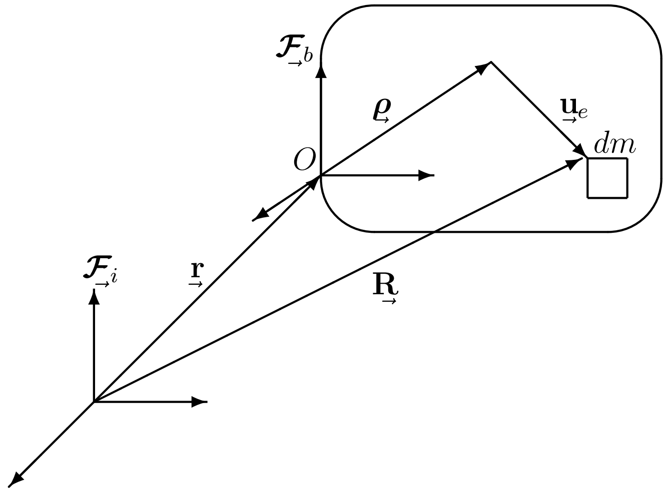
6.10 Unconstrained Elastic Bodies

Consider the following free elastic body:



Let $\boldsymbol{\mathcal{F}}_b$ be a body-fixed frame attached to the body at O. The absolute velocity and angular velocity of $\boldsymbol{\mathcal{F}}_b$ with respect to an inertial frame $\boldsymbol{\mathcal{F}}_i$ are given by

$$\begin{aligned} \dot{\underset{\rightarrow}{\mathbf{r}}} = \underset{\rightarrow}{\mathbf{v}} &= \underset{\rightarrow}{\mathcal{F}}_b^T \mathbf{v} \\ \underset{\rightarrow}{\boldsymbol{\omega}} &= \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\omega} \end{aligned}$$

The undeformed position of the mass element dm and its deformation are described by

$$\begin{aligned} \underset{\rightarrow}{\boldsymbol{\varrho}} &= \underset{\rightarrow}{\mathcal{F}}_b^T \boldsymbol{\rho} \\ \underset{\rightarrow}{\mathbf{u}}_e &= \underset{\rightarrow}{\mathcal{F}}_b^T \mathbf{u}_e(\boldsymbol{\rho}, t) \end{aligned}$$

The absolute position of dm is

$$\underset{\rightarrow}{\mathbf{R}}(\underset{\rightarrow}{\boldsymbol{\varrho}}, t) = \underset{\rightarrow}{\mathbf{r}}(t) + \underset{\rightarrow}{\boldsymbol{\varrho}}(t) + \underset{\rightarrow}{\mathbf{u}}_e(\boldsymbol{\rho}, t) \tag{1}$$

where we take as boundary conditions

$$\underset{\rightarrow}{\mathbf{u}}_e(\mathbf{0}, t) = \underset{\rightarrow}{\mathbf{0}} \tag{2}$$

$$\underset{\rightarrow}{\boldsymbol{\nabla}} \times \underset{\rightarrow}{\mathbf{u}}_e(\mathbf{0}, t) = \underset{\rightarrow}{\mathbf{0}} \tag{3}$$

The velocity of dm is

$$\begin{aligned}\underset{\rightarrow}{\mathbf{V}} = \dot{\underset{\rightarrow}{\mathbf{R}}} &= \dot{\underset{\rightarrow}{\mathbf{r}}} + \dot{\underset{\rightarrow}{\boldsymbol{\rho}}} + \dot{\underset{\rightarrow}{\mathbf{u}}}_e \\ &= \underset{\rightarrow}{\mathbf{v}} + (\overset{\circ}{\underset{\rightarrow}{\boldsymbol{\rho}}} + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{\rho}}) + (\overset{\circ}{\underset{\rightarrow}{\mathbf{u}}}_e + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathbf{u}}_e)\end{aligned}$$

Note that $\overset{\circ}{\underset{\rightarrow}{\boldsymbol{\rho}}} = \underset{\rightarrow}{\mathbf{0}}$. Writing $\underset{\rightarrow}{\mathbf{V}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T \mathbf{V}$ and expressing each term in $\boldsymbol{\mathcal{F}}_b$ gives

$$\mathbf{V} = \mathbf{v}(t) - \boldsymbol{\rho}^{\times}\boldsymbol{\omega}(t) + \dot{\mathbf{u}}_e(\boldsymbol{\rho}, t) - \mathbf{u}_e^{\times}\boldsymbol{\omega}(t) \tag{4}$$

For small elastic deformations and small angular velocities, it is defensable to neglect the last term. This will not substantially alter the form of the equations. We assume a Ritz expansion for the elastic deformation field as

$$\mathbf{u}_e(\boldsymbol{\rho}, t) = \sum_{\alpha=1}^{N_e} \boldsymbol{\psi}_{e\alpha}(\boldsymbol{\rho}) q_{e\alpha}(t) \tag{5}$$

where each basis function $\boldsymbol{\psi}_{e\alpha}$ satisfies the boundary conditions in (2) and (3). We call this a *constrained modal expansion* and a suitable choice for the $\boldsymbol{\psi}_{e\alpha}$ are the constrained mode shapes. With this expansion and the neglect of the indicated term the velocity field can be written as

$$\begin{aligned}\mathbf{V}(\boldsymbol{\rho}, t) &= [\mathbf{1} \;\; -\boldsymbol{\rho}^{\times}] \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix} + \dot{\mathbf{u}}_e(\boldsymbol{\rho}, t) \\ &= \boldsymbol{\Psi}_r \boldsymbol{\nu}(t) + \sum_{\alpha=1}^{N_e} \boldsymbol{\psi}_{e\alpha}(\boldsymbol{\rho}) \dot{q}_{e\alpha}(t)\end{aligned}$$

where

$$\boldsymbol{\Psi}_r = [\mathbf{1} \;\; -\boldsymbol{\rho}^{\times}], \quad \boldsymbol{\nu} = \mathrm{col}\{\mathbf{v}, \boldsymbol{\omega}\} \tag{6}$$

Using this expression for the velocity of $dm = \sigma(\boldsymbol{\rho})dV$, the total kinetic energy can be written as

$$\begin{aligned}T &= \tfrac{1}{2}\int_V \underset{\rightarrow}{\mathbf{V}} \cdot \underset{\rightarrow}{\mathbf{V}}\, dm \\ &= \tfrac{1}{2}\int_V \mathbf{V}^T\mathbf{V}\, \sigma(\boldsymbol{\rho})\, dV \\ &= \tfrac{1}{2}\int_V [\boldsymbol{\nu}^T \boldsymbol{\Psi}_r^T + \sum_{\alpha=1}^{N_e} \boldsymbol{\psi}_{e\alpha}^T \dot{q}_{e\alpha}][\boldsymbol{\Psi}_r \boldsymbol{\nu} + \sum_{\beta=1}^{N_e} \boldsymbol{\psi}_{e\beta}\dot{q}_{e\beta}]\, dm \\ &= \tfrac{1}{2}\Bigg\{\boldsymbol{\nu}^T \int_V \begin{bmatrix} \mathbf{1} & -\boldsymbol{\rho}^{\times} \\ \boldsymbol{\rho}^{\times} & -\boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times} \end{bmatrix} dm\, \boldsymbol{\nu}\end{aligned}$$

$$
\begin{aligned}
&+ \quad \boldsymbol{\nu}^T \sum_{\alpha=1}^{N_e} \int_V \begin{bmatrix} \mathbf{1} \\ \boldsymbol{\rho}^\times \end{bmatrix} \boldsymbol{\psi}_{e\alpha} \, dm \dot{q}_{e\alpha} \\
&+ \quad \sum_{\alpha=1}^{N_e} \dot{q}_{e\alpha} \int_V \boldsymbol{\psi}_{e\alpha}^T [\mathbf{1} \;\; -\boldsymbol{\rho}^\times] \, dm \, \boldsymbol{\nu} \\
&+ \quad \sum_{\alpha=1}^{N_e} \sum_{\beta=1}^{N_e} \int_V \boldsymbol{\psi}_{e\alpha}^T \boldsymbol{\psi}_{e\beta} \, dm \dot{q}_{e\alpha} \dot{q}_{e\beta} \Big\}
\end{aligned}
$$

We can simplify the appearance of this expression by defining the following matrices:

$$
\mathbf{M}_{rr} = \begin{bmatrix} m\mathbf{1} & -\mathbf{c}^\times \\ \mathbf{c}^\times & \mathbf{J} \end{bmatrix} \tag{7}
$$

is the rigid mass matrix and

$$
\begin{aligned}
m &= \int_V dm \\
\mathbf{c} &= \int_V \boldsymbol{\rho} \, dm \\
\mathbf{J} &= -\int_V \boldsymbol{\rho}^\times \boldsymbol{\rho}^\times \, dm
\end{aligned}
$$

The elastic mass matrix is

$$
\mathbf{M}_{ee} = \mathrm{matrix}\{\int_V \boldsymbol{\psi}_{e\alpha}^T \boldsymbol{\psi}_{e\beta} \, dm\} \tag{8}
$$

and we define the rigid-elastic coupling matrix by

$$
\mathbf{M}_{re} = \begin{bmatrix} \mathrm{row}\{\mathbf{P}_\alpha\} \\ \mathrm{row}\{\mathbf{H}_\alpha\} \end{bmatrix} \tag{9}
$$

where

$$
\mathbf{P}_\alpha = \int_V \boldsymbol{\psi}_{e\alpha} \, dm
$$

are the modal momentum coefficients and

$$
\mathbf{H}_\alpha = \int_V \boldsymbol{\rho}^\times \boldsymbol{\psi}_{e\alpha} \, dm
$$

are the modal angular momentum coefficients. With these definitions, the kinetic energy becomes

$$
T = \tfrac{1}{2} [\boldsymbol{\nu}^T \;\; \dot{\mathbf{q}}_e^T] \begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{re}^T & \mathbf{M}_{ee} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu} \\ \dot{\mathbf{q}}_e \end{bmatrix}
$$

where $\mathbf{q}_e = \mathrm{col}\{q_{e\alpha}\}$. Clearly, the mass matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{re}^T & \mathbf{M}_{ee} \end{bmatrix} \tag{10}$$

is symmetric and it is positive definite since $T > 0$ provided that $\boldsymbol{\nu}$ and $\dot{\mathbf{q}}_e$ are not both identically zero.

Strain Energy

Recall that the strain energy of an elastic body is given by

$$U = \tfrac{1}{2} \int_{\mathcal{V}} \boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} \, dV$$

Using the expansion in (5), the strains can be written as

$$\boldsymbol{\varepsilon}(\mathbf{u}_e) = \boldsymbol{\varepsilon}(\sum_\alpha \boldsymbol{\psi}_{e\alpha} q_{e\alpha}) = \sum_\alpha \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{e\alpha}) q_{e\alpha}$$

Inserting this into the strain energy yields

$$\begin{aligned} U &= \tfrac{1}{2} \sum_\alpha \sum_\beta \underbrace{\int_V \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{e\alpha}) \mathbf{E} \boldsymbol{\varepsilon}(\boldsymbol{\psi}_{e\beta}) \, dV}_{K_{ee,\alpha\beta}} q_{e\alpha} q_{e\beta} \\ &= \tfrac{1}{2} \mathbf{q}_e^T \mathbf{K}_{ee} \mathbf{q}_e \end{aligned}$$

Consistent with the definition of the mass matrix in (10) we define

$$\mathbf{K} = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{ee} \end{bmatrix} = \mathbf{K}^T \geq \mathbf{O} \tag{11}$$

Equations of Motion

Note that $\boldsymbol{\nu}^T = [\mathbf{v}^T \ \ \boldsymbol{\omega}^T]$ are not integrable coordinates. However, writing

$$\underset{\rightarrow}{\mathbf{r}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}{}_i^T \mathbf{r}$$

and establishing an Euler sequence $\boldsymbol{\theta}$ which parametrizes $\mathbf{C}_{bi}(\boldsymbol{\theta})$ yields the rigid coordinates

$$\mathbf{q}_r = \mathrm{col}\{\mathbf{r}, \boldsymbol{\theta}\}$$

with

$$\boldsymbol{\nu} = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{bi}(\boldsymbol{\theta}) & \mathbf{O} \\ \mathbf{O} & \mathbf{S}(\boldsymbol{\theta}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\boldsymbol{\theta}} \end{bmatrix}$$

The generalized coordinates for the entire system become

$$\mathbf{q} = \mathrm{col}\{\mathbf{q}_r, \mathbf{q}_e\}$$

In order to simplify things, let us assume that $\{\dot{\mathbf{r}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}\}$ are small and hence so are $\mathbf{v}$ and $\boldsymbol{\omega}$. This allows us to make the following approximations:

$$\begin{aligned} \mathbf{C}_{bi} &\doteq \mathbf{1} - \boldsymbol{\theta}^{\times}, \quad \boldsymbol{\omega} = \dot{\boldsymbol{\theta}} \\ \boldsymbol{\nu} &\doteq \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} = \dot{\mathbf{q}}_r \end{aligned}$$

Therefore, the above equations allow us to write the energies as

$$T = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} \tag{12}$$

$$U = \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \tag{13}$$

The position of the mass element dm is

$$\begin{aligned} \underset{\rightarrow}{\mathbf{R}} &= \underset{\rightarrow}{\mathbf{r}} + \underset{\rightarrow}{\boldsymbol{\rho}} + \underset{\rightarrow}{\mathbf{u}}_e \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_i^T[\mathbf{r} + \mathbf{C}_{bi}^T(\boldsymbol{\rho} + \mathbf{u}_e)] \\ &\doteq \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_i^T[\mathbf{r} + (\mathbf{1} + \boldsymbol{\theta}^{\times})(\boldsymbol{\rho} + \mathbf{u}_e)] \\ &\doteq \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_i^T[\boldsymbol{\rho} + \mathbf{r} - \boldsymbol{\rho}^{\times}\boldsymbol{\theta} + \mathbf{u}_e] \end{aligned}$$

where $\mathbf{u}_e = \sum_\alpha \boldsymbol{\psi}_{e\alpha} q_{e\alpha}(t)$. Hence, a virtual displacment of dm is given by

$$\delta\underset{\rightarrow}{\mathbf{R}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_i^T[\delta\mathbf{r} - \boldsymbol{\rho}^{\times}\delta\boldsymbol{\theta} + \sum_\alpha \boldsymbol{\psi}_{e\alpha}\delta q_{e\alpha}]$$

The virtual work performed by $\underset{\rightarrow}{\mathbf{f}}_e(\underset{\rightarrow}{\boldsymbol{\rho}}, t) = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b^T\mathbf{f}_e$ is

$$\begin{aligned} \delta W_e &= \int_V \underset{\rightarrow}{\mathbf{f}}_e \cdot \delta\underset{\rightarrow}{\mathbf{R}}\, dV \\ &= \int_V \mathbf{f}_e^T[\delta\mathbf{r} - \boldsymbol{\rho}^{\times}\delta\boldsymbol{\theta} + \sum_\alpha \boldsymbol{\psi}_{e\alpha}\delta q_{e\alpha}]\, dV \\ &= \delta\mathbf{r}^T\int_V \mathbf{f}_e\, dV + \delta\boldsymbol{\theta}^T\int_V \boldsymbol{\rho}^{\times}\mathbf{f}_e\, dV + \sum_\alpha \delta q_{e\alpha}\int_V \boldsymbol{\psi}_{e\alpha}^T\mathbf{f}_e\, dV \\ &= [\delta\mathbf{r}^T\ \delta\boldsymbol{\theta}^T\ \delta\mathbf{q}_e^T]\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \\ \widehat{\mathbf{f}}_e \end{bmatrix} \\ &= \delta\mathbf{q}^T\widehat{\mathbf{f}} \end{aligned} \tag{14}$$

Here,

$$\hat{\mathbf{f}} = \mathrm{col}\{\mathbf{F}, \mathbf{G}, \hat{\mathbf{f}}_e\}, \hat{\mathbf{f}}_e = \mathrm{col}\{\hat{f}_{e\alpha}\}$$

where

$$\begin{aligned}
\mathbf{F} &= \int_V \mathbf{f}_e \, dV = \text{total force on } \mathcal{B} \\
\mathbf{G} &= \int_V \boldsymbol{\rho}^{\times} \mathbf{f}_e \, dV = \text{total torque on } \mathcal{B} \text{ (about } O\text{)} \\
\hat{f}_{e\alpha} &= \int_V \boldsymbol{\psi}_{e\alpha}^T \mathbf{f}_e \, dV
\end{aligned}$$

Applying Lagrange's equations to the energy expressions in Eqs. (12), (13), and (14) yields the motion equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \hat{\mathbf{f}} \tag{15}$$

or

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{re}^T & \mathbf{M}_{ee} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{f}}_r \\ \hat{\mathbf{f}}_e \end{bmatrix}, \quad \hat{\mathbf{f}}_r = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \tag{16}$$

6.11 Unconstrained Modes

Consider the eigenproblem corresponding to (15):

$$-\omega_\alpha^2 \mathbf{M}\mathbf{q}_\alpha + \mathbf{K}\mathbf{q}_\alpha = \mathbf{0} \tag{17}$$

Corresponding to nonzero ω_α are a series of orthonormal eigenvectors as before:

$$\mathbf{q}_\alpha^T \mathbf{M} \mathbf{q}_\beta = \delta_{\alpha\beta} \tag{18}$$

$$\mathbf{q}_\alpha^T \mathbf{K} \mathbf{q}_\beta = \omega_\alpha^2 \delta_{\alpha\beta} \tag{19}$$

where $\alpha, \beta = 1 \cdots N_e$.

Since $\mathbf{K}$ is merely positive-semidefinite, there are zero-frequency (rigid body) modes satisfying:

$$\mathbf{K}\mathbf{Q}_r = \mathbf{O} \tag{20}$$

Given the form of $\mathbf{K}$, $\mathbf{Q}_r$ has the form

$$\mathbf{Q}_r = \begin{bmatrix} \mathbf{1}_{6\times 6} \\ \mathbf{O} \end{bmatrix}$$

corresponding to three translational and three rotational rigid-body modes. Using (20) in conjunction with the eigenproblem in (17) yields

$$\begin{aligned} \mathbf{q}_\alpha^T \mathbf{K} \mathbf{Q}_r &= \mathbf{O}, \quad \alpha = 1, 2, 3, \cdots && (21) \\ \mathbf{q}_\alpha^T \mathbf{M} \mathbf{Q}_r &= \mathbf{O}, \quad \alpha = 1, 2, 3, \cdots && (22) \\ \mathbf{Q}_r^T \mathbf{M} \mathbf{Q}_r &= \mathbf{M}_{rr} && (23) \end{aligned}$$

Let us expand the solution of (15) in terms of the rigid and vibration modes:

$$\begin{aligned} \mathbf{q}(t) &= \mathbf{Q}_r \boldsymbol{\eta}_r(t) + \sum_{\alpha=1}^{N_e} \mathbf{q}_\alpha \eta_\alpha(t), \quad \boldsymbol{\eta}_r(t) \mathrm{col}\{\mathbf{r}_0, \boldsymbol{\theta}_0\} \\ \Rightarrow \begin{bmatrix} \mathbf{r}(t) \\ \boldsymbol{\theta}(t) \\ \mathbf{q}_e(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{r}_0(t) \\ \boldsymbol{\theta}_0(t) \\ \mathbf{0} \end{bmatrix} + \sum_{\alpha=1}^{N_e} \begin{bmatrix} \mathbf{r}_\alpha \\ \boldsymbol{\theta}_\alpha \\ \widehat{\mathbf{q}}_{e\alpha} \end{bmatrix} \eta_\alpha(t) \end{aligned}$$

Substituting this into (15) and premultiplying by $\mathbf{Q}_r^T$ and $\mathbf{q}_\alpha^T$ gives the unconstrained modal equations

$$\begin{aligned} \mathbf{M}_{rr} \ddot{\boldsymbol{\eta}}_r &= \widehat{\mathbf{f}}_r && (24) \\ \ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha &= \mathbf{q}_\alpha^T \widehat{\mathbf{f}}, \quad \alpha = 1, 2, 3, \ldots && (25) \end{aligned}$$

Note that the first of these would yield the body's motion if it were rigid. For a flexible body, we see that the vibration modes contribute to the "rigid" coordinates $\mathbf{q}_r(t)$.

Example. Longitudinal Vibrations of an Elastic Rod