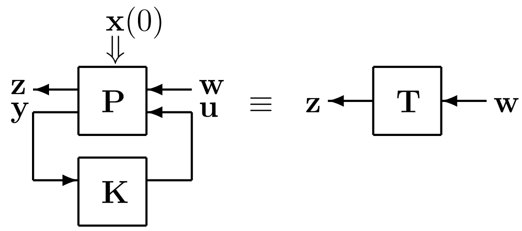
7 Optimal Control

7.1 General Control Formulation

Consider the following block diagram:



The signals are as follows:

$$
\begin{aligned}
\mathbf{z} &= \text{regulated outputs} \\
\mathbf{w} &= \text{exogenous inputs} \\
\mathbf{y} &= \text{measurements} \\
\mathbf{u} &= \text{controller output}
\end{aligned}
$$

$\mathbf{P}$ is called the generalized plant and will be typically described by equations of the form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\
\mathbf{z} &= \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\
\mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})
\end{aligned}
$$

If $\mathbf{P}$ is linear and time invariant, then these equations take the form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u} \\
\mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u}
\end{aligned}
\tag{1}
$$

$\mathbf{K}$ is the controller and is typically selected to minimize the effect of $\mathbf{w}$ or $\mathbf{x}(0)$ on $\mathbf{z}$. Some specific possibilities:
(a) For specific $\mathbf{w}$ or initial conditions $\mathbf{x}(0)$ select $\mathbf{K}$ to minimize some performance index involving $\mathbf{z}$.
(b) For $\mathbf{x}(0) = \mathbf{0}$, find $\mathbf{K}$ to minimize the L_2-gain of the closed-loop system:

$$
\mathcal{J}_\infty = \sup_{\mathbf{0} \neq \mathbf{w} \in L_2} \frac{||\mathbf{z}||_2}{||\mathbf{w}||_2} = ||\mathbf{T}|| \tag{2}
$$

Another possibility is to minimize the size of the impulse response:

$$\mathcal{J}_2 = \sqrt{\sum_i ||\mathbf{z}^{(i)}||_2^2} \tag{3}$$

where $\mathbf{z}^{(i)}$ is the response when $w_j = 0, j \neq i$, and $w_i = \delta(t)$, *i.e.*, the i^{th} input is the Dirac delta function.

We begin with approach (a).

7.2 A General Optimal Control Problem

Assume that a system is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4}$$

We seek the control input $\mathbf{u}$ which minimizes the performance index

$$\mathcal{J}[\mathbf{x}, \mathbf{u}] = \psi[\mathbf{x}(t_f)] + \int_0^{t_f} \phi(\mathbf{x}, \mathbf{u})\, dt \tag{5}$$

Our approach is to minimize $\mathcal{J}$ with Eq. (4) as a constraint. Let us introduce Lagrange multipliers $\boldsymbol{\lambda} = \text{col}\{\lambda_1, \ldots, \lambda_n\}$ and consider the augmented performance index

$$\mathcal{J}[\mathbf{x}, \mathbf{u}] = \psi[\mathbf{x}(t_f)] + \int_0^{t_f} [\phi(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^T(\mathbf{f}(\mathbf{x}, \mathbf{u}) - \dot{\mathbf{x}})]\, dt \tag{6}$$

Integrating the last term by parts gives

$$-\int_0^{t_f} \boldsymbol{\lambda}^T \dot{\mathbf{x}}\, dt = -\boldsymbol{\lambda}^T(t)\mathbf{x}(t)|_{t=0}^{t=t_f} + \int_0^{t_f} \dot{\boldsymbol{\lambda}}^T \mathbf{x}\, dt$$

Therefore,

$$\begin{aligned} \mathcal{J}[\mathbf{x}, \mathbf{u}] &= [\psi[\mathbf{x}(t)] - \boldsymbol{\lambda}^T(t)\mathbf{x}(t)]_{t=t_f} + \boldsymbol{\lambda}^T(0)\mathbf{x}(0) \\ &= + \int_0^{t_f} [H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}}^T \mathbf{x}]\, dt \end{aligned} \tag{7}$$

where we have defined the Hamiltonian

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = \phi(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \mathbf{u})$$

Consider

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}^*(t) + \delta\mathbf{x}(t) \\ \mathbf{u}(t) &= \mathbf{u}^*(t) + \delta\mathbf{u}(t) \end{aligned}$$

where $\mathbf{u}^*(t)$ is the optimal control and

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*), \quad \mathbf{x}^*(0) = \mathbf{x}_0$$

The trajectories $\mathbf{x}(t)$ and $\mathbf{u}(t)$ have been assumed to be perturbed slightly from the optimal ones. Since $\mathbf{x}(t)$ must be admissible,

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \delta\mathbf{x}(0) = \mathbf{0}$$

We seek the following expansion:

$$\mathcal{J}[\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}] = \mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] + \delta\mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] + \cdots$$

where $\delta\mathcal{J}$ contains first order terms in $\delta\mathbf{x}$ and $\delta\mathbf{u}$. If $(\mathbf{x}^*, \mathbf{u}^*)$ is optimal, then the necessary condition for minimum $\mathcal{J}$ is

$$\delta\mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] = 0 \tag{8}$$

Using (7),

$$\begin{aligned}
\mathcal{J}[\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}] &= [\psi[\mathbf{x}^* + \delta\mathbf{x}] - \boldsymbol{\lambda}^T(\mathbf{x}^* + \delta\mathbf{x})]_{t=t_f} + \boldsymbol{\lambda}^T(0)[\mathbf{x}^*(0) + \delta\mathbf{x}(0)] \\
&\quad + \int_0^{t_f} [H(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}}^T(\mathbf{x}^* + \delta\mathbf{x})]\, dt \qquad (9)
\end{aligned}$$

The following Taylor expansions can be made:

$$\begin{aligned}
\psi[\mathbf{x}^* + \delta\mathbf{x}] &= \psi[\mathbf{x}^*] + \delta\mathbf{x}^T \frac{\partial\psi}{\partial\mathbf{x}}[\mathbf{x}^*] + \cdots \\
H(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}, \boldsymbol{\lambda}) &= H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) + \delta\mathbf{x}^T \frac{\partial H}{\partial\mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \\
&\quad + \delta\mathbf{u}^T \frac{\partial H}{\partial\mathbf{u}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) + \cdots
\end{aligned}$$

Introducing these into (9) gives

$$\begin{aligned}
\mathcal{J}[\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}] &= \mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] + \left[\delta\mathbf{x}^T \left(\frac{\partial\psi}{\partial\mathbf{x}}[\mathbf{x}^*] - \boldsymbol{\lambda}\right)\right]_{t=t_f} \\
&\quad + \int_0^{t_f} [\delta\mathbf{x}^T \frac{\partial H}{\partial\mathbf{x}} + \delta\mathbf{u}^T \frac{\partial H}{\partial\mathbf{u}} + \delta\mathbf{x}^T \dot{\boldsymbol{\lambda}}]\, dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta\mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] &= \left[\delta\mathbf{x}^T \left(\frac{\partial\psi}{\partial\mathbf{x}}[\mathbf{x}^*] - \boldsymbol{\lambda}\right)\right]_{t=t_f} \\
&\quad + \int_0^{t_f} \left[\delta\mathbf{x}^T \left(\frac{\partial H}{\partial\mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}}\right) + \delta\mathbf{u}^T \frac{\partial H}{\partial\mathbf{u}}\right] dt \qquad (10)
\end{aligned}$$

Since, $H = \phi + \boldsymbol{\lambda}^T \mathbf{f}$, it follows that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \Rightarrow \dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}$$

and

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \boldsymbol{\lambda}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \tag{11}$$

For minimum $\mathcal{J}$, $\delta\mathcal{J}[\mathbf{x}^*, \mathbf{u}^*] = 0$ for all admissible $\delta\mathbf{x}$ and $\delta\mathbf{u}$. Therefore, (10) implies that

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}), \quad \boldsymbol{\lambda}(t_f) = \frac{\partial \psi}{\partial \mathbf{x}}[\mathbf{x}^*(t_f)] \tag{12}$$

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) = \mathbf{0} \tag{13}$$

Eqs. (11), (12), and (13) determine $\mathbf{u}^*$. Eq. (13) implies that we should minimize H with respect to $\mathbf{u}$ to determine $\mathbf{u}^*$ in terms of $\mathbf{x}$ and $\boldsymbol{\lambda}$. This is a special form of Pontryagin's mimimum principle: To minimize $\mathcal{J}$, minimize the Hamiltonian.

Once, $\mathbf{u}^*$ is determined as a function of $\mathbf{x}^*$ and $\boldsymbol{\lambda}$, we need to solve the following two-point boundary value problem:

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{14}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -\left[\frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*) + \frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*)\boldsymbol{\lambda}\right], \quad \boldsymbol{\lambda}(t_f) = \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) \tag{15}$$

The above problem must typically be solved numerically and leads to an open-loop strategy for the optimal control $\mathbf{u}^*$. However, there is an important case where an analytical solution can be obtained.

7.3 The Linear Quadratic Regulator (LQR) Problem

The plant dynamics are now assumed to be linear:

$$\dot{\mathbf{x}} = \underbrace{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}_{\mathbf{f}(\mathbf{x}, \mathbf{u})}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{16}$$

The performance index is given by

$$\mathcal{J} = \underbrace{\tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f)}_{\psi[\mathbf{x}(t_f)]} + \int_0^{t_f} [\underbrace{\tfrac{1}{2}(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})}_{\phi(\mathbf{x},\mathbf{u})}]\, dt$$

We assume that $\mathbf{S}$, $\mathbf{Q}$, and $\mathbf{R}$ are symmetric and positive definite. The Hamiltonian for this problem is given by

$$\begin{aligned} H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) &= \phi + \boldsymbol{\lambda}^T\mathbf{f} \\ &= \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \tfrac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \end{aligned}$$

Dropping the $(\,)^*$ notation, the two-point boundary value problem in (14) and (15) becomes

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{17}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} = -[\mathbf{Q}\mathbf{x} + \mathbf{A}^T\boldsymbol{\lambda}], \quad \boldsymbol{\lambda}(t_f) = \mathbf{S}\mathbf{x}(t_f) \tag{18}$$

Minimizing the Hamiltonian with respect to $\mathbf{u}$ gives

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{R}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0}$$

$$\Rightarrow \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(t) \tag{19}$$

Substituting (19) into (17) gives

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{20}$$

$$\dot{\boldsymbol{\lambda}} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^T\boldsymbol{\lambda}, \quad \boldsymbol{\lambda}(t_f) = \mathbf{S}\mathbf{x}(t_f) \tag{21}$$

or

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}$$

The boundary condition in (21) suggests a possible solution:

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t), \quad \mathbf{P}(t_f) = \mathbf{S} \tag{22}$$

where $\mathbf{P}$ is an $n \times n$ matrix to be determined. This is called the sweep solution or Riccati transformation. Eq. (22) implies that

$$\begin{aligned} \dot{\boldsymbol{\lambda}} &= \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}\dot{\mathbf{x}} \\ &= \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \\ &= \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}\mathbf{A}\mathbf{x} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda} \end{aligned}$$

But, $\dot{\boldsymbol{\lambda}} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^T\boldsymbol{\lambda}$ and $\boldsymbol{\lambda} = \mathbf{P}\mathbf{x}$. Therefore,

$$-\mathbf{Q}\mathbf{x} - \mathbf{A}^T\mathbf{P}\mathbf{x} = \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}\mathbf{A}\mathbf{x} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}$$

or

$$[\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}]\mathbf{x}(t) = \mathbf{0}$$

Since this must hold for all $\mathbf{x}(t)$, we conclude that

$$-\dot{\mathbf{P}}(t) = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q}, \quad \mathbf{P}(t_f) = \mathbf{S}$$

This is a matrix differential equation for $\mathbf{P}(t)$ which must be solved backward in time from $t = t_f$ to $t = 0$. It is called the Riccati equation. Note that $\mathbf{P}(t) = \mathbf{P}^T(t)$ since they satisfy the same equation. Combining the sweep solution in (22) with (19) gives

$$\begin{aligned} \mathbf{u}(t) &= -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(t) \\ &= \underbrace{-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)}_{\mathbf{F}(t)}\mathbf{x}(t) \end{aligned}$$

Therefore, the optimal control is time-varying state feedback.

It can be shown that as $t_f \to \infty$, $\mathbf{P}(t) \to \bar{\mathbf{P}}$, a constant matrix which is the solution of the algebraic Riccati equation

$$\bar{\mathbf{P}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{P}} - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{P}} + \mathbf{Q} = \mathbf{O} \tag{23}$$

Therefore, on an infinite time interval,

$$\mathbf{F} = -\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{P}} \tag{24}$$

is also constant. The closed-loop system is given by

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{x}$$

If $(\mathbf{A}, \mathbf{B})$ is controllable, then $(\mathbf{A} + \mathbf{B}\mathbf{F})$ has eigenvalues with negative real parts.

7.4 Application to Flexible Spacecraft Control

Recall that the modal motion equations for a generic flexible spacecraft model were

$$\begin{aligned} \mathbf{M}_{rr}\ddot{\boldsymbol{\eta}}_r &= \mathbf{B}_r\mathbf{u} \\ \ddot{\boldsymbol{\eta}}_e + \boldsymbol{\Omega}_e^2\boldsymbol{\eta}_e &= \mathbf{B}_e\mathbf{u} \end{aligned}$$

where

$$\boldsymbol{\eta}_e = \mathrm{col}\{\eta_\alpha\}, \quad \boldsymbol{\Omega}_e = \mathrm{diag}\{\omega_\alpha\}, \quad \mathbf{B}_e = \mathrm{col}\{\mathbf{b}_\alpha^T\}$$

A suitable choice for the state vector is

$$\mathbf{x} = \mathrm{col}\{\dot{\boldsymbol{\eta}}_r, \boldsymbol{\eta}_r, \dot{\boldsymbol{\eta}}_e, \boldsymbol{\Omega}_e\boldsymbol{\eta}_e\} \tag{25}$$

The total energy in terms of the original physical coordinates is

$$T + U = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}$$

The physical coordinates may be related to the modal coordinates by

$$\mathbf{q} = \mathbf{Q}_r\boldsymbol{\eta}_r + \sum_\alpha \mathbf{q}_\alpha\eta_\alpha$$

where

$$\begin{aligned} \mathbf{q}_\alpha^T\mathbf{M}\mathbf{q}_\beta &= \delta_{\alpha\beta},\ \mathbf{Q}_r^T\mathbf{M}\mathbf{Q}_r = \mathbf{M}_{rr},\ \mathbf{Q}_r^T\mathbf{M}\mathbf{q}_\alpha = \mathbf{O} \\ \mathbf{q}_\alpha^T\mathbf{K}\mathbf{q}_\beta &= \omega_\alpha^2\delta_{\alpha\beta},\ \mathbf{K}\mathbf{Q}_r = \mathbf{O} \end{aligned}$$

Therefore, the energy can be expressed in terms of the modal coordinates as

$$T + U = \tfrac{1}{2}\dot{\boldsymbol{\eta}}_r^T\mathbf{M}_{rr}\dot{\boldsymbol{\eta}}_r + \tfrac{1}{2}\dot{\boldsymbol{\eta}}_e^T\dot{\boldsymbol{\eta}}_e + \tfrac{1}{2}\boldsymbol{\eta}_e^T\boldsymbol{\Omega}_e^2\boldsymbol{\eta}_e$$

In the LQR problem, we minimize

$$\mathcal{J} = \tfrac{1}{2}\int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\, dt$$

A suitable choice for $\mathbf{Q}$ is

$$\mathbf{Q} = \mathrm{diag}\{\mathbf{M}_{rr}, \mathbf{K}_p, \mathbf{1}, \mathbf{1}\}, \quad \mathbf{K}_p = \mathbf{K}_p^T > \mathbf{O}$$

so that

$$\tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} = T + U + \tfrac{1}{2}\boldsymbol{\eta}_r^T\mathbf{K}_p\boldsymbol{\eta}_r$$

For $\mathbf{R}$, we can take

$$\mathbf{R} = \rho\, \mathrm{diag}\{1/u_{i,max}^2\},\ \rho > 0$$

where $-u_{i,max} \leq u_i(t) \leq u_{i,max}$ defines the actuator range.

7.5 Optimal State Estimation

We have presented an optimal way of determining the feedback gain $\mathbf{F}$ in the state feedback controller $\mathbf{u} = \mathbf{F}\mathbf{x}$. Recall that when $\mathbf{x}$ is unavailable, it has been suggested that it be replaced with an estimate $\hat{\mathbf{x}}$ which can be generated with an observer:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{26}$$

Is there a sophisticated way to generate the gain matrix $\mathbf{L}$?

One technique lies in describing the system in a stochastic framework. Here, we assume that the system is described by the state-space equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{w}(t) \tag{27}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}(t) \tag{28}$$

Here, $\mathbf{w}$ and $\mathbf{v}$ are zero-mean, Gaussian signals. They are also assumed to be white noise which means that their covariances are given by

$$\begin{aligned} \mathcal{E}\{\mathbf{w}(t)\mathbf{w}^T(\tau)\} &= \mathbf{Q}_w\delta(t-\tau) \\ \mathcal{E}\{\mathbf{v}(t)\mathbf{v}^T(\tau)\} &= \mathbf{Q}_v\delta(t-\tau) \end{aligned}$$

Here, $\mathcal{E}\{\cdot\}$ denotes the expected value. Also, $\mathbf{Q}_w$ and $\mathbf{Q}_v$ are symmetric, positive-definite matrices which measure the intensity of the white noises. The signal $\mathbf{w}$ can be thought of as a disturbance and $\mathbf{v}$ can be thought of as a sensor noise.

We desire to find the observer which minimizes the asymptotic covariance of the estimation error, *i.e.*

$$\lim_{t\to\infty} \mathcal{E}\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^T[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]\}$$

The optimal observer is known as the Kalman filter. It is given by (26) where

$$\mathbf{L} = -\mathbf{P}_e\mathbf{C}^T\mathbf{Q}_v^{-1} \tag{29}$$

and $\mathbf{P}_e$ is the solution of the Riccati equation

$$\mathbf{P}_e\mathbf{A}^T + \mathbf{A}\mathbf{P}_e - \mathbf{P}_e\mathbf{C}^T\mathbf{Q}_v^{-1}\mathbf{C}\mathbf{P}_e + \mathbf{Q}_w = \mathbf{O}$$

This presupposes that $(\mathbf{C}, \mathbf{A})$ is observable.

7.6 The Linear-Quadratic-Gaussian (LQG) Problem

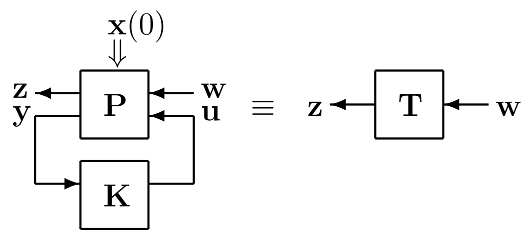
Assume that the system is described by the model in (27) where it is assumed that only the measurements $\mathbf{y}$ are available for feedback. We wish to find the control input which minimizes

$$J = \mathcal{E}\left\{ \lim_{T\to\infty} \frac{1}{2T} \int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\, dt \right\} \tag{30}$$

The solution to this problem is $\mathbf{u} = \mathbf{F}\hat{\mathbf{x}}$ where $\mathbf{F}$ is given by (24) and $\hat{\mathbf{x}}$ is determined by the state estimator in (26). The gain matrix $\mathbf{L}$ is given by (29).

7.7 The $\mathcal{H}_\infty$ Control Problem

Recall the following block diagram: Assume that $\mathbf{P}$ is described by



$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u} \\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u} \end{aligned} \tag{31}$$

We wish to select $\mathbf{K}$ to make

$$\mathcal{J}_\infty = \sup_{\mathbf{0}\neq\mathbf{w}\in L_2} \frac{||\mathbf{z}||_2}{||\mathbf{w}||_2} = ||\mathbf{T}|| < \gamma \tag{32}$$

for prescribed $\gamma > 0$. In practice, γ can be systematically reduced until no solution can be found. For scalar $T(s)$, we have shown that

$$||T|| = ||T(s)||_\infty = \sup_{\omega\in\Re} |T(j\omega)|$$

The corresponding result for multivariable systems is

$$||\mathbf{T}|| = ||\mathbf{T}(s)||_\infty = \sup_{\omega\in\Re} \sigma_{\max}\{T(j\omega)\}$$

Here $\sigma_{\max}\{\cdot\} = \max_i \sigma_i\{\cdot\}$ where σ_i are the singular values of a matrix. They are defined by

$$\sigma_i\{\mathbf{A}\} = \sqrt{\lambda_i\{\mathbf{A}^H\mathbf{A}\}}$$

where λ_i are the eigenvalues and H denotes the Hermitian (complex-conjugate transpose).

Here we simply note that under appropriate assumptions on the matrices in (31), one can determine a controller $\mathbf{K}(s) = \mathbf{F}_\infty(s\mathbf{1} - \mathbf{A}_\infty)^{-1}\mathbf{L}_\infty$ that satisfies (32). This requires solving two Riccati equations which are similar to those encountered in the LQG problem.

7.8 Positive Real Design

Recall that a SISO LTI system is passive if its transfer function is positive real, *i.e.*,

$$G(s) \text{ is analytic for } \Re e\{s\} > 0 \text{ and } \Re e\{G(j\omega)\} \geq 0$$

We would like to give a state-space characterization of passivity. Assume that a system with input $\mathbf{u}$ and $\mathbf{y}$ is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{33}$$

It corresponds to the transfer function $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$.

Claim. The system in (33) is passive if there exists $\mathbf{P} = \mathbf{P}^T > \mathbf{O}$ and $\mathbf{Q} = \mathbf{Q}^T \geq \mathbf{O}$ such that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}, \quad \mathbf{P}\mathbf{B} = \mathbf{C}^T \tag{34}$$

Proof. Consider

$$V(t) = \tfrac{1}{2}\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

We have

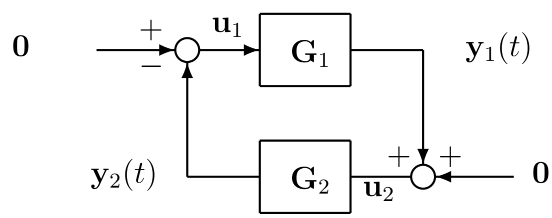
$$\begin{aligned}
\dot{V} &= \tfrac{1}{2}\mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} \\
&= \mathbf{x}^T\mathbf{P}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \\
&= \tfrac{1}{2}\mathbf{x}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{x} + \mathbf{x}^T\mathbf{C}^T\mathbf{u} \\
&= -\tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{y}^T\mathbf{u}
\end{aligned}$$

Integrating both sides from $t = 0$ to $t = T$ while taking $V(0) = 0$ gives

$$\int_0^T \mathbf{y}^T \mathbf{u}\, dt = \tfrac{1}{2} \int_0^T \mathbf{x}^T \mathbf{Q} \mathbf{x}\, dt + V(T) \geq 0$$

Hence, the mapping from $\mathbf{u}$ to $\mathbf{y}$ is passive. □

Therefore, $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$ is positive real if (34) is satisfied for appropriate $\mathbf{P}$ and $\mathbf{Q}$. If the positive semidefinite requirement on $\mathbf{Q}$ is strengthened to positive definiteness, then we say that $\mathbf{G}(s)$ is strictly positive real (SPR). This corresponds to a system that is somewhere between passive and strictly passive. It turns out that a SPR system always stabilizes a passive one. We will not prove this but instead prove that two SPR systems connected in negative feedback are globally asymptotically stable.



Let $\mathbf{G}_i(s) = \mathbf{C}_i(s\mathbf{1} - \mathbf{A}_i)^{-1}\mathbf{B}_i$, $i = 1, 2$, and assume that there exists positive definite $\mathbf{P}_i$ and $\mathbf{Q}_i$ such that

$$\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P}_i = -\mathbf{Q}_i, \quad \mathbf{P}_i\mathbf{B}_i = \mathbf{C}_i^T$$

Claim. The negative feedback interconnection of two SPR systems is globally asymptotically stable.

Proof. Let $\mathbf{x}_i$ be the state vector of $\mathbf{G}_i$ so that

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i, \quad \mathbf{y}_i = \mathbf{C}_i\mathbf{x}_i$$

Consider

$$V(t) = \tfrac{1}{2}\mathbf{x}_1^T\mathbf{P}_1\mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}_2^T\mathbf{P}_2\mathbf{x}_2$$

as a Lyapunov function candidate. Therefore,

$$\begin{aligned}
\dot{V} &= \mathbf{x}_1^T\mathbf{P}_1\dot{\mathbf{x}}_1 + \mathbf{x}_2^T\mathbf{P}_2\dot{\mathbf{x}}_2 \\
&= \mathbf{x}_1^T\mathbf{P}_1(\mathbf{A}_1\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}_1) + \mathbf{x}_2^T\mathbf{P}_2(\mathbf{A}_2\mathbf{x}_2 + \mathbf{B}_2\mathbf{u}_2) \\
&= \tfrac{1}{2}\mathbf{x}_1^T(\mathbf{P}_1\mathbf{A}_1 + \mathbf{A}_1^T\mathbf{P}_1)\mathbf{x}_1 + \mathbf{x}_1^T\mathbf{C}_1^T\mathbf{u}_1 \\
&+ \tfrac{1}{2}\mathbf{x}_2^T(\mathbf{P}_2\mathbf{A}_2 + \mathbf{A}_2^T\mathbf{P}_2)\mathbf{x}_2 + \mathbf{x}_2^T\mathbf{C}_2^T\mathbf{u}_2 \\
&= -\tfrac{1}{2}\mathbf{x}_1^T\mathbf{Q}_1\mathbf{x}_1 + \mathbf{y}_1^T\mathbf{u}_1 - \tfrac{1}{2}\mathbf{x}_2^T\mathbf{Q}_2\mathbf{x}_2 + \mathbf{y}_2^T\mathbf{u}_2
\end{aligned}$$

Since $\mathbf{u}_1 = -\mathbf{y}_2$ and $\mathbf{u}_2 = \mathbf{y}_1$, we have $\mathbf{y}_1^T\mathbf{u}_1 + \mathbf{y}_2^T\mathbf{u}_2 = 0$. Therefore,

$$\dot{V} = -\tfrac{1}{2}\mathbf{x}_1^T\mathbf{Q}_1\mathbf{x}_1 - \tfrac{1}{2}\mathbf{x}_2^T\mathbf{Q}_2\mathbf{x}_2$$

which completes the proof. $\square$

With some additional arguments, it can be shown that the feedback system is also globally asymptotically stable when $\mathbf{Q}_1$ is merely positive semidefinite and hence $\mathbf{G}_1$ is a passive but not SPR system. We have shown that flexible mechanical systems with collocated rate sensors and force actuators are passive regardless of the natural frequencies, mode shapes, and the number of modelled modes. Hence, such systems can be robustly stabilized using an SPR controller. We now present a systematic way of designing such a controller.

Given $\mathbf{G}_1(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}$ we want to design $\mathbf{G}_2(s) = \mathbf{C}_c(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{B}_c$ to be SPR.

Step 1. Pick $\mathbf{C}_c = -\mathbf{F}$ to minimize the performance index

$$\mathcal{J} = \tfrac{1}{2}\int_0^\infty (\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt$$

where $\mathbf{Q}_1$ and $\mathbf{R}$ are symmetric and positive definite. Then take $\mathbf{A}_c = \mathbf{A} - \mathbf{B}\mathbf{C}_c$ which has eigenvalues with negative real parts.

Step 2. Given $\mathbf{Q}_2 = \mathbf{Q}_2^T > \mathbf{O}$, solve the Lyapunov equation

$$\mathbf{P}\mathbf{A}_c + \mathbf{A}_c^T\mathbf{P} = -\mathbf{Q}_2$$

for the symmetric positive-definite matrix $\mathbf{P}$.

Step 3. Choose $\mathbf{B}_c$ to satisfy $\mathbf{P}\mathbf{B}_c = \mathbf{C}_c^T$ or $\mathbf{B}_c = \mathbf{P}^{-1}\mathbf{C}_c^T$.

Note that $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{R}$ are free design parameters. Also the controller does not have the form of an observer-based compensator since $\mathbf{A}_c \neq \mathbf{A} - \mathbf{B}\mathbf{C}_c - \mathbf{B}_c\mathbf{C}$ (the $\mathbf{B}_c\mathbf{C}$ term is missing).