A parallel solution adaptive method for radiative heat transfer using a Newton-Krylov approach

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ABSTRACT
The discrete ordinates method (DOM) and finite-volume method (FVM) are used extensively to solve the radiative transfer equation (RTE) in furnaces and combusting mixtures due to their balance between numerical efficiency and accuracy. These methods are typically solved using space-marching techniques since they converge rapidly for constant coefficient spatial discretization schemes and non-scattering media. However, space-marching methods lose their effectiveness when applied to scattering media or are used in combination with high-resolution limited total-variation-diminishing (TVD) schemes. They are also not well-suited to complex geometries and are not readily implemented within parallel CFD algorithms. A pseudo-time marching algorithm is therefore proposed herein to solve the DOM or FVM equations on multi-block body-fitted meshes using a highly scalable parallel-implicit solution approach in conjunction with high-resolution TVD spatial discretization. Adaptive mesh refinement (AMR) is also employed to properly capture disparate solution scales with a reduced number of grid points. The scheme is assessed in terms of discontinuity-capturing capabilities, spatial and angular solution accuracy, scalability, and serial performance through comparisons to other commonly employed solution techniques. The proposed algorithm is shown to possess excellent parallel scaling characteristics and can be readily applied to problems involving complex geometries.

1 INTRODUCTION
Two proven effective numerical techniques for solving the radiative transfer equation (RTE) are the discrete ordinates method (DOM) [1] and finite-volume method (FVM) [2]. Both techniques discretize the solid angle describing the direction of propagation to create a set of partial differential equations (PDEs) with only spatial coordinates as independent variables. As a result, standard finite-difference or finite-volume spatial discretization techniques that are commonly used in CFD can be applied.

Due to the linear upwind nature of the RTE, space-marching techniques are commonly applied to solve both the DOM and FVM equations. For non-scattering media and constant coefficient spatial discretization schemes, a converged solution can be obtained in one full sweep of the domain as the intensity at the downstream boundaries is related directly to upstream quantities. However, constant coefficient schemes are either overly dissipative or unstable [3]. Bounded high-resolution schemes developed for CFD were found to offer increased accuracy and oscillation-free solutions when applied to the DOM [3–5]. However, these schemes are considerably more expensive and standard space-marching techniques cannot cope with the local non-linearities that they introduce.

Space-marching algorithms perform poorly when applied to scattering media with high optical thickness where strong coupling exists between the intensities in different directions. Such computations can require a large number of iterations to converge, are prone to unphysical oscillations, and can fail to converge. Several researchers have applied explicit [6–8] and implicit [9, 10] time-evolution solution techniques to cope with arbitrary meshes and scattering media. However, the convergence characteristics of explicit solution algorithms are generally poor and the implicit approaches mentioned here are not directly compatible with standard parallel solution techniques.

Aside from the challenges associated with the treatment of complex geometries and scattering media, parallel implementations of conventional space-marching techniques are also problematic due to their inherently serial nature. This serial nature makes space-marching
techniques unattractive for large-scale parallel CFD solution algorithms. Several investigators have developed spatial-domain-based parallel algorithms for the DOM or FVM either based on semi-implicit iterative matrix solution techniques [11, 12] or optimized sweeping strategies [13–15]. Moderate parallel efficiencies are typically achieved.

In the present study, a cell-centred, flux-based, upwind finite-volume discretization procedure is applied to solve the PDEs resulting from the DOM and FVM. Multiple solution scales associated with optically thin and thick media are accurately resolved using a block-based adaptive mesh refinement (AMR) algorithm. To cope with non-linearities introduced by the spatial discretization and equation coupling, a parallel implicit time-evolution/relaxation procedure is employed.

2 NUMERICAL METHOD

The RTE for a monochromatic beam of light in the direction \( \hat{s} \) is given by

\[
\frac{1}{c} \frac{\partial I}{\partial t} + \hat{s} \cdot \nabla I = \kappa I_b - (\kappa + \sigma_s) I + \sigma_a \frac{\sum_{n=1}^{M} \omega_n I_m}{4\pi} \Phi(\hat{s}, \hat{s}') d\Omega',
\]

where \( c \) is the speed of light in a vacuum, \( I \) is the intensity in the direction of \( \hat{s} \), \( I_b \) is the blackbody radiative intensity, \( \kappa \) and \( \sigma_a \) are the absorption and scattering coefficients, \( \Phi(\hat{s}', \hat{s}) \) is the scattering phase function, and \( \Omega \) is the solid angle. The phase function describes the probability that a ray traveling in direction \( \hat{s}' \) will be scattered into the direction \( \hat{s} \). Applying the DOM to the RTE yields [1]

\[
\frac{1}{c} \frac{\partial I_m}{\partial t} + \hat{s}_m \cdot \nabla I_m = \kappa I_b - (\kappa + \sigma_s) I_m + \sigma_a \sum_{n=1}^{M} w_n I_n \Phi(\hat{s}_m, \hat{s}_m) \tag{2}
\]

where the subscript \( m \) denotes the discrete ordinate direction, \( M \) is the total number of ordinate directions, \( I_m \) is the intensity in the \( m \)-th direction, and \( \hat{s}_m \) and \( w_m \) are the ordinate direction vector and associated quadrature weight. The FVM equations are obtained by discretizing the solid angle into \( M \) control-angle elements and integrating the RTE over each element to give [2]

\[
\frac{1}{c} \frac{\partial I_m}{\partial t} + s_m \cdot \nabla I_m = \kappa I_b - (\kappa + \sigma_s) I_m + \sigma_a \sum_{n=1}^{M} \omega_n I_n \Phi_{mn} \Delta \Omega_n \tag{3}
\]

where the control-angle-averaged direction cosine vector, \( s_m \), and phase function, \( \Phi_{mn} \), are defined as

\[
\begin{align*}
\Phi_{mn} &= \frac{1}{\Delta \Omega_m \Delta \Omega_n} \int_{\Delta \Omega_n} \int_{\Delta \Omega_m} \Phi(\hat{s}', \hat{s}) d\Omega' d\Omega, \tag{4} \\
\mathbf{s}_m &= \frac{1}{\Delta \Omega_m} \int_{\Delta \Omega_m} \hat{s} d\Omega 
\end{align*}
\]

Equation (3) was derived assuming piecewise-constant intensity over each control angle and therefore \( I_m \) represents the average intensity in \( m \)-th control angle.

Let us now consider a two-dimensional Cartesian coordinate system. Both Eqs. (2) and (3) can be reformulated into a weak-conservation form since \( \hat{s}_m \) and \( s_m \) are independent of spatial location. The resulting equation is given by

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S \tag{6}
\]

where \( U = I_m \) is the solution vector, \( F = c \mu_t I_m \) is the flux in the \( x \)-direction, \( G = c \eta_m I_m \) is the flux in the \( y \)-direction, \( S = S_m \) is the source vector, and \( m = 1, \ldots, M \). The source term \( S_m \) is defined as the right-hand side of either Eqs. (2) or (3) for the DOM or FVM, respectively, multiplied by the speed of light \( c \). The directional coefficients \( \mu_m \) and \( \eta_m \) are either the direction cosines defined by the numerical quadrature scheme or the \( x \) and \( y \) components of \( s_m \).

2.1 Finite-volume spatial discretization

The proposed scheme for the RTE uses an upwind finite-volume spatial discretization procedure in conjunction with limited linear solution reconstruction to solve Eq. (6) [16]. It is applied on a multi-block mesh composed of arbitrary quadrilateral cells. Applied to cell \((i,j)\), the resulting system of coupled semi-discrete ODEs is given by

\[
\frac{\partial \mathbf{U}_{ij}}{\partial t} = -\frac{1}{\Delta t} \sum_k \left( \mathbf{F}_{k} \cdot \hat{n}_k \Delta l_k \right)_{ij} + \mathbf{S}(\mathbf{U}_{ij}) \tag{7}
\]

where \( \mathbf{U}_{ij} \) is the cell-average solution state, \( \mathbf{A}_{ij} \) is the area of the computational cell, \( \mathbf{F}(\mathbf{U}) = (\mathbf{F}, \mathbf{G}) \) is the flux dyad, \( \hat{n}_k \) and \( \Delta l_k \) are the normal and length of the \( k \)-th cell face. For Eq. (7), it has been assumed that \( \frac{1}{\Delta t} \int \mathbf{S}(\mathbf{U}) dA \approx \mathbf{S}(\mathbf{U}) \).

In two-dimensions, given the left and right solution states, \( \mathbf{U}_L \) and \( \mathbf{U}_R \), the numerical flux at the cell interface is \( \mathbf{F} \cdot \hat{n} = f(\mathbf{U}_L, \mathbf{U}_R, \hat{n}) \) where \( f \) is the upwind solution flux in a direction aligned along the face normal \( \hat{n} \). For this work, second-order spatial accuracy is
achieved by interpolating the solution state at the cell face between the two neighboring cells. Monotonicity is ensured using limiters to control gradients locally and damp any over- and under-shoots [16]. The linear least-squares reconstruction error minimization technique of Barth and Fredrickson [17] is used to evaluate the cell solution gradients. Slope limiting is performed with a slope limiter specifically designed for use in multiple dimensions [18].

2.2 Inexact Newton method

For cases involving scattering, the equations for radiative transport in each direction (Eq. (6)) become coupled through the source terms and this coupling increases with optical thickness. This angular coupling combined with the non-linearities introduced by the spatial discretization scheme can be problematic for space-marching techniques. Newton’s method is a robust and efficient nonlinear solution technique and is applied in this work to overcome these difficulties. It is used to relax the semi-discrete form of the governing equations to steady-state such that

\[ \mathbf{R}(\mathbf{U}) = \frac{d\bar{\mathbf{U}}}{dr} = 0 \]  

(8)

This particular implementation follows the algorithm developed previously by Groth and Northrup [19] specifically for use on large multi-processor parallel clusters. The implementation makes use of a Jacobian-free inexact Newton method coupled with an iterative Krylov subspace linear solver. In Newton’s method, a solution to Eq. (8) is sought by iteratively solving a sequence of linear systems given an initial estimate, \( \mathbf{U}^0 \). Successively improved estimates are obtained by solving

\[ \left( \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \right)^n \Delta \mathbf{U}^n = \mathbf{J}(\mathbf{U}^n) \Delta \mathbf{U}^n = -\mathbf{R}(\mathbf{U}^n) \]  

(9)

where \( \mathbf{J} = \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \) is the residual Jacobian. The improved solution at step \( n \) is then determined from

\[ \mathbf{U}^{n+1} = \mathbf{U}^n + \Delta \mathbf{U}^n \]  

(10)

The Newton iterations proceed until some desired reduction of the norm of the residual is achieved and the condition \( \| \mathbf{R}(\mathbf{U}^n) \| < \varepsilon \| \mathbf{R}(\mathbf{U}^0) \| \) is met. The tolerance, \( \varepsilon \), used in this work was \( 10^{-10} \).

For a system of nonlinear equations, each step of Newton’s method requires the solution of the linear problem, \( \mathbf{J} \mathbf{x} = \mathbf{b} \), where \( \mathbf{x} = \Delta \mathbf{U} \) and \( \mathbf{b} = -\mathbf{R}(\mathbf{U}) \). This system is solved using the generalized minimal residual (GMRES) technique [20]. GMRES is particularly attractive because the matrix \( \mathbf{J} \) is not explicitly formed, reducing the required storage. Termination generally only requires solving the linear system to some specified tolerance, \( \| \mathbf{R}^n + \mathbf{J}^n \Delta \mathbf{U}^n \|_2 < \zeta \| \mathbf{R}(\mathbf{U}^n) \|_2 \), where \( \zeta \) is typically in the range \( 0.1 - 0.5 \) [21]. We use a restarted version of the GMRES algorithm here, GMRES(\( m \)), that minimizes storage by restarting every \( m \) iterations.

Right preconditioning \( \mathbf{J} \) is performed to help facilitate the solution of the linear system without affecting the solution residual, \( \mathbf{b} \). A combination of an additive Schwarz global preconditioner and a block incomplete lower-upper (BILU) local preconditioner is used which is easily implemented in the block-based AMR scheme. The local preconditioner is based on block ILU(p) factorization [20] of the Jacobian for the first order approximation of each domain. The level of fill, \( p \), was maintained at zero in order to minimize storage requirements.

2.3 Parallel adaptive mesh refinement

A flexible block-based AMR scheme is adopted here to limit the number of necessary computational cells by dynamically adapting the mesh to meet solution requirements. Details of the scheme and its implementation in parallel are described by [22]. In the proposed parallel implementation, block-based domain decomposition is applied to a body-fitted quadrilateral mesh. The resulting grid blocks are organized in a hierarchical quad-tree data structure to facilitate automatic solution-directed mesh adaptation with physics-based criteria.

To decrease the overall computational time, integration of the governing equations is performed in parallel. An even distribution of solution blocks is generally sought on homogeneous architectures while a weighted distribution is permissible for computations performed on heterogeneous systems such as networked workstations or computational grids. To ensure efficient load balancing, blocks are organized using a Morton ordering space filling curve which co-locates nearest neighbors on the same processor [23]. The proposed AMR scheme was implemented using the message passing interface (MPI) library [24].

Ghost cells which surround the solution block and overlap cells on neighboring blocks are used to share solution content through inter-block communication. The conservation principle of the solution scheme is retained across blocks with resolution changes by us-
The first test problem considered was a two-dimensional unit square enclosure defined on $-0.5 \leq x \leq 0.5$ and $-0.5 \leq y \leq 0.5$ with a transparent medium.

For this case, all walls are black and cold ($b_w = 0$) except the bottom wall where $b_w = 1$. This case is similar to the one previously studied by Coelho [5] who employed a variety of high-resolution schemes and compared the resulting solution errors. The problem is re-considered here only to verify that the current implementation properly captures solution discontinuities. In this test case, radiation propagating along a single direction with direction cosines corresponding to $\mu = \eta = \sqrt{2}/2$ is computed. The domain is discretized into 61 by 61 uniform cells and the solution computed using various schemes. The discretization schemes tested include the first-order upwind and second-order unlimited approximations, and the TVD schemes of Venkatakrishnan [26], Barth and Jespersen [27], Van Albada et al. [28], and Van Leer [29]. The computed intensity along the $y$-direction at $x = 0$ is compared with the exact solution in Fig. 1. As expected, the upwind scheme is overly dissipative while unphysical oscillations are obtained using the second-order unlimited scheme. Limiting the second-order solution offers considerable improvement, with the Venkatakrishnan and Barth-Jespersen limiters outperforming the Van Albada and Van Leer limiters. The limiter of Venkatakrishnan was selected for all remaining test cases for its superior performance characteristics.

### 3.2 Discretization accuracy

The proposed scheme was analyzed in terms of spatial and angular accuracy through comparison with exact solutions for the RTE in rectangular enclosures. Similar to the previous test case, the enclosure for this study was taken to consist of a unit square except with all walls cold and black and containing a hot, absorbing-emitting medium with $\kappa = 10 \text{ m}^{-1}$. Exact solutions were previously derived by Cheng [30] for this particular problem. The overall error between numerical and exact solutions was defined by the change in the two-norm of the error in direction-integrated radiative intensity.

The change in error between numerical and exact solutions of the RTE with increasing number of directions of propagation is illustrated in Fig. 2a. Here, solutions were obtained on a fixed spatial grid of 64 by 64 uniform cells with either the DOM or FVM and varying levels of angular resolution. Quadrature rules used for the DOM include the $S_2$, $S_4$, $S_6$, and $S_8$ schemes of Lathrop and Carlson [31] as well as the $T_1$, $T_2$, $T_3$, and $T_4$ schemes of Thurgood et al. [32]. For the FVM, uniform angular meshes were employed over the hemisphere ($0 \leq \theta \leq \pi/2, 0 \leq \psi \leq 2\pi$) with $1 \times 2, 2 \times 4, 4 \times 8$, and $8 \times 16$ control angles in the polar and azimuthal directions, respectively. The figure confirms that reductions in error are achieved by increasing the angular resolution and that both DOM quadrature rules provide larger rates of decrease with number of directions than the FVM.

The accuracy of the spatial discretization was also assessed using a procedure similar to the one used for the angular discretization analysis. However, numerical solutions were compared to exact solutions for the DOM equations themselves instead of exact solutions of the RTE. These are spatially exact solutions to the angular approximation introduced by the DOM and therefore errors in the numerical solution are attributed to the spatial discretization only. Exact solu
tions for the DOM in rectangular enclosures were previously presented by Jessee et al. [33]. A unit square enclosure with cold and black walls that contains a hot, absorbing-emitting medium with $\kappa = 0.01 \text{ m}^{-1}$ was modelled. Results for the effect of grid resolution on the error are presented in Fig. 2b for both the upwind and TVD spatial discretizations of the $S_6$ DOM equations. The figure indicates that only first order-accuracy is achieved for both schemes in the asymptotic regime are 0.80 and 1.13 for the upwind and TVD schemes, respectively. Despite this lack of improvement in the order of accuracy over the upwind scheme as the mesh is refined, the TVD scheme still provides a far more accurate solution with fewer grid points.

### 3.3 Adaptive mesh refinement

To demonstrate the advantages of mesh refinement and the ability of the proposed method to deal with complex geometry, the AMR scheme was applied to a realistic test case which possessed both steep gradients and curved boundaries. The test case consisted of a discontinuous absorbing-emitting medium confined between two concentric circular enclosures, shown schematically in Fig. 3a. Both walls are black with an emissivity of $\varepsilon_{w1} = \varepsilon_{w2} = 1$, inner wall temperature of $T_{w1} = 100 \text{ K}$, and outer wall temperature of $T_{w2} = 0 \text{ K}$. The walls are located at $r_{w1} = 0.1 \text{ m}$ and $r_{w2} = 2.0 \text{ m}$ while the medium inside the enclosure is discontinuous at $r_3 = 0.59 \text{ m}$. The temperature and absorption coefficient for the inner gas are $T_{g1} = 100 \text{ K}$ and $\kappa_{g1} = 10 \text{ m}^{-1}$ while they are $T_{g2} = 0 \text{ K}$ and $\kappa_{g2} = 100 \text{ m}^{-1}$ for the outer surrounding gas, respectively.

Radiative heat transfer between the two concentric cylinders was studied numerically by solving the RTE using the proposed Newton-Krylov-Schwarz (NKS) algorithm and the Venkatakrishnan TVD spatial discretization scheme. The FVM was used to discretize the angular coordinate and divide the hemisphere into 4 control angles in the polar and 24 in the azimuthal direction. The circular computational domain was subdivided into an initial non-uniform, body-fitted mesh with two equally-sized blocks and 256 total cells. The AMR criterion employed here was based on the maximum $\nabla I_m$.

Computed contours for the direction-integrated intensity, $G$, along with the mesh block boundaries after 8 levels of refinement are provided in Fig. 3a. Block boundaries for an intermediate mesh refinement level, level 3, are also provided in the figure to illustrate the AMR process. The AMR algorithm correctly identified the large gradient in $G$ at $r = 0.59 \text{ m}$ produced by the discontinuity and refined the mesh in the corresponding location. A large improvement in the solution accuracy as the mesh is initially refined is observed in Fig. 3b, which depicts the effect of mesh resolution on the radial profile for $G$.

### 3.4 Parallel performance

The parallel performance of the algorithm applied to the square enclosure case was assessed by examining both the strong and weak scaling properties. These two properties are a measure of the ability to demonstrate a proportionate increase in parallel speedup with more processors. For the strong scaling test, the problem size is held fixed while the number of processors used
to perform the computation is varied. Weak scaling is measured by holding the work load per processor fixed and varying the problem size with the number of processors. These two scaling properties are measured by the parallel speedup $S_p = t_1 / t_p$ and efficiency $\eta_p = S_p / p$, where $t_1$ and $t_p$ are the total wall times required to solve the problem with 1 and $p$ processors, respectively.

### 3.4.1 Strong scaling.

Strong scaling was measured by dividing a uniform mesh of 512 by 512 cells amongst a number of equally sized blocks and solving the problem in parallel. The test was first carried out using two different meshes divided into a fixed number of blocks with either 32 by 32 or 64 by 64 cells. This allowed the work load per processor to be varied by changing the number of blocks assigned to each processor without affecting the partitioning of the mesh. As a result, only the effect of inter-processor communication on efficiency is observed since the number of residual evaluations to achieve converged solutions does not change. Each solution was said to have converged when the two-norm of the residual was reduced by 10 orders of magnitude from the original. Spatial discretization was performed using the Venkatakrishnan TVD scheme, angular discretization was performed with the $S_6$ quadrature scheme, and ILU(0) was used as the block preconditioner. GMRES was restarted every 20 inner iterations and terminated after 80 iterations or when the residual for the linear problem was reduced by one order of magnitude. The resulting relationship between parallel speedup, efficiency, and number of processors is shown in Fig. 4a for the two meshes with different fixed block sizes. Excellent parallel performance is achieved with an efficiency greater than 85% on 256 processors. A slight deviation from ideal speedup begins to occur as the number of processors is increased beyond 16 which is magnified as more processors are used.

The strong scaling test was performed a third time with the same uniform mesh of 512 by 512 cells as the one used previously except the mesh partitioning was varied such that only one block was assigned to each processor. As a result, the mesh was divided into smaller blocks as more processors were used. This test not only measures the scalability of the particular implementation as the previous test did, but also the scalability of the overall algorithm. It includes the negative effect of partitioning on convergence which results from global Schwarz preconditioning in addition to the effect of communication. The results from this test are illustrated in Fig. 4a along with those obtained using the fixed block sizes. The results indicate similar performance to the fixed block size cases with a slight improvement observed when using fewer than 256 processors.

### 3.4.2 Weak scaling.

Weak scaling performance of the proposed RTE solution algorithm is observed in Fig. 4b for two different block sizes of 32 by 32 and 64 by 64 cells. It was obtained by assigning each processor a single block and either iterating for a fixed number of Newton steps...
or until the solutions were fully converged. Solutions were deemed fully converged when the residual was reduced by ten orders of magnitude. Excellent performance for the fixed iteration case is observed up to 1024 processors (blocks) with a parallel efficiency greater than 90% relative to 64 processors. This indicates that the parallel implementation effectively minimizes the necessary inter-processor communication. However, efficiencies of only 50% and 20% were achieved on 256 and 1024 processors, respectively, when the solutions were fully converged. This drastic loss of performance is largely attributed to the effect of mesh partitioning and size on convergence history.

3.5 Serial performance

A final test was carried out to compare the serial performance of the proposed solver with other standard space-marching solution techniques discussed in this work. Two unit square enclosures were studied. The first enclosure was the same one studied in Sect. 3.2 containing a purely absorbing medium while the second enclosure contained a purely scattering medium. For the second enclosure, all walls are cold ($T = 0$ K) and black except the bottom wall which is hot and black. The emissivity of all walls was set to one. The medium is also cold but scatters photons according to the high back-scattering B2 phase function described by Kim and Lee [34]. The effect of optical thickness, spatial discretization, and mesh resolution on overall performance for each solver is discussed for both test cases.

All of the solution methods considered in the serial performance assessment made use of the $S_6$ DOM quadrature and solutions were obtained for three single-block mesh sizes: 32 by 32, 64 by 64, and 128 by 128 uniformly spaced cells. The proposed NKS scheme with a GMRES tolerance of 0.1 was used in combination with the upwind and Venkatakrishnan TVD discretizations. In addition to this, the standard space-marching solution technique outlined by Carlson and Lathrop [1] was employed with a variety of finite-volume schemes. Solutions were obtained using the space-marching technique with the upwind, central, CLAM, and the genuinely multidimensional (GM) [9] schemes. The high-resolution CLAM and GM schemes were implemented using the deferred correction procedure of Khosla and Rubin [35]. An explicit time-marching algorithm was also tested in this study for comparison purposes despite the poor performance characteristics of these types of solvers. The explicit solver makes use of FAS multigrid with a regular V-cycle and the five-stage optimally smoothing relaxation scheme of Van Leer et al. [36]. The iterations with the multigrid and Newton-Krylov solvers were stopped when residuals were reduced by 10 orders of magnitude while the convergence criterion for the space-marching solver was $\max (|\Delta I_m|) < 10^{-5}$ where $\max (|\Delta I_m|)$ is the maximum absolute change in spectral intensity between iterations.

The resulting CPU times required to solve each case using the various methods are provided in Table 1 for the purely absorbing and purely scattering media. For the absorbing case, the resulting upwind nature of the governing equations is easily handled by the space-marching technique. Both upwind and central schemes require only one iteration in each direction to reduce the residual below the specified convergence criterion. Despite the added iterations required by the
Table 1: CPU times (s) for square enclosure with absorbing-emitting and scattering medium.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>(\kappa = 0.01\ m^{-1})</th>
<th>(\kappa = 10.0\ m^{-1})</th>
<th>(\sigma_s = 0.01\ m^{-1})</th>
<th>(\sigma_s = 10.0\ m^{-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32x32</td>
<td>128x128</td>
<td>32x32</td>
<td>128x128</td>
</tr>
<tr>
<td>NKS (Upwind)</td>
<td>1.6</td>
<td>78.1</td>
<td>1.0</td>
<td>38.2</td>
</tr>
<tr>
<td>NKS (Limited)</td>
<td>1.9</td>
<td>90.7</td>
<td>2.5</td>
<td>52.1</td>
</tr>
<tr>
<td>Space-March (Upwind)</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>Space-March (Central)</td>
<td>0.0</td>
<td>0.5</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>Space-March (CLAM)</td>
<td>0.6</td>
<td>31.3</td>
<td>0.4</td>
<td>19.5</td>
</tr>
<tr>
<td>Space-March (GM)</td>
<td>0.3</td>
<td>14.5</td>
<td>0.3</td>
<td>11.4</td>
</tr>
<tr>
<td>FAS Multigrid (Upwind)</td>
<td>12.9</td>
<td>495.5</td>
<td>3.3</td>
<td>88.6</td>
</tr>
<tr>
<td>FAS Multigrid (Limited)</td>
<td>48.7</td>
<td>1612.2</td>
<td>25.4</td>
<td>313.5</td>
</tr>
</tbody>
</table>

space-marching solver when using either of the two high-resolution schemes, they still outperform the proposed solution algorithm. The combined NKS and TVD scheme is a factor of 3.2 and 4.7 times slower on average than the CLAM scheme for the optically thin (\(\kappa = 0.01\ m^{-1}\)) and thick (\(\kappa = 10.0\ m^{-1}\)) cases, respectively. The GM scheme performs slightly better than the CLAM scheme. Multigrid performs the poorest, requiring excessively long solution times. The solution time required for all schemes is observed to decrease with optical thickness.

The Newton-Krylov algorithm compared more favourably to the space-marching method when scattering was introduced. Comparing the high-resolution schemes, the Newton algorithm is at least twice as fast as the CLAM scheme and 1.3 times faster on average than the GM scheme for the optically thick case (\(\sigma_s = 10.0\ m^{-1}\)). However, similar results to those obtained for the purely absorbing case are observed near the optically thin limit (\(\sigma_s = 0.01\ m^{-1}\)). It is interesting to note that the convergence characteristics of both time-marching solvers improve as the optical thickness is increased while they decrease for the space-marching solvers. This loss of performance for the space-marching solver is due to the high back-scattering nature of the medium, which requires additional passes to propagate scattered rays in all directions. As the optical thickness increases, the coupling between intensities strengthens and is easily taken into account by the proposed scheme. Similar to the purely absorbing case, convergence is much more rapid for large optical thickness (\(\sigma_s = 10.0\ m^{-1}\)).

4 CONCLUSION

The proposed algorithm displayed excellent scaling characteristics with greater than 85 % parallel efficiency up to 256 processors. Decreasing the block size had a small negative effect on convergence due to the additional partitioning used in the Schwarz preconditioner. However, the number of residual evaluations required to obtain a converged solution can increase significantly with problem size and condition number of the system. Comparing the CPU times required for several different solution techniques and high-resolution schemes, the proposed algorithm outperformed standard TVD space-marching methods by at least a factor of two for strongly scattering media. This favorable performance was not observed for weakly scattering and purely absorbing media. Nonetheless, the algorithm proves promising for large-scale computations of more realistic cases having complex geometry that must be solved in parallel.

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