Chapter 1 Supplement: More on Euler Angles

It is a fact that any attitude can be obtained by three successive principal rotations, called an Euler sequence, provided no two adjacent rotations are about the same axis. In the book, we examined one example of an Euler sequence. In this supplement, we present a general treatment.

1 Rotation Matrix

Let i, j and k denote the first, second and third principal rotation axes in the rotation sequence, respectively. Also, let θ_1, θ_2 and θ_3 denote the first, second and third rotation angles in the sequence. Note that $i, j, k \in \{x, y, z\}$, and that $i \neq j$, $j \neq k$. We call this an i-j-k rotation sequence. Note that often times, an Euler rotation sequence is specified in terms of the numbers 1, 2, 3. Namely, 1 denotes a principal x rotation, 2 denotes a principal y rotation and 3 denotes a principal z rotation. Thus, a 1-2-3 rotation sequence is the same as a xyz rotation sequence.

We label the initial and final frames in the rotation sequence by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. We also define two other frames, an intermediate frame $\mathcal{F}_{if}$, and a transformed frame $\mathcal{F}_{tf}$. Specifically, $\mathcal{F}_{if}$ is the frame obtained from $\mathcal{F}_1$ after the first rotation (i) in the sequence, $\mathcal{F}_{tf}$ is the frame obtained from $\mathcal{F}_{if}$ after the second rotation (j) in the sequence, and the final frame is obtained from $\mathcal{F}_{tf}$ from after the final (k) rotation in the sequence.

1.1 Successive i, j, k Rotations about the Principal Axes of Successive Frames (Conventional)

The convention in the aerospace community for Euler rotation sequences is to perform the principal rotations about the coordinate axes of the successive frames. Specifically, $\mathcal{F}_{if}$ is obtained by starting at $\mathcal{F}_1$, and rotating about the i-axis of $\mathcal{F}_1$, through angle θ_1. $\mathcal{F}_{tf}$ is subsequently obtained by starting at $\mathcal{F}_{if}$, and rotating about the j-axis of $\mathcal{F}_{if}$, through angle θ_2. Finally, $\mathcal{F}_2$ is obtained by starting at $\mathcal{F}_{tf}$, and rotating about the k-axis of $\mathcal{F}_{tf}$, through angle θ_3. As such, we have

$$\mathbf{C}_{if,1} = \mathbf{C}_i(\theta_1), \;\; \mathbf{C}_{tf,if} = \mathbf{C}_j(\theta_2), \;\; \mathbf{C}_{2,tf} = \mathbf{C}_k(\theta_3).$$

Consequently, by (1.19) in the book, we have

$$\mathbf{C}_{21} = \mathbf{C}_{2,tf}\mathbf{C}_{tf,if}\mathbf{C}_{if,1},$$

which leads to

$$\mathbf{C}_{21} = \mathbf{C}_k(\theta_3)\mathbf{C}_j(\theta_2)\mathbf{C}_i(\theta_1). \tag{1}$$

From (1), it is clear that when the sequence of principal rotations are performed about the coordinate axes of the successive frames, the resulting rotation matrix is obtained by multiplying the individual principal rotation matrices in the order the rotations were performed, from right to left.

1.2 Successive i, j, k Rotations about the Principal Axes of $\mathcal{F}_1$

In some applications, it is more convenient to define Euler sequences in terms of successive rotations about the principal axes of the first frame $\mathcal{F}_1$. That is, the intermediate frame $\mathcal{F}_{if}$ is obtained by starting at $\mathcal{F}_1$, and rotating about the i-axis of $\mathcal{F}_1$, through angle θ_1. $\mathcal{F}_{tf}$ is subsequently obtained by starting at $\mathcal{F}_{if}$, and rotating about the j-axis of $\mathcal{F}_1$, through angle θ_2. Finally, $\mathcal{F}_2$ is obtained by starting at $\mathcal{F}_{tf}$, and rotating about the k-axis of $\mathcal{F}_1$, through angle θ_3.

In this case, the expression for the overall rotation in (1) is not valid. We shall now derive the appropriate expression. First of all, we immediately have

$$\mathbf{C}_{if,1} = \mathbf{C}_i(\theta_1). \tag{2}$$

Now we examine the second rotation in the sequence. The axis of rotation to get from $\mathcal{F}_{if}$ to $\mathcal{F}_{tf}$ is given by

$$\vec{\mathbf{a}} = \mathcal{F}_1^T \mathbf{e}_j = \mathcal{F}_{if}^T \mathbf{C}_{if,1} \mathbf{e}_j,$$

where $\mathbf{e}_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{e}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{e}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, are the coordinates of the unit vectors $\vec{\mathbf{x}}_1$, $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{z}}_1$ defining frame $\mathcal{F}_1$. Therefore, from (1.26) in the book, we have

$$\begin{aligned}
\mathbf{C}_{tf,if} &= \cos\theta_2 \mathbf{1} + (1-\cos\theta_2)\mathbf{C}_{if,1}\mathbf{e}_j\mathbf{e}_j^T\mathbf{C}_{if,1}^T - \sin\theta_2 \mathbf{C}_{if,1}\mathbf{e}_j^\times \mathbf{C}_{if,1}^T, \\
&= \mathbf{C}_{if,1}\left(\cos\theta_2 \mathbf{1} + (1-\cos\theta_2)\mathbf{e}_j\mathbf{e}_j^T - \sin\theta_2 \mathbf{e}_j^\times\right)\mathbf{C}_{if,1}^T, \\
&= \mathbf{C}_{if,1}\mathbf{C}_j(\theta_2)\mathbf{C}_{if,1}^T.
\end{aligned}$$

Note that we have used the fact that $(\mathbf{Ca})^\times = \mathbf{C}\mathbf{a}^\times\mathbf{C}^T$ for any $\mathbf{a} \in R^3$, and any rotation matrix $\mathbf{C}$ (see (1.21) in the book), and that $\mathbf{C}\mathbf{C}^T = \mathbf{C}^T\mathbf{C} = \mathbf{1}$. Next we obtain the composite rotational transformation from $\mathcal{F}_1$ to $\mathcal{F}_{tf}$ as

$$\begin{aligned}
\mathbf{C}_{tf,1} &= \mathbf{C}_{tf,if}\mathbf{C}_{if,1}, \\
&= \mathbf{C}_{if,1}\mathbf{C}_j(\theta_2)\mathbf{C}_{if,1}^T\mathbf{C}_{if,1}, \\
&= \mathbf{C}_i(\theta_1)\mathbf{C}_j(\theta_2),
\end{aligned} \tag{3}$$

where we have made use of (2). We now examine the third rotation in the sequence. The axis of rotation to get from $\mathcal{F}_{tf}$ to $\mathcal{F}_2$ is given by

$$\vec{\mathbf{a}} = \mathcal{F}_1^T \mathbf{e}_k = \mathcal{F}_{tf}^T \mathbf{C}_{tf,1} \mathbf{e}_k.$$

In a similar manner, we obtain from (1.26) in the book,

$$\mathbf{C}_{2,tf} = \mathbf{C}_{tf,1}\mathbf{C}_k(\theta_3)\mathbf{C}_{tf,1}^T,$$

which leads to the composite rotation from $\mathcal{F}_1$ to $\mathcal{F}_2$, given by

$$\mathbf{C}_{21} = \mathbf{C}_{2,tf}\mathbf{C}_{tf,1} = \mathbf{C}_{tf,1}\mathbf{C}_k(\theta_3),$$

and from (3), we finally obtain

$$\mathbf{C}_{21} = \mathbf{C}_i(\theta_1)\mathbf{C}_j(\theta_2)\mathbf{C}_k(\theta_3). \tag{4}$$

From (4), it is clear that when the sequence of principal rotations are performed about the coordinate axes of the first frame ($\mathcal{F}_1$), the resulting rotation matrix is obtained by multiplying the individual principal rotation matrices in the order the rotations were performed, from left to right. **That is, the order of multiplication is the opposite of that for principal rotations performed about the coordinate axes of successive frames (the conventional approach)**.

Interestingly, comparing (1) with (4), it can be seen that a conventional i-j-k Euler rotation sequence is equivalent to a k-j-i Euler rotation sequence with rotations performed about the principal axes of $\mathcal{F}_1$, with the same angles of rotation.

2 Angular Velocity in Terms of Euler Angle Rates

Making use of the fact that angular velocities are additive, we can now easily obtain an expression for the angular velocity in terms of the Euler angle rates.

2.1 Successive i, j, k Rotations about the Principal Axes of Successive Frames (Conventional)

In this section, we shall treat the conventional Euler rotation sequences (about successive principal axes), with resulting rotation matrix given by (1). Recall that the Euler sequence involved sequential transformations between the frames $\mathcal{F}_1$, $\mathcal{F}_{if}$, $\mathcal{F}_{tf}$ and $\mathcal{F}_2$.

By additivity of angular velocities (1.56) in the book, we have

$$\vec{\boldsymbol{\omega}}_{21} = \vec{\boldsymbol{\omega}}_{2,tf} + \vec{\boldsymbol{\omega}}_{tf,if} + \vec{\boldsymbol{\omega}}_{if,1}, \tag{5}$$

where $\vec{\boldsymbol{\omega}}_{21}$ is the angular velocity of the final frame $\mathcal{F}_2$ relative to the initial frame $\mathcal{F}_1$, $\vec{\boldsymbol{\omega}}_{2,tf}$ is the angular velocity of the final frame $\mathcal{F}_2$ relative to the transformed frame $\mathcal{F}_{tf}$, $\vec{\boldsymbol{\omega}}_{tf,if}$ is the angular velocity of the transformed frame $\mathcal{F}_{tf}$ relative to the intermediate frame $\mathcal{F}_{if}$, and $\vec{\boldsymbol{\omega}}_{if,1}$ is the angular velocity of the intermediate frame $\mathcal{F}_{if}$ relative to the initial frame $\mathcal{F}_1$.

Since each frame in the Euler rotation sequence is obtained by a principal rotation from the previous frame in the sequence, it is trivial to obtain the individual angular velocities on the right-hand side of (5). Note that the i-axis of frame $\mathcal{F}_1$ coincides with the i-axis of frame $\mathcal{F}_{if}$, This is because $\mathcal{F}_{if}$ is obtained by starting at $\mathcal{F}_1$, and then rotating it about the i-axis of frame $\mathcal{F}_1$. Likewise, the j-axes of frames $\mathcal{F}_{if}$ and $\mathcal{F}_{tf}$ coincide, as do the k-axes of frames $\mathcal{F}_{tf}$ and $\mathcal{F}_2$. Therefore, we have

$$\vec{\boldsymbol{\omega}}_{if,1} = \dot{\theta}_1 \vec{\mathbf{i}}_1 = \dot{\theta}_1 \vec{\mathbf{i}}_{if} = \mathcal{F}_{if}^T \mathbf{e}_i \dot{\theta}_1, \tag{6}$$

$$\vec{\boldsymbol{\omega}}_{tf,if} = \dot{\theta}_2 \vec{\mathbf{j}}_{if} = \dot{\theta}_2 \vec{\mathbf{j}}_{tf} = \mathcal{F}_{tf}^T \mathbf{e}_j \dot{\theta}_2, \tag{7}$$

$$\vec{\boldsymbol{\omega}}_{2,tf} = \dot{\theta}_3 \vec{\mathbf{k}}_{tf} = \dot{\theta}_3 \vec{\mathbf{k}}_2 = \mathcal{F}_2^T \mathbf{e}_k \dot{\theta}_3. \tag{8}$$

where $\mathbf{e}_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{e}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{e}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, are the coordinates of the unit vectors $\vec{\mathbf{x}}_a$, $\vec{\mathbf{y}}_a$ and $\vec{\mathbf{z}}_a$ defining the arbitrary (but right-handed) frame $\mathcal{F}_a$. Let us now write $\vec{\boldsymbol{\omega}}_{21}$ in $\mathcal{F}_2$ coordinates as

$$\vec{\boldsymbol{\omega}}_{21} = \mathcal{F}_2^T \boldsymbol{\omega}_{21}.$$

Substituting (6), (7) and (8) into (5), and applying the necessary rotational transformations, we have

$$\mathcal{F}_2^T \boldsymbol{\omega}_{21} = \mathcal{F}_2^T \mathbf{e}_k \dot{\theta}_3 + \mathcal{F}_2^T \mathbf{C}_{2,tf} \mathbf{e}_j \dot{\theta}_2 + \mathcal{F}_2^T \mathbf{C}_{2,if} \mathbf{e}_i \dot{\theta}_1.$$

Noting that $\mathbf{C}_{2,tf} = \mathbf{C}_k(\theta_3)$, and $\mathbf{C}_{2,if} = \mathbf{C}_k(\theta_3)\mathbf{C}_j(\theta_2)$, we can now extract the equation for $\boldsymbol{\omega}_{21}$ as

$$\boldsymbol{\omega}_{21} = \dot{\theta}_3 \mathbf{e}_k + \dot{\theta}_2 \mathbf{C}_k(\theta_3)\mathbf{e}_j + \dot{\theta}_1 \mathbf{C}_k(\theta_3)\mathbf{C}_j(\theta_2)\mathbf{e}_i. \tag{9}$$

Equation (9) can be written compactly as

$$\boldsymbol{\omega}_{21} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}, \tag{10}$$

where

$$\mathbf{S}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{C}_k(\theta_3)\mathbf{C}_j(\theta_2)\mathbf{e}_i & \mathbf{C}_k(\theta_3)\mathbf{e}_j & \mathbf{e}_k \end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}.$$

Equation (10) can in turn be inverted to obtain the Euler angle rates in terms of the angular velocity

$$\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}(\boldsymbol{\theta})\boldsymbol{\omega}_{21}. \tag{11}$$

The matrix $\mathbf{S}(\boldsymbol{\theta})$ is invertible everywhere, except at the singularity of the Euler rotation sequence. Hence, equation (11) is only valid when $\boldsymbol{\theta}$ is not at the singularity.

2.2 Successive i, j, k Rotations about the Principal Axes of $\mathcal{F}_1$

We could repeat the analysis from the previous section to obtain the kinematic equations for an i-j-k rotation sequence with all successive rotations being performed about the principal axes of the first frame, $\mathcal{F}_1$. However, we shall make use of the observation obtained in Section 3.2.2 that such a rotation sequence is equivalent to a conventional k-j-i rotation sequence, with the same angles of rotation. Therefore, from (9), the kinematics of an i-j-k rotation sequence with rotations performed about the principal axes of $\mathcal{F}_1$ (and corresponding rotation matrix given by (4)), has the kinematical equation

$$\boldsymbol{\omega}_{21} = \dot{\theta}_1\mathbf{e}_i + \dot{\theta}_2\mathbf{C}_i(\theta_1)\mathbf{e}_j + \dot{\theta}_3\mathbf{C}_i(\theta_1)\mathbf{C}_j(\theta_2)\mathbf{e}_k. \tag{12}$$