_n}\sum_{i=1}^{n} h_i^2 \leq \int_{-\ell/2}^{\ell/2} h_s^2\,\mathrm{d}x, \qquad \ell_n \triangleq \int_{V_i}\mathrm{d}x = \ell/n \tag{82}$$

and based on the properties of the Riemann integral, we expect equality as $n \to \infty$. The adjoint of the input operator, from Eq. (19), is given by

$$\mathcal{K}^*\mathbf{w} = \mathrm{col}\left\{\begin{bmatrix} h_i\left(\dfrac{\partial w_1}{\partial x}\right)_{x=x_i} \\ h_i\left(\dfrac{\partial w_2}{\partial x}\right)_{x=x_i}\end{bmatrix}\right\} \in R^{2n} \equiv \mathcal{U} \tag{83}$$

which should be compared with Eq. (70).

When $h_T = 0$, the quantity corresponding to $\mathbf{S}_{\theta\theta}$ (the distributed case) is

$$\mathbf{S}_{\theta\theta}^{(n)} = (\mathcal{K}^*\tilde{x}\mathbf{A})^T(\mathcal{K}^*\tilde{x}\mathbf{A}) = \mathbf{A}^T\left(\sum_{i=1}^{n} h_i^2\right)\mathbf{A}$$

and making use of Eqs. (74) and (82), we have the inequality

$$\mathbf{S}_{\theta\theta}^{(n)} \leq \ell_N \mathbf{S}_{\theta\theta}$$

The factor ℓ_n in the right expression relates the dimensions of the distributed and pointwise cases. The inequality reveals that, from the point of view of relative controllability, the pointwise case approaches the distributed case from *below*. The reader should note that as $n \to \infty$, $\ell_n \to 0$ and, therefore, $\mathbf{S}_{\theta\theta}^{(n)} \to \mathbf{O}$. It is inappropriate that the measure of controllabilty provided by $\mathbf{S}_{\theta\theta}^{(n)}$ should tend to zero as the number of gyros increases with h_A fixed. Rather, one would prefer a measure that tended to the distributed result. Hence, we propose that $\mathbf{S}_{\theta\theta}^{(n)}/\ell_n$ would be a better measure of relative controllability when assessing the influence of the number of gyros. It has the same dimension as $\mathbf{S}_{\theta\theta}$ and as $n \to \infty$, $\mathbf{S}_{\theta\theta}^{(n)}/\ell_n \to \mathbf{S}_{\theta\theta}$, which is the desired result.

The pointwise definition of $S_{\alpha\alpha}$, from Eqs. (49) and (83), is

$$S_{\alpha\alpha}^{(n)} = \sum_{i=1}^{n} h_i^2\left[\left(\frac{\partial u_{1,\alpha}}{\partial x}\right)^2 + \left(\frac{\partial u_{2,\alpha}}{\partial x}\right)^2\right]_{x=x_i}$$

and like $\mathbf{S}_{\theta\theta}^{(n)}$, $S_{\alpha\alpha}^{(n)} \to 0$ as $n \to \infty$, which is misleading as a measure of controllability. Therefore, we define the pointwise controllability norms $\mathbf{C}_\alpha^{(n)}$ by

$$[\mathbf{C}_\alpha^{(n)}]^2 \triangleq \frac{1}{\ell_n\omega_\alpha^2}[S_{\alpha\alpha}^{(n)} + S_{-\alpha,-\alpha}^{(n)}]$$

We emphasize that, when calculating this quantity, the mode shapes and frequencies are functions of n. The factor $1/\ell_n$, once again, relates the dimension of the distributed and pointwise cases. Given the way in which the pointwise gyricity distribution is determined, the modes tend toward their distributed counterparts as $n \to \infty$. We claim that as $n \to \infty$—i.e.,

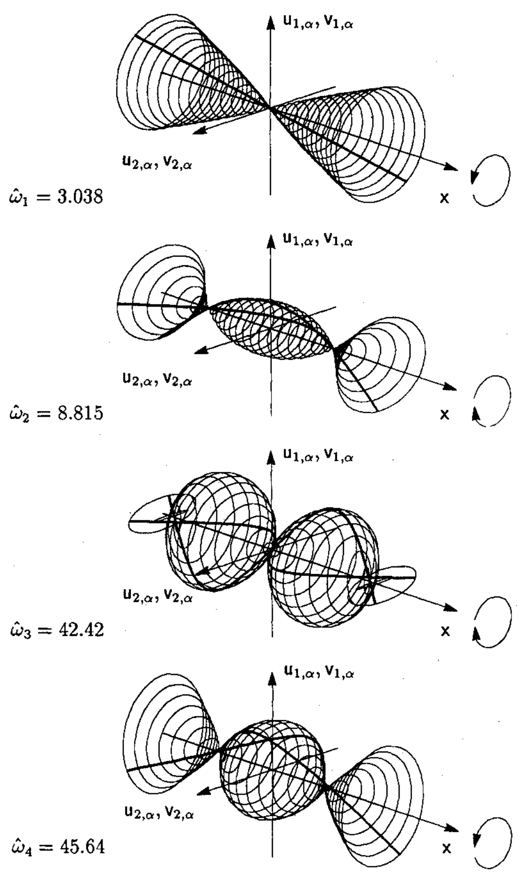

Fig. 3 Gyroelastic mode pairs for symmetric gyricity distribution.

Table 1 Controllability norms for distributed gyricity

Mode pair (α)	1	2	3	4	5	6	7	8
$\hat{\mathbf{C}}_\alpha$	2.157	0.672	0.271	0.241	0.148	0.205	0.131	0.134

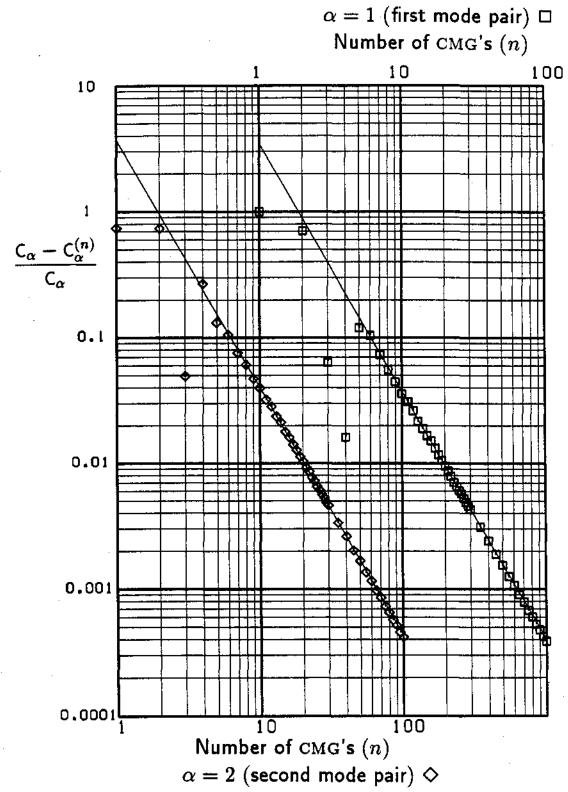

Fig. 4 Controllability norms for pointwise gyricity.

we take many small gyros—the pointwise controllability norm $C_\alpha^{(n)}$ tends toward the continuum result.

To see this numerically, let us turn to Fig. 4, where the quantities $(C_\alpha - C_\alpha^{(n)})/C_\alpha$ are plotted against n for the first and second mode pairs. The gyricity distribution is given by Eq. (80) and the net gyricity is $\hbar_A = 1.0$. From the plot, we acquire the approximate asymptotic relationship

$$\frac{C_\alpha^{(n)}}{C_\alpha} \doteq 1 - \frac{\kappa_\alpha}{n^2}, \qquad \kappa_\alpha > 0, \quad \alpha = 1,2$$

This behavior is representative of typical mode pairs and a general gyricity distribution. Convergence to the distributed result is from below, which demonstrates that distributed gyricity represents an ideal limiting case. Furthermore, the difference between the two diminishes in proportion to the square of the number of devices.

Now focusing on observability, consider the mode shapes of Fig. 3. We shall assume a single-point sensor ($M = 1$), which senses the beam's velocity at location $x = x_s$ in the direction $\mathbf{n} = [n_1 \;\; n_2]^T$. The observation kernels in Eq. (57) are $\mathcal{Y}_1 = \mathbf{n}^T\delta(x - x_s)$, $\mathcal{Z}_1 = \mathbf{O}$. The first and third modes are not observable if $x_s = 0$ since they possess a node in all directions $\mathbf{n}$. If $\mathcal{Y}_1 = \mathbf{n}^T\delta(x - x_s)\tilde{\nabla}$, $\mathcal{Z}_1 = \mathbf{0}$ (a singular angular rate sensor) and $x_s = 0$, then the second and fourth mode pairs are not observable since these modes are locally "flat" at the origin.

VII. Conclusions

Complete conditions for controllability and observability of gyroelastic structures have been expressed in terms of the system modal parameters. A very general approach yielded conditions for control that stems from a pointwise distribution of control moment gyros and for the distributed analog. The results can be applied to general actuator systems by proper formation of the input operator and its adjoint. We showed that controllability for distributed gyricity corresponds to the limit of a sequence of pointwise controllability problems. The relationship between the two from the standpoint of relative controllability was investigated in our numerical example. The conditions for attitude controllability were investigated in some detail because they present the greatest analytical challenge. In the beam example, the form of the conditions depends on whether or not a net momentum bias exists. The concept of a gyroelastic node was defined and, in the example, its importance for observability was illustrated graphically.

Appendix

The solution of Eq. (43) can be written as

$$\mathbf{x}(t) = \mathcal{W}(t)\boldsymbol{v}, \qquad \mathcal{W}(t)[\cdot] \triangleq \int_0^t e^{\mathcal{A}(t-\tau)}\boldsymbol{B}[\cdot]\,\mathrm{d}t, \qquad \boldsymbol{v}\in\mathcal{U}_{ad}$$

where $\mathcal{W}$ is a mapping from $\mathcal{U}_{ad}$ into the state space $\mathcal{X}$. Its adjoint is defined by

$$\mathbf{x}^T\mathcal{W}(t)\boldsymbol{v} = \int_0^t \langle \mathcal{W}^*(t)\mathbf{x},\boldsymbol{v}\rangle_{\mathcal{U}}\,\mathrm{d}\tau, \qquad \forall \mathbf{x}\in\mathcal{X},\ \forall \boldsymbol{v}\in\mathcal{U}_{ad}$$

and, therefore, $\mathcal{W}^*(t) = \boldsymbol{B}^*\exp[\mathcal{A}^T(t-\tau)]$, where $\boldsymbol{B}^*$ is defined so that

$$\mathbf{x}^T\boldsymbol{B}\boldsymbol{v} = \langle \boldsymbol{B}^*\mathbf{x},\ \boldsymbol{v}\rangle_{\mathcal{U}}$$

The complement of the reachable subspace in Eq. (45) is defined as

$$\mathfrak{R}^\perp = \{\boldsymbol{\xi}\in\mathcal{X}\,|\,\boldsymbol{\xi}^T\mathbf{x}_d = 0, \quad \forall \mathbf{x}_d\in\mathfrak{R}\}$$

and controllability as defined by $\mathfrak{R} = \mathcal{X}$ is equivalent to requiring that $\mathfrak{R}^\perp = \{\mathbf{0}\}$. Given the definition of $\mathfrak{R}$, $\mathfrak{R}^\perp$ can be characterized further by

$$\mathfrak{R}^\perp = \{\boldsymbol{\xi}\in\mathcal{X}\,|\,\boldsymbol{\xi}^T\mathcal{W}(T)\boldsymbol{v} = 0, \qquad \forall T(0<T<\infty),\ \forall \boldsymbol{v}\in\mathcal{U}_{ad}\}$$

But

$$\boldsymbol{\xi}^T\mathcal{W}(T)\boldsymbol{v} = \int_0^T \langle \mathcal{W}^*(T)\boldsymbol{\xi},\ \boldsymbol{v}\rangle_{\mathcal{U}}\,\mathrm{d}\tau$$

and $\boldsymbol{\xi}\in\mathfrak{R}^\perp$ if, and only if, $\boldsymbol{\xi}\in\mathfrak{N}\{\mathcal{W}^*(T)\}$, where $\mathfrak{N}\{\cdot\}$ denotes the null space. A simple proof shows that $\mathfrak{N}\{\mathcal{W}^*(T)\} = \mathfrak{N}\{\mathcal{W}(T)\mathcal{W}^*(T)\}$, and, therefore, controllability is equivalent to $\mathfrak{N}\{\mathcal{W}(T)\mathcal{W}^*(T)\} = \{\mathbf{0}\}$ $(0<T<\infty)$, or, alternatively, the matrix $\mathbf{X}(T)$ defined in Eq. (46) is positive definite for all finite, positive T.

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