2 Rigid Body Dynamics

2.1 Dynamics of a Particle

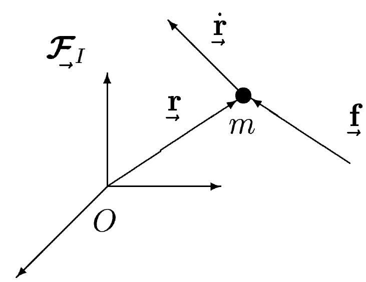
We consider a particle of mass m which is located at a position $\underset{\to}{\mathbf{r}}(t)$. It is subject to a force $\underset{\to}{\mathbf{f}}(t)$. The momentum of the particle is $\underset{\to}{\mathbf{p}} = m\underset{\to}{\dot{\mathbf{r}}}$ Newton's second law of motion states that

$$\underset{\to}{\dot{\mathbf{p}}} = \underset{\to}{\mathbf{f}}$$

or

$$m\underset{\to}{\ddot{\mathbf{r}}} = \underset{\to}{\mathbf{f}} \tag{1}$$

if it is assumed that m is constant.



The *angular momentum* (about O) of the particle is defined as:

$$\underset{\to}{\mathbf{h}}_o \triangleq \underset{\to}{\mathbf{r}} \times \underset{\to}{\mathbf{p}} = m\underset{\to}{\mathbf{r}} \times \underset{\to}{\dot{\mathbf{r}}}$$

The time derivative of the angular momentum is given by

$$\begin{aligned} \underset{\to}{\dot{\mathbf{h}}}_o &= m\underset{\to}{\dot{\mathbf{r}}} \times \underset{\to}{\dot{\mathbf{r}}} + m\underset{\to}{\mathbf{r}} \times \underset{\to}{\ddot{\mathbf{r}}} \\ &= \underset{\to}{\mathbf{r}} \times \underset{\to}{\mathbf{f}} \end{aligned}$$

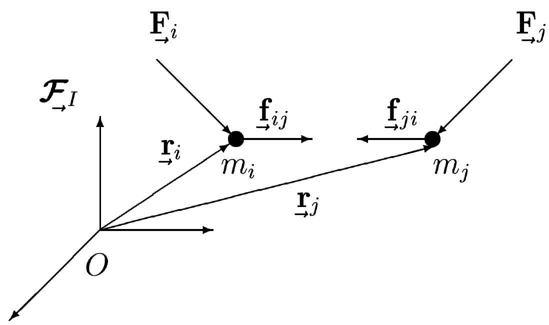
Hence,

$$\underset{\to}{\dot{\mathbf{h}}}_o = \underset{\to}{\mathbf{G}}_o \ , \quad \underset{\to}{\mathbf{G}}_o \triangleq \underset{\to}{\mathbf{r}} \times \underset{\to}{\mathbf{f}} \tag{2}$$

The quantity $\underset{\to}{\mathbf{G}}_o$ is the torque (about O) produced by $\underset{\to}{\mathbf{f}}$. In words, the time derivative of the angular momentum is equal to the external torque.

2.2 Dynamics of a System of Particles

Consider a system of particles with masses m_i, $i = 1 \cdots N$, and positions $\underset{\rightarrow}{\mathbf{r}}_i$.



The forces acting on m_i can be decomposed into two groups:

(a) the external force $\underset{\rightarrow}{\mathbf{F}}_i$;

(b) the internal forces stemming from the other $N-1$ masses. We denote these by $\underset{\rightarrow}{\mathbf{f}}_{ij}$ and take $\underset{\rightarrow}{\mathbf{f}}_{ii} = \mathbf{0}$. Hence, $\underset{\rightarrow}{\mathbf{f}}_{ij}$ is the force exerted on m_i by m_j. The use of Newton's third law asserts that

$$\underset{\rightarrow}{\mathbf{f}}_{ij} = -\underset{\rightarrow}{\mathbf{f}}_{ji} \tag{3}$$

We define the location of the *centre of mass* by

$$\underset{\rightarrow}{\mathbf{r}}_c = \left(\sum_{i=1}^{N} m_i \underset{\rightarrow}{\mathbf{r}}_i\right) / \sum_{i=1}^{N} m_i = \frac{\sum_{i=1}^{N} m_i \underset{\rightarrow}{\mathbf{r}}_i}{m} \tag{4}$$

Newton's second law applied to the ith mass gives:

$$m_i \ddot{\underset{\rightarrow}{\mathbf{r}}}_i = \underset{\rightarrow}{\mathbf{F}}_i + \sum_{j=1}^{N} \underset{\rightarrow}{\mathbf{f}}_{ij} \tag{5}$$

Now, add together the above relation for each of the N masses:

$$\sum_{i=1}^{N} m_i \ddot{\underset{\rightarrow}{\mathbf{r}}}_i = \sum_{i=1}^{N} \underset{\rightarrow}{\mathbf{F}}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \underset{\rightarrow}{\mathbf{f}}_{ij}$$

The last term vanishes by virtue of Eq. (3); therefore,

$$\sum_{i=1}^{N} m_i \ddot{\underset{\rightarrow}{\mathbf{r}}}_i = \sum_{i=1}^{N} \underset{\rightarrow}{\mathbf{F}}_i \tag{6}$$

Using the definition of the centre of mass, (4), we then have

$$m \ddot{\underset{\rightarrow}{\mathbf{r}}}_c = \underset{\rightarrow}{\mathbf{F}}\,, \quad \underset{\rightarrow}{\mathbf{F}} \triangleq \sum_{i=1}^{N} \underset{\rightarrow}{\mathbf{F}}_i \tag{7}$$

which has the same form as Newton's second law for a particle.

Now let us introduce the angular momentum of the system. For the ith mass,

$$\underrightarrow{\mathbf{h}}_i = m_i \underrightarrow{\mathbf{r}}_i \times \dot{\underrightarrow{\mathbf{r}}}_i$$

and for the entire system, the angular momentum (about O) is

$$\underrightarrow{\mathbf{h}}_o \triangleq \sum_{i=1}^{N} m_i \underrightarrow{\mathbf{r}}_i \times \dot{\underrightarrow{\mathbf{r}}}_i \tag{8}$$

Taking the time derivative of both sides gives,

$$\dot{\underrightarrow{\mathbf{h}}}_o = \sum_{i=1}^{N} m_i \underrightarrow{\mathbf{r}}_i \times \ddot{\underrightarrow{\mathbf{r}}}_i \tag{9}$$

Now, let us use the expression (5) to get

$$\dot{\underrightarrow{\mathbf{h}}}_o = \sum_{i=1}^{N} \underrightarrow{\mathbf{r}}_i \times (\underrightarrow{\mathbf{F}}_i + \sum_{j=1}^{N} \underrightarrow{\mathbf{f}}_{ij})$$

But,

$$\sum_{i=1}^{N} \underrightarrow{\mathbf{r}}_i \times \sum_{j=1}^{N} \underrightarrow{\mathbf{f}}_{ij} = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\underrightarrow{\mathbf{r}}_i - \underrightarrow{\mathbf{r}}_j) \times \underrightarrow{\mathbf{f}}_{ij}$$

If we assume that $\underrightarrow{\mathbf{f}}_{ij}$ acts along the line connecting m_j to m_i, $(\underrightarrow{\mathbf{r}}_i - \underrightarrow{\mathbf{r}}_j)$, then the above double summation must vanish.

Therefore, the time derivative of the system angular momentum satisfies:

$$\dot{\underrightarrow{\mathbf{h}}}_o = \underrightarrow{\mathbf{G}}_o \,, \quad \underrightarrow{\mathbf{G}}_o \triangleq \sum_{i=1}^{N} \underrightarrow{\mathbf{r}}_i \times \underrightarrow{\mathbf{F}}_i \tag{10}$$

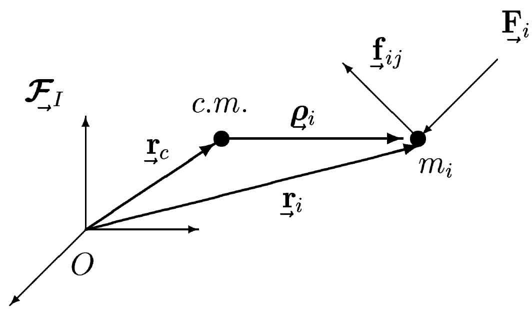
In words, the sum of the torques about O is equal to the time derivative of the system angular momentum about O.

Now, we would like to express the motion of the particle in terms of position relative to the centre of mass. Let us write

$$\underrightarrow{\mathbf{r}}_i = \underrightarrow{\mathbf{r}}_c + \underrightarrow{\boldsymbol{\rho}}_i \tag{11}$$

Using (8) and (11), the system angular momentum can be expressed as follows:

$$\begin{aligned}
\underrightarrow{\mathbf{h}}_o &= \sum_{i=1}^{N} m_i \underrightarrow{\mathbf{r}}_i \times \dot{\underrightarrow{\mathbf{r}}}_i \\
&= \sum_{i=1}^{N} m_i \left[(\underrightarrow{\mathbf{r}}_c + \underrightarrow{\boldsymbol{\rho}}_i) \times (\dot{\underrightarrow{\mathbf{r}}}_c + \dot{\underrightarrow{\boldsymbol{\rho}}}_i) \right] \\
&= (\sum_{i=1}^{N} m_i) \underrightarrow{\mathbf{r}}_c \times \dot{\underrightarrow{\mathbf{r}}}_c \; + \; \underrightarrow{\mathbf{r}}_c \times (\sum_{i=1}^{N} m_i \dot{\underrightarrow{\boldsymbol{\rho}}}_i) \\
&\quad + (\sum_{i=1}^{N} m_i \underrightarrow{\boldsymbol{\rho}}_i) \times \dot{\underrightarrow{\mathbf{r}}}_c \; + \; \sum_{i=1}^{N} m_i \underrightarrow{\boldsymbol{\rho}}_i \times \dot{\underrightarrow{\boldsymbol{\rho}}}_i
\end{aligned}$$



The second and third terms must vanish by virtue of the fact that

$$\sum_{i=1}^{N} m_i \underrightarrow{\boldsymbol{\rho}}_i = \sum_{i=1}^{N} m_i \dot{\underrightarrow{\boldsymbol{\rho}}}_i = \underrightarrow{\mathbf{0}}$$

since the $\underrightarrow{\boldsymbol{\rho}}_i$ are defined with respect to the centre of mass. Therefore,

$$\underrightarrow{\mathbf{h}}_o = m\underrightarrow{\mathbf{r}}_c \times \dot{\underrightarrow{\mathbf{r}}}_c \ + \ \underrightarrow{\mathbf{h}}_c \tag{12}$$

where

$$\underrightarrow{\mathbf{h}}_c \triangleq \sum_{i=1}^{N} m_i \underrightarrow{\boldsymbol{\rho}}_i \times \dot{\underrightarrow{\boldsymbol{\rho}}}_i$$

is the angular momentum with respect to the centre of mass. The total angular momentum with respect to O is equal to the angular momentum of the centre of mass plus the the angular momentum with respect to the centre of mass.

Now, take the time-derivative of the above expression and use (7)

$$\begin{aligned} \dot{\underrightarrow{\mathbf{h}}}_o &= m\underrightarrow{\mathbf{r}}_c \times \ddot{\underrightarrow{\mathbf{r}}}_c \ + \ \dot{\underrightarrow{\mathbf{h}}}_c \\ &= \underrightarrow{\mathbf{r}}_c \times \underrightarrow{\mathbf{F}} \ + \ \dot{\underrightarrow{\mathbf{h}}}_c \end{aligned} \tag{13}$$

But, from (10),

$$\begin{aligned} \dot{\underrightarrow{\mathbf{h}}}_o = \underrightarrow{\mathbf{G}}_o &= \sum_{i=1}^{N} \underrightarrow{\mathbf{r}}_i \times \underrightarrow{\mathbf{F}}_i \\ &= \sum_{i=1}^{N} (\underrightarrow{\mathbf{r}}_c + \underrightarrow{\boldsymbol{\rho}}_i) \times \underrightarrow{\mathbf{F}}_i \\ &= \underrightarrow{\mathbf{r}}_c \times \underrightarrow{\mathbf{F}} \ + \ \sum_{i=1}^{N} \underrightarrow{\boldsymbol{\rho}}_i \times \underrightarrow{\mathbf{F}}_i \end{aligned} \tag{14}$$

Comparing (13) and (14), we must have

$$\dot{\underrightarrow{\mathbf{h}}}_c = \underrightarrow{\mathbf{G}}_c \ , \quad \underrightarrow{\mathbf{G}}_c \triangleq \sum_{i=1}^{N} \underrightarrow{\boldsymbol{\rho}}_i \times \underrightarrow{\mathbf{F}}_i \tag{15}$$

Hence, the torque about the centre of mass is equal to the time derivative of the angular momentum with respect to the centre of mass.

Summary

translational equation:

$$m\ddot{\underset{\to}{\mathbf{r}}}_c = \underset{\to}{\mathbf{F}} = \sum_{i=1}^{N} \underset{\to}{\mathbf{F}}_i \tag{7}$$

rotational equation:

$$\dot{\underset{\to}{\mathbf{h}}}_c = \underset{\to}{\mathbf{G}}_c \tag{15}$$

where

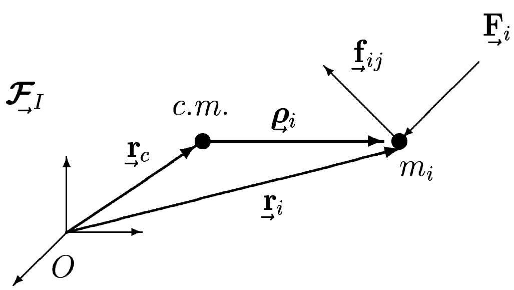
$$\underset{\to}{\mathbf{h}}_c = \sum_{i=1}^{N} m_i \underset{\to}{\boldsymbol{\varrho}}_i \times \dot{\underset{\to}{\boldsymbol{\varrho}}}_i\ , \quad \underset{\to}{\mathbf{G}}_c = \sum_{i=1}^{N} \underset{\to}{\boldsymbol{\varrho}}_i \times \underset{\to}{\mathbf{F}}_i$$

2.3 Rigid Body Dynamics

Translational Dynamics

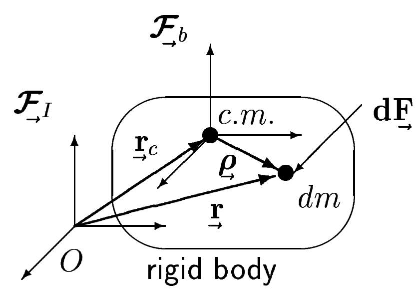
Definition. A *rigid body* is a continuum in which the distance between any two points on the body remains fixed.

We shall take a rigid body to be the limiting case of a system of particles in which the number of particles becomes infinitely large and their masses become infinitesimal. Therefore,



$$\begin{aligned}
\underset{\to}{\boldsymbol{\varrho}}_i &\to \underset{\to}{\boldsymbol{\varrho}} \\
m_i &\to dm = \sigma(\underset{\to}{\boldsymbol{\varrho}})\, dV \\
\underset{\to}{\mathbf{F}}_i &\to \mathbf{d}\underset{\to}{\mathbf{F}} = \underset{\to}{\mathbf{f}}(\underset{\to}{\boldsymbol{\varrho}})\, dV \\
\sum_{i=1}^{N} &\to \int_V
\end{aligned}$$

Here, dV is an infinitesimal volume element and $\sigma(\underset{\to}{\boldsymbol{\varrho}})$ is the mass density at a point $\underset{\to}{\boldsymbol{\varrho}}$. The force per unit volume is $\underset{\to}{\mathbf{f}}$.



For a system of point masses, the translational equation is from §2, Eq. (6):

$$\sum_{i=1}^{N} m_i \ddot{\underset{\rightarrow}{\mathbf{r}}}_i = \sum_{i=1}^{N} \underset{\rightarrow}{\mathbf{F}}_i$$

Making the changes indicated above, it becomes:

$$\int_V \ddot{\underset{\rightarrow}{\mathbf{r}}}\, dm = \int_V \underset{\rightarrow}{\mathbf{f}}\, dV \tag{16}$$

From §2, Eq. (4), the centre of mass for a system of particles, $\underset{\rightarrow}{\mathbf{r}}_c$, is defined by

$$m\underset{\rightarrow}{\mathbf{r}}_c = \sum_{i=1}^{N} m_i \underset{\rightarrow}{\mathbf{r}}_i$$

In the continuum case, this becomes

$$m\underset{\rightarrow}{\mathbf{r}}_c = \int_V \underset{\rightarrow}{\mathbf{r}}\, dm \tag{17}$$

Differentiating this twice with respect to time allows Eq. (1) to be written as

$$m\ddot{\underset{\rightarrow}{\mathbf{r}}}_c = \underset{\rightarrow}{\mathbf{F}}\,, \quad \underset{\rightarrow}{\mathbf{F}} \triangleq \int_V \underset{\rightarrow}{\mathbf{f}}\, dV \tag{18}$$

where $\underset{\rightarrow}{\mathbf{F}}$ is the total force acting on the rigid body

Eq. (18) is of the same form as the point mass result. The rigid body behaves (in translation) like a particle if its mass is lumped at the centre of mass. This lends credence to the two body solution of §3 when applied to finite-size space vehicles. The justification for treating the other body (*i.e.*, the earth) as a point mass will be given in §7.

The Angular Momentum of a Rigid Body

The angular momentum of a system of particles (with respect to the *c.m.*) is given by

$$\underset{\rightarrow}{\mathbf{h}}_c = \sum_{i=1}^{N} m_i \underset{\rightarrow}{\boldsymbol{\varrho}}_i \times \dot{\underset{\rightarrow}{\boldsymbol{\varrho}}}_i$$

Letting the number of particles $N \to \infty$, this becomes

$$\underset{\rightarrow}{\mathbf{h}}_c = \int_V \underset{\rightarrow}{\boldsymbol{\varrho}} \times \dot{\underset{\rightarrow}{\boldsymbol{\varrho}}}\, dm \tag{19}$$

The time derivative of $\underset{\rightarrow}{\boldsymbol{\varrho}}$ (as seen in the inertial frame) can be written as follows:

$$\dot{\underset{\rightarrow}{\boldsymbol{\varrho}}} = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{\varrho}}} + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{\varrho}} \tag{20}$$

where $\underset{\rightarrow}{\boldsymbol{\omega}}$ is the angular velocity of the rigid body with respect to the inertial reference frame. Because of the rigid body hypothesis, $\overset{\circ}{\underset{\rightarrow}{\boldsymbol{\rho}}} = \underset{\rightarrow}{\mathbf{0}}$. Therefore, the angular momentum becomes

$$
\begin{aligned}
\underset{\rightarrow}{\mathbf{h}}_c &= \int_V \underset{\rightarrow}{\boldsymbol{\rho}} \times (\underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{\rho}})\, dm \\
&= -\int_V \left(\underset{\rightarrow}{\boldsymbol{\rho}} \times \underset{\rightarrow}{\boldsymbol{\rho}} \times \underset{\rightarrow}{\boldsymbol{\omega}}\right) dm
\end{aligned} \tag{21}
$$

Let us express the remaining quantities in a *body-fixed frame.* For the duration of the notes, this frame will be represented as

$$
\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b = \begin{bmatrix} \underset{\rightarrow}{\mathbf{b}}_1 \\ \underset{\rightarrow}{\mathbf{b}}_2 \\ \underset{\rightarrow}{\mathbf{b}}_3 \end{bmatrix}
$$

Furthermore, the origin of this frame is the centre of mass. The position vector, angular velocity, and angular momentum become:

$$
\underset{\rightarrow}{\boldsymbol{\rho}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{\rho}\,, \quad \underset{\rightarrow}{\boldsymbol{\omega}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \boldsymbol{\omega}\,, \quad \underset{\rightarrow}{\mathbf{h}}_c = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \mathbf{h}_c \tag{22}
$$

An omitted subscript on component matrices shall designate the body-fixed frame. Using the above definitions in (6), we get the following component form:

$$
\mathbf{h}_c = -\int_V \left(\boldsymbol{\rho}^\times \boldsymbol{\rho}^\times \boldsymbol{\omega}\right) dm = \left[-\int_V \boldsymbol{\rho}^\times \boldsymbol{\rho}^\times\, dm\right] \boldsymbol{\omega} \tag{23}
$$

It can be verified by direct expansion that

$$
-\boldsymbol{\rho}^\times \boldsymbol{\rho}^\times = (\boldsymbol{\rho}^T \boldsymbol{\rho})\mathbf{1} - \boldsymbol{\rho}\boldsymbol{\rho}^T
$$

The quantity

$$
\begin{aligned}
\mathbf{I} &\triangleq \int_V \left[(\boldsymbol{\rho}^T \boldsymbol{\rho})\mathbf{1} - \boldsymbol{\rho}\boldsymbol{\rho}^T\right] dm \\
&= \int_V \begin{bmatrix} (\rho_2^2 + \rho_3^2) & -\rho_1\rho_2 & -\rho_1\rho_3 \\ -\rho_2\rho_1 & (\rho_1^2 + \rho_3^2) & -\rho_2\rho_3 \\ -\rho_3\rho_1 & -\rho_3\rho_2 & (\rho_1^2 + \rho_2^2) \end{bmatrix} \sigma(\rho_1, \rho_2, \rho_3)\, dV
\end{aligned} \tag{24}
$$

is called the *moment of inertia matrix.*

Therefore, we can write the system angular momentum as

$$
\begin{aligned}
\mathbf{h}_c &= \mathbf{I}\boldsymbol{\omega} \qquad (25) \\
&= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{21} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\end{aligned}
$$

The Inertia Matrix

(1) It can be defined with respect to any body-fixed coordinate system. However, in order to write the angular momentum in the form used above, (25), this frame must have its origin at the centre of mass.

(2) The inertia matrix is real and symmetric:

$$\mathbf{I} = \mathbf{I}^T$$

Furthermore, it is *positive-definite*:

$$\mathbf{x}^T \mathbf{I} \mathbf{x} > 0 \;\; (\mathbf{x} \neq \mathbf{0})$$

These two properties ensure that the eigenvalues of $\mathbf{I}$ are real and positive.

The moment of inertia matrix is a constant property of a rigid body. It fulfills the same role in rotational dynamics as the mass, m, does in translational dynamics.

Rotational Transformation Theorem

Consider two frames $\boldsymbol{\mathcal{F}}_1$ and $\boldsymbol{\mathcal{F}}_2$ which have their origins at the centre of mass of a rigid body. They are related by a rotation matrix:

$$\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2 = \mathbf{C}_{21} \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1$$

Recall that

$$\mathbf{C}_{21}^T = \mathbf{C}_{21}^{-1} = \mathbf{C}_{12}$$

A point on the rigid body can be expressed in either frame:

$$\underset{\rightarrow}{\boldsymbol{\rho}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1^T \boldsymbol{\rho}_1 = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \boldsymbol{\rho}_2$$

The inertia matrix can be defined with respect to both frames as follows:

$$\begin{aligned}
\mathbf{I}_1 &= \int_V \left[(\boldsymbol{\rho}_1^T \boldsymbol{\rho}_1)\mathbf{1} - \boldsymbol{\rho}_1 \boldsymbol{\rho}_1^T \right] dm \\
\mathbf{I}_2 &= \int_V \left[(\boldsymbol{\rho}_2^T \boldsymbol{\rho}_2)\mathbf{1} - \boldsymbol{\rho}_2 \boldsymbol{\rho}_2^T \right] dm
\end{aligned}$$

We desire a relationship between these two expressions. Since, $\boldsymbol{\rho}_2 = \mathbf{C}_{21}\boldsymbol{\rho}_1$, we can write

$$\begin{aligned}
\mathbf{I}_2 &= \int_V \left[(\mathbf{C}_{21}\boldsymbol{\rho}_1)^T (\mathbf{C}_{21}\boldsymbol{\rho}_1)\mathbf{1} - (\mathbf{C}_{21}\boldsymbol{\rho}_1)(\mathbf{C}_{21}\boldsymbol{\rho}_1)^T \right] dm \\
&= \int_V \left[(\boldsymbol{\rho}_1^T \mathbf{C}_{21}^T \mathbf{C}_{21} \boldsymbol{\rho}_1)\mathbf{1} - (\mathbf{C}_{21}\boldsymbol{\rho}_1 \boldsymbol{\rho}_1^T \mathbf{C}_{21}^T) \right] dm
\end{aligned}$$

But,

$$\begin{aligned}
(\boldsymbol{\rho}_1^T \mathbf{C}_{21}^T \mathbf{C}_{21} \boldsymbol{\rho}_1)\mathbf{1} &= \boldsymbol{\rho}_1^T \boldsymbol{\rho}_1 \mathbf{1} \\
&= \boldsymbol{\rho}_1^T \boldsymbol{\rho}_1 \mathbf{C}_{21} \mathbf{C}_{21}^T \\
&= \mathbf{C}_{21} \boldsymbol{\rho}_1^T \boldsymbol{\rho}_1 \mathbf{C}_{21}^T
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbf{I}_2 &= \mathbf{C}_{21} \int_V \left[(\boldsymbol{\rho}_1^T \boldsymbol{\rho}_1) \mathbf{1} - \boldsymbol{\rho}_1 \boldsymbol{\rho}_1^T \right] dm \, \mathbf{C}_{21}^T \\
&= \mathbf{C}_{21} \mathbf{I}_1 \mathbf{C}_{21}^T \qquad (26)
\end{aligned}
$$

which is the rotational transformation theorem.

Principal Axes

Definition. A body-fixed frame in which the moment of inertia matrix is diagonal, *i.e.*,

$$
\mathbf{I} \equiv \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}
$$

is called a principal axes frame. The diagonal elements of $\mathbf{I}$ are called the principal moments of inertia.

If the moment of inertial matrix is not diagonal, then the principal moments of inertia can be determined by considering the eigenvalue problem for the inertia matrix:

$$
\lambda_i \mathbf{e}_i = \mathbf{I} \mathbf{e}_i \ , \quad i = 1, 2, 3
$$

Since $\mathbf{I}$ is symmetric, the eigenvectors are orthonormal:

$$
\mathbf{e}_i^T \mathbf{e}_j = \delta_{ij} \triangleq \begin{cases} 1 \ , & i = j \\ 0 \ , & i \neq j \end{cases}
$$

Here, δ_{ij} is the Kronecker delta. The matrix form of this relationship is

$$
\mathbf{E}^T \mathbf{E} = \mathbf{1} \ , \quad \mathbf{E} = [\mathbf{e}_1 \ \ \mathbf{e}_2 \ \ \mathbf{e}_3]
$$

The matrix $\mathbf{E}$ is the *eigenmatrix* of $\mathbf{I}$. The eigenvectors also satisfy

$$
\mathbf{e}_i^T \mathbf{I} \mathbf{e}_j = \lambda_i \delta_{ij}
$$

which can be compactly written as

$$
\boldsymbol{\Lambda} = \mathbf{E}^T \mathbf{I} \mathbf{E} \ , \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

Since $\mathbf{E}^{-1} = \mathbf{E}^T$, the above equation implies that the inertia matrix can be written as

$$
\mathbf{I} = \mathbf{E} \boldsymbol{\Lambda} \mathbf{E}^T
$$

Comparing this equation with (26), we can conclude that the $\mathbf{E}^T$ is the rotation matrix from the original frame used to calculate $\mathbf{I}$ to the principal axis frame in which the inertia matrix is diagonal.

Since $\mathbf{I}$ is positive-definite, the eigenvalues are positive. The eigenvalues are the principal moments of inertia.

Rotational Kinetic Energy

The rotational kinetic energy of a rigid body is given by

$$T = \frac{1}{2}\int_V \dot{\underset{\rightarrow}{\boldsymbol{\rho}}} \cdot \dot{\underset{\rightarrow}{\boldsymbol{\rho}}}\, dm \tag{27}$$

Substituting (20) (noting that $\overset{\circ}{\underset{\rightarrow}{\boldsymbol{\rho}}} = \underset{\rightarrow}{\mathbf{0}}$) gives

$$\begin{aligned} T &= \frac{1}{2}\int_V (\boldsymbol{\omega}^\times \boldsymbol{\rho})^T (\boldsymbol{\omega}^\times \boldsymbol{\rho})\, dm \\ &= \frac{1}{2}\int_V (\boldsymbol{\rho}^\times \boldsymbol{\omega})^T (\boldsymbol{\rho}^\times \boldsymbol{\omega})\, dm \\ &= \frac{1}{2}\boldsymbol{\omega}^T \left[-\int_V \boldsymbol{\rho}^\times \boldsymbol{\rho}^\times\, dm \right] \boldsymbol{\omega} \\ &= \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} \end{aligned} \tag{28}$$

In a principal axis frame this takes on the simple form

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{29}$$

2.4 Euler's Equation

It has been demonstrated that

$$\dot{\underset{\rightarrow}{\mathbf{h}}}_c = \underset{\rightarrow}{\mathbf{G}}_c$$

where $\underset{\rightarrow}{\mathbf{h}}_c$ is the angular momentum (with respect to the centre of mass) and $\underset{\rightarrow}{\mathbf{G}}_c$ is the total external torque about the *c.m.* For a rigid body, the former can be expressed in a body-fixed frame $\underset{\rightarrow}{\mathbf{F}}_b$ as

$$\underset{\rightarrow}{\mathbf{h}}_c = \underset{\rightarrow}{\mathbf{F}}_b^T \mathbf{I}\boldsymbol{\omega}$$

Now,

$$\dot{\underset{\rightarrow}{\mathbf{h}}}_c = \overset{\circ}{\underset{\rightarrow}{\mathbf{h}}}_c + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathbf{h}}_c \tag{30}$$

and hence

$$\overset{\circ}{\underset{\rightarrow}{\mathbf{h}}}_c + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathbf{h}}_c = \underset{\rightarrow}{\mathbf{G}}_c \tag{31}$$

If all quantities are expressed in the body-fixed frame $\boldsymbol{\mathcal{F}}_b$, then

$$\dot{\mathbf{h}}_c + \boldsymbol{\omega}^\times \mathbf{h}_c = \mathbf{G}_c \tag{32}$$

Substituting (10), $\mathbf{h}_c = \mathbf{I}\boldsymbol{\omega}$, into this equation gives *Euler's Equation*:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} = \mathbf{G}_c \tag{33}$$

If we assume that $\boldsymbol{\mathcal{F}}_b$ is a principal axis frame, then we can write the above in component form as

$$\boxed{\begin{array}{rcl} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &=& G_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &=& G_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &=& G_3 \end{array}} \tag{34}$$

where we have used

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \mathbf{G}_c = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

The solution of these equations for $\boldsymbol{\omega}$ combined with a solution of

$$\dot{\mathbf{C}}_{bi} = -\boldsymbol{\omega}^{\times}\mathbf{C}_{bi}$$

for $\mathbf{C}_{bi}$ (the rotation matrix relating vectors in an inertial frame $\boldsymbol{\mathcal{F}}_i$ to the body frame $\boldsymbol{\mathcal{F}}_b$) yields a complete description for the attitude.

Lastly, we note that the external torque can be written as

$$\underrightarrow{\mathbf{G}}_c = \int_V \underrightarrow{\boldsymbol{\rho}} \times \underrightarrow{\mathbf{f}}\, dV = \underrightarrow{\boldsymbol{\mathcal{F}}}_b^T \mathbf{G}_c\,, \quad \mathbf{G}_c = \int_V \boldsymbol{\rho}^{\times}\mathbf{f}\, dV$$

where $\mathbf{f}$ are the components of the external force per unit volume, $\underrightarrow{\mathbf{f}}$, expressed in $\boldsymbol{\mathcal{F}}_b$.