9 Controller Discretization

In most applications, a control system is implemented in a digital fashion on a computer. This implies that the measurements that are supplied to the control system must be sampled. If the sampler has period T, then the sampled value of the measurements are denoted by

$$\mathbf{y}_k = \mathbf{y}(t_k), \quad t_k = kT, \ k = 0, 1, 2, 3, \cdots \tag{1}$$

The output from the controller will take on discrete values which we denote by $\mathbf{u}_k$. The actual input to the plant be controlled is usually obtained by passing the sequence $\mathbf{u}_k$ through a zero-order hold (ZOH). The output of the ZOH is given by

$$\mathbf{u}(t) = \mathbf{u}_k, \ \ t_k \le t < t_{k+1}$$

In this section we would like to determine mathematical models for the relationship between $\mathbf{u}_k$ and $\mathbf{y}_k$ and establish a technique for digital implementation of a controller design.

9.1 Discrete-Time Plant Model

Assume that the plant to be controlled is described by a state-space model of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \ \ \mathbf{y} = \mathbf{C}\mathbf{x} \tag{2}$$

Let us first of all note that the sampled value of the output $\mathbf{y}$ satisfies

$$\mathbf{y}_k = \mathbf{y}(t_k) = \mathbf{C}\mathbf{x}(t_k) = \mathbf{C}\mathbf{x}_k$$

Given the plant model in (2), the solution is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

Let $t_0 = t_k$, $t = t_{k+1}$, and $t_{k+1} - t_k = T$. Therefore,

$$\begin{aligned}
\mathbf{x}_{k+1} &= e^{\mathbf{A}T}\mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau \\
&= e^{\mathbf{A}T}\mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}\, d\tau \mathbf{u}_k
\end{aligned}$$

Letting, $\tau' = t_{k+1} - \tau$, we have

$$\int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}\, d\tau = \int_T^0 e^{\mathbf{A}\tau'}(-d\tau')\mathbf{B} = \int_0^T e^{\mathbf{A}\tau} d\tau \mathbf{B}$$

Therefore,

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_d \mathbf{x}_k + \mathbf{B}_d \mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_d \mathbf{x}_k \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mathbf{C}_d &= \mathbf{C} \\ \mathbf{A}_d &= e^{\mathbf{A}T} \\ \mathbf{B}_d &= \int_0^T e^{\mathbf{A}\tau}\, d\tau \mathbf{B} \end{aligned}$$

This is termed the ZOH equivalent of the state-space system.

9.2 Stability of LTI Discrete-Time Systems

Consider

$$\mathbf{x}_{k+1} = \mathbf{A}_d \mathbf{x}_k, \quad \mathbf{x}_0 \text{ given}$$

It is readily verified that the solution is

$$\mathbf{x}_k = \mathbf{A}_d^k \mathbf{x}_0$$

For simplicity, assume that $\mathbf{A}_d$ has distinct eigenvalues and permits the eigen-decomposition

$$\mathbf{A}_d = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$$

Therefore,

$$\begin{aligned} \mathbf{A}_d^k &= (\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1})(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1})\cdots(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}) \\ &= \mathbf{E}\mathbf{\Lambda}^k\mathbf{E}^{-1} \end{aligned}$$

and

$$\mathbf{x}_k = \mathbf{E}\mathbf{\Lambda}^k\mathbf{E}^{-1}\mathbf{x}_0$$

Letting $\hat{\mathbf{x}}_k = \mathbf{E}^{-1}\mathbf{x}_k$, it follows that

$$\hat{\mathbf{x}}_k = \mathbf{\Lambda}^k \hat{\mathbf{x}}_0$$

where $\mathbf{\Lambda}^k = \text{diag}\{\lambda_i^k\}$. If

$$|\lambda_i| < 1, \quad i = 1, \cdots, n \tag{4}$$

then

$$\begin{aligned}
&\Rightarrow \quad \lambda_i^k \rightarrow 0 \text{ as } k \rightarrow \infty \\
&\Rightarrow \quad \mathbf{\Lambda}^k \rightarrow \mathbf{O} \text{ as } k \rightarrow \infty \\
&\Rightarrow \quad \mathbf{x}_k = \mathbf{E}\hat{\mathbf{x}}_k \rightarrow \mathbf{0} \text{ as } k \rightarrow \infty
\end{aligned}$$

If $|\lambda_i| > 1$ for some i, then

$$\begin{aligned}
&\Rightarrow \quad \lambda_i^k \rightarrow \infty \text{ as } k \rightarrow \infty \\
&\Rightarrow \quad ||\mathbf{x}_k|| \rightarrow \infty \text{ as } k \rightarrow \infty
\end{aligned}$$

Theorem 1. If the eigenvalues of $\mathbf{A}_d$ lie within the open unit disk of the complex plane, the discrete-time system is asymptotically stable.

Theorem 2. If any of the eigenvalues of $\mathbf{A}_d$ lie outside the unit disk of the complex plane, then the discrete-time system is unstable.

Consider the ZOH equivalent in (3) and assume that $\mathbf{A}$ has distinct eigenvalues. If $\mathbf{A}$ has the eigendecomposition

$$\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}, \quad \mathbf{\Lambda} = \text{diag}\{\lambda_i\}, \quad \lambda_i = \sigma_i \pm j\omega_i$$

then

$$\mathbf{A}_d = e^{\mathbf{A}T} = \mathbf{E}e^{\mathbf{\Lambda}T}\mathbf{E}^{-1}$$

Therefore

$$\lambda\{\mathbf{A}_d\} = e^{\lambda_i T} = e^{\sigma_i T}e^{j\omega_i T}$$

Now, $|e^{\lambda_i T}| = |e^{\sigma_i T}|$. Hence if $\sigma_i < 0$, then $|e^{\lambda_i T}| < 1$. Therefore, the ZOH equivalent of an asymptotically stable continuous-time system is also asymptotically stable.

9.3 Digital Control Design

There are two broad approaches to digital control design:

(i) direct discrete design using the ZOH equivalent as a model;

(ii) discretization of a continuous-time design based on the continuous-time plant model.

Initially we consider (ii) using the bilinear transformation (also called Tustin's rule or the trapezoidal rule).

Assume that we have a continuous-time controller with the following model:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \mathbf{y} = \mathbf{C}\mathbf{x} \tag{5}$$

We want to replace this with difference equations that relate the sampled values $\mathbf{y}_k$ and $\mathbf{u}_k$. Since $\mathbf{y}(t_k) = \mathbf{C}\mathbf{x}(t_k)$ we have

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

Since $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, it follows that

$$\mathbf{x}_{k+1} - \mathbf{x}_k = \int_{t_k}^{t_{k+1}} \dot{\mathbf{x}}\, dt = \mathbf{A}\int_{t_k}^{t_{k+1}} \mathbf{x}\, dt + \mathbf{B}\int_{t_k}^{t_{k+1}} \mathbf{u}\, dt$$

Now, approximate the integrals on the right-hand side using the trapezoidal rule:

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} \mathbf{x}\, dt &= \frac{T}{2}(\mathbf{x}_{k+1} + \mathbf{x}_k) \\
\int_{t_k}^{t_{k+1}} \mathbf{u}\, dt &= \frac{T}{2}(\mathbf{u}_{k+1} + \mathbf{u}_k)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{x}_{k+1} - \mathbf{x}_k &= \mathbf{A}\frac{T}{2}[\mathbf{x}_{k+1} + \mathbf{x}_k] + \mathbf{B}\frac{T}{2}[\mathbf{u}_{k+1} + \mathbf{u}_k] \\
\Rightarrow [1 - \mathbf{A}\frac{T}{2}]\mathbf{x}_{k+1} &= [1 + \mathbf{A}\frac{T}{2}]\mathbf{x}_k + \mathbf{B}\frac{T}{2}[\mathbf{u}_{k+1} + \mathbf{u}_k] \\
\Rightarrow \mathbf{x}_{k+1} &= \underbrace{[1 - \mathbf{A}\frac{T}{2}]^{-1}[1 + \mathbf{A}\frac{T}{2}]}_{\hat{\mathbf{A}}_d}\mathbf{x}_k + \underbrace{[1 - \mathbf{A}\frac{T}{2}]^{-1}\mathbf{B}\frac{T}{2}}_{\hat{\mathbf{B}}_d}[\mathbf{u}_{k+1} + \mathbf{u}_k]
\end{aligned}$$

Consider the difference equations

$$\begin{aligned}
\mathbf{z}_{k+1} &= \hat{\mathbf{A}}_d\mathbf{z}_k + \hat{\mathbf{B}}_d\mathbf{u}_k \\
\mathbf{z}_{k+2} &= \hat{\mathbf{A}}_d\mathbf{z}_{k+1} + \hat{\mathbf{B}}_d\mathbf{u}_{k+1}
\end{aligned}$$

Note that setting $\mathbf{x}_k = \mathbf{z}_k + \mathbf{z}_{k+1}$ and adding the two equations gives the desired one for $\mathbf{x}_k$. Hence

$$\begin{aligned}
\mathbf{y}_k = \mathbf{C}\mathbf{x}_k &= \mathbf{C}(\mathbf{z}_k + \mathbf{z}_{k+1}) \\
&= \mathbf{C}\mathbf{z}_k + \mathbf{C}(\hat{\mathbf{A}}_d\mathbf{z}_k + \hat{\mathbf{B}}_d\mathbf{u}_k) \\
&= \underbrace{\mathbf{C}(\mathbf{1} + \hat{\mathbf{A}}_d)}_{\hat{\mathbf{C}}_d}\mathbf{z}_k + \underbrace{\mathbf{C}\hat{\mathbf{B}}_d}_{\hat{\mathbf{D}}_d}\mathbf{u}_k
\end{aligned}$$

Hence the discrete equivalent of the state-space model in Eq. (5) is

$$\mathbf{z}_{k+1} = \hat{\mathbf{A}}_d\mathbf{z}_k + \hat{\mathbf{B}}_d\mathbf{u}_k \tag{6}$$

$$\mathbf{y}_k = \hat{\mathbf{C}}_d\mathbf{z}_k + \hat{\mathbf{D}}_d\mathbf{u}_k \tag{7}$$

where $\hat{\mathbf{A}}_d$, $\hat{\mathbf{B}}_d$, $\hat{\mathbf{C}}_d$, and $\hat{\mathbf{D}}_d$ are defined as above.

9.4 Closed-Loop Discrete-Time Stability Analysis

Assume the plant is described by its ZOH equivalent:

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d\mathbf{u}_k \\
\mathbf{y}_k &= \mathbf{C}_d\mathbf{x}_k
\end{aligned}$$

Assume that the controller is described by a discrete equivalent of the form

$$\begin{aligned}
\mathbf{z}_{k+1} &= \hat{\mathbf{A}}_d\mathbf{z}_k + \hat{\mathbf{B}}_d\mathbf{y}_k \\
-\mathbf{u}_k &= \hat{\mathbf{C}}_d\mathbf{z}_k + \hat{\mathbf{D}}_d\mathbf{y}_k
\end{aligned}$$

Closing the loop gives:

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}_d\mathbf{x}_k - \mathbf{B}_d\hat{\mathbf{C}}_d\mathbf{z}_k - \mathbf{B}_d\hat{\mathbf{D}}_d\mathbf{y}_k \\
&= (\mathbf{A}_d - \mathbf{B}_d\hat{\mathbf{D}}_d\mathbf{C}_d)\mathbf{x}_k - \mathbf{B}_d\hat{\mathbf{C}}_d\mathbf{z}_k
\end{aligned}$$

Also,

$$\mathbf{z}_{k+1} = \hat{\mathbf{A}}_d\mathbf{z}_k + \hat{\mathbf{B}}_d\mathbf{C}_d\mathbf{x}_k$$

Combining these two gives

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A}_d - \mathbf{B}_d\hat{\mathbf{D}}_d\mathbf{C}_d & -\mathbf{B}_d\hat{\mathbf{C}}_d \\ \hat{\mathbf{B}}_d\mathbf{C}_d & \hat{\mathbf{A}}_d \end{bmatrix}}_{\mathbf{A}_{comp}} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix}$$

For stability $\lambda\{\mathbf{A}_{comp}\}$ must lie within the unit disc.

9.5 Discrete-Time Optimal Control

Assume that the system to be controlled is described by difference equations of the form

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \quad \mathbf{x}_0 = \mathbf{c} \tag{8}$$

Consider a performance index of the form

$$\mathcal{J} = \tfrac{1}{2}\mathbf{x}_N^T\mathbf{S}\mathbf{x}_N + \tfrac{1}{2}\sum_{k=0}^{N-1}\mathbf{x}_k^T\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k \tag{9}$$

where

$$\mathbf{R} = \mathbf{R}^T > \mathbf{O}, \quad \mathbf{Q} = \mathbf{Q}^T \geq \mathbf{O}, \quad \mathbf{S} = \mathbf{S}^T \geq \mathbf{O},$$

We would like to determine $\mathbf{u}_k$, $k = 0, 1, 2, \cdots, N-1$ to minimize $\mathcal{J}$.

The state equation is treated as a constraint which we rewrite as

$$\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k - \mathbf{x}_{k+1} = \mathbf{0} \tag{10}$$

Let us adjoin (10) to (9) using Lagrange multipliers $\boldsymbol{\lambda}_k$, $k = 1, \cdots, N$:

$$L = \mathcal{J} + \sum_{k=0}^{N-1}\boldsymbol{\lambda}_{k+1}^T(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k - \mathbf{x}_{k+1}) \tag{11}$$

Necessary conditions for optimality are

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}_k} &= \mathbf{0}, \quad k = 1, \cdots, N \\
\frac{\partial L}{\partial \mathbf{u}_k} &= \mathbf{0}, \quad k = 1, \cdots, N-1 \\
\frac{\partial L}{\partial \boldsymbol{\lambda}_k} &= \mathbf{0}, \quad k = 1, \cdots, N
\end{aligned}$$

We have

$$L = \tfrac{1}{2}\mathbf{x}_N^T\mathbf{S}\mathbf{x}_N + \sum_{k=0}^{N-1}[\tfrac{1}{2}\mathbf{x}_k^T\mathbf{Q}\mathbf{x}_k + \tfrac{1}{2}\mathbf{u}_k^T\mathbf{R}\mathbf{u}_k + \boldsymbol{\lambda}_{k+1}^T(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k - \mathbf{x}_{k+1})]$$

Hence the optimality conditions become

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}_N} &= \mathbf{S}\mathbf{x}_N - \boldsymbol{\lambda}_N = \mathbf{0} \\
\frac{\partial L}{\partial \mathbf{x}_k} &= \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T\boldsymbol{\lambda}_{k+1} - \boldsymbol{\lambda}_k = \mathbf{0}, \quad k = 1, \cdots, N-1
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{u}_k} &= \mathbf{R}\mathbf{u}_k + \mathbf{B}^T\boldsymbol{\lambda}_{k+1} = \mathbf{0}, \quad k = 1, \cdots, N-1 \\
\frac{\partial L}{\partial \boldsymbol{\lambda}_k} &= \mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1} - \mathbf{x}_k = \mathbf{0}, \quad k = 1, \cdots, N
\end{aligned}$$

We can rewrite these as

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \quad \mathbf{x}_0 = \mathbf{c} && (12) \\
\boldsymbol{\lambda}_k &= \mathbf{A}^T\boldsymbol{\lambda}_{k+1} + \mathbf{Q}\mathbf{x}_k, \quad \boldsymbol{\lambda}_N = \mathbf{S}\mathbf{x}_N && (13) \\
\mathbf{u}_k &= -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}_{k+1} && (14)
\end{aligned}$$

Substituting (14) into (12) leads to

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}_{k+1}, \quad \mathbf{x}_0 = \mathbf{c} \\
\boldsymbol{\lambda}_k &= \mathbf{A}^T\boldsymbol{\lambda}_{k+1} + \mathbf{Q}\mathbf{x}_k, \quad \boldsymbol{\lambda}_N = \mathbf{S}\mathbf{x}_N && (15)
\end{aligned}$$

This is a discrete two-point boundary value problem. Since $\boldsymbol{\lambda}_N = \mathbf{S}\mathbf{x}_N$, let us assume

$$\boldsymbol{\lambda}_k = \mathbf{P}_k\mathbf{x}_k, \quad k = 1, \cdots, N \tag{16}$$

with $\mathbf{P}_N = \mathbf{S}$. Substituting (16) into (14) gives:

$$\begin{aligned}
\mathbf{u}_k &= -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{x}_{k+1} \\
&= -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k) \\
\Rightarrow (\mathbf{R} + \mathbf{B}^T\mathbf{P}_{k+1}\mathbf{B})\mathbf{u}_k &= -\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{A}\mathbf{x}_k \\
\Rightarrow \mathbf{u}_k &= -\hat{\mathbf{R}}_{k+1}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{A}\mathbf{x}_k \\
&= \mathbf{F}_k\mathbf{x}_k, \quad \mathbf{F}_k = -\hat{\mathbf{R}}_{k+1}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{A}
\end{aligned}$$

where

$$\hat{\mathbf{R}}_{k+1} = \mathbf{R} + \mathbf{B}^T\mathbf{P}_{k+1}\mathbf{B}$$

Hence, the optimal control is state feedback.

In order to determine $\mathbf{P}_k$, let us substitute (16) into (15):

$$\begin{aligned}
\mathbf{P}_k\mathbf{x}_k &= \mathbf{A}^T\mathbf{P}_{k+1}\mathbf{x}_{k+1} + \mathbf{Q}\mathbf{x}_k \\
&= \mathbf{A}^T\mathbf{P}_{k+1}(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k) + \mathbf{Q}\mathbf{x}_k \\
&= \mathbf{A}^T\mathbf{P}_{k+1}\mathbf{A}\mathbf{x}_k - \mathbf{A}^T\mathbf{P}_{k+1}\mathbf{B}\hat{\mathbf{R}}_{k+1}^{-1}\mathbf{B}^T\mathbf{P}_{k+1}\mathbf{A}\mathbf{x}_k + \mathbf{Q}\mathbf{x}_k
\end{aligned}$$

Since this must hold for all $\mathbf{x}_k$, the coefficient matrix on each side must match. Therefore,

$$\mathbf{P}_k = \mathbf{A}^T(\mathbf{P}_{k+1} - \mathbf{P}_{k+1}\mathbf{B}\hat{\mathbf{R}}_{k+1}^{-1}\mathbf{B}^T\mathbf{P}_{k+1})\mathbf{A} + \mathbf{Q} \tag{17}$$

This is called the discrete-time Riccati equation. It can be solved backwards given the terminal condition $\mathbf{P}_N = \mathbf{S}$. When $\mathbf{P}_k$, $k = N, N-1, \cdots, 0$ is known, the optimal feedback gains $\mathbf{F}_k$ can be determined.