4 Euler Parameters (Quaternions)

4.1 Kinematical Relationships

Euler's Theorem: The most general motion of a rigid body with one point fixed is a rotation (ϕ) about an axis $\mathbf{a} = [a_1\ a_2\ a_3]^T$ ($\mathbf{a}^T\mathbf{a} = 1$) through that point.

Recall that the rotation matrix corresponding to this situation is

$$\mathbf{C}(\mathbf{a}, \phi) = \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times \tag{1}$$

Therefore, $\mathbf{C}$ can be parametrized by the four quantities $\{\mathbf{a}, \phi\}$.

Consider the rotation matrix corresponding to two consecutive roations:

$$\mathbf{C}(\mathbf{a}_3, \phi_3) = \mathbf{C}(\mathbf{a}_2, \phi_2)\mathbf{C}(\mathbf{a}_1, \phi_1)$$

Expanding the product eventually leads to

$$\cos\frac{\phi_3}{2} = \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\mathbf{a}_1^T\mathbf{a}_2 \tag{2}$$

$$\sin\frac{\phi_3}{2}\mathbf{a}_3 = \mathbf{a}_1\sin\frac{\phi_1}{2}\cos\frac{\phi_2}{2} + \mathbf{a}_2\cos\frac{\phi_1}{2}\sin\frac{\phi_2}{2} + \mathbf{a}_1^\times\mathbf{a}_2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2} \tag{3}$$

This suggests that the combinations

$$\boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{\phi}{2}, \quad \eta = \varepsilon_4 = \cos\frac{\phi}{2} \tag{4}$$

are useful. Note that

$$\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} + \eta^2 = 1$$

The quantities $\{\boldsymbol{\varepsilon}, \eta\}$ are called Euler parameters.

From (1),

$$\begin{aligned}
\mathbf{C} &= (2\cos^2\frac{\phi}{2} - 1)\mathbf{1} + 2\sin^2\frac{\phi}{2}\mathbf{a}\mathbf{a}^T - 2\cos\frac{\phi}{2}\sin\frac{\phi}{2}\mathbf{a}^\times \\
&= (1 - 2\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - 2\eta\boldsymbol{\varepsilon}^\times \\
&= \begin{bmatrix} 1 - 2(\varepsilon_2^2 + \varepsilon_3^2) & 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\eta) & 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\eta) \\ 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\eta) & 1 - 2(\varepsilon_1^2 + \varepsilon_3^2) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\eta) \\ 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\eta) & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\eta) & 1 - 2(\varepsilon_1^2 + \varepsilon_2^2) \end{bmatrix}
\end{aligned} \tag{5}$$

Given **C** we can extract the Euler parameters using the following relations:

$$\begin{aligned} C_{11} + C_{22} + C_{33} &= 3 - 4(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) \\ &= 3 - 4(1 - \eta^2) \\ &= -1 + 4\eta^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \eta &= \pm\tfrac{1}{2}\sqrt{1 + C_{11} + C_{22} + C_{33}} \\ \boldsymbol{\varepsilon} &= \frac{1}{4\eta}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \qquad (\eta \neq 0) \end{aligned}$$

Taking the positive root in the first of these corresponds to $0 \le \phi \le \pi$. If $\eta = 0$,

$$\begin{aligned} \mathbf{C} &= -\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T \\ &= \begin{bmatrix} 2\varepsilon_1^2 - 1 & 2\varepsilon_1\varepsilon_2 & 2\varepsilon_1\varepsilon_3 \\ 2\varepsilon_1\varepsilon_2 & 2\varepsilon_2^2 - 1 & 2\varepsilon_2\varepsilon_3 \\ 2\varepsilon_1\varepsilon_3 & \varepsilon_2\varepsilon_3 & 2\varepsilon_3^2 - 1 \end{bmatrix} \end{aligned} \tag{6}$$

Hence,

$$\varepsilon_i = \pm\sqrt{\tfrac{1}{2}(C_{ii} + 1)}, \quad i = 1, 2, 3$$

and the signs are resolved by considering the off-diagonal terms in (6).

From (2) and (3),

$$\eta_3 = \eta_1\eta_2 - \boldsymbol{\varepsilon}_1^T\boldsymbol{\varepsilon}_2 \tag{7}$$

$$\boldsymbol{\varepsilon}_3 = \eta_2\boldsymbol{\varepsilon}_1 + \eta_1\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_1^\times\boldsymbol{\varepsilon}_2 \tag{8}$$

These can be written in two different ways:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\varepsilon}_3 \\ \eta_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} \eta_2\mathbf{1} - \boldsymbol{\varepsilon}_2^\times & \boldsymbol{\varepsilon}_2 \\ -\boldsymbol{\varepsilon}_2^T & \eta_2 \end{bmatrix}}_{\mathbf{Q}(\boldsymbol{\varepsilon}_2, \eta_2)} \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \eta_1 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \eta_1\mathbf{1} + \boldsymbol{\varepsilon}_1^\times & \boldsymbol{\varepsilon}_1 \\ -\boldsymbol{\varepsilon}_1^T & \eta_1 \end{bmatrix}}_{\bar{\mathbf{Q}}(\boldsymbol{\varepsilon}_1, \eta_1)} \begin{bmatrix} \boldsymbol{\varepsilon}_2 \\ \eta_2 \end{bmatrix} \end{aligned}$$

Rate Kinematics

Using,

$$\boldsymbol{\omega}^{\times} = -\dot{\mathbf{C}}\mathbf{C}^T$$

in conjunction with (1) for $\mathbf{C}(\mathbf{a}, \phi)$ leads to

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{a} - (1 - \cos\phi)\mathbf{a}^{\times}\dot{\mathbf{a}} + \sin\phi\dot{\mathbf{a}}$$

If $\mathbf{a}$ is constant, this reduces to $\boldsymbol{\omega} = \dot{\phi}\mathbf{a}$ as expected. It can also be shown that

$$\begin{aligned} \dot{\phi} &= \mathbf{a}^T\boldsymbol{\omega} \\ \dot{\mathbf{a}} &= \tfrac{1}{2}[\mathbf{a}^{\times} - \cot\frac{\phi}{2}\mathbf{a}^{\times}\mathbf{a}^{\times}]\boldsymbol{\omega} \end{aligned}$$

Using these results in conjunction with (4) leads to

$$\begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\eta} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -\boldsymbol{\omega}^{\times} & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix} \tag{9}$$

4.2 Quaternion-Based Attitude Control

Let $\mathbf{C}(\mathbf{a}, \phi)$ denote the true attitude, $\mathbf{C}(\mathbf{a}_d, \phi_d)$ denote the desired attitude, and $\mathbf{C}(\mathbf{a}_e, \phi_e)$ denote the "error" attitude. Therefore,

$$\mathbf{C}(\mathbf{a}, \phi) = \mathbf{C}(\mathbf{a}_e, \phi_e)\mathbf{C}(\mathbf{a}_d, \phi_d)$$

or

$$\begin{aligned} \mathbf{C}(\mathbf{a}_e, \phi_e) &= \mathbf{C}(\mathbf{a}, \phi)\mathbf{C}^T(\mathbf{a}_d, \phi_d) \\ &= \mathbf{C}(\mathbf{a}, \phi)\mathbf{C}(\mathbf{a}_d, -\phi_d) \end{aligned}$$

In terms of quaternions, we have

$$\begin{bmatrix} \boldsymbol{\varepsilon}_e \\ \eta_e \end{bmatrix} = \bar{\mathbf{Q}}(-\boldsymbol{\varepsilon}_d, \eta_d)\begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix} = \begin{bmatrix} \eta_d\mathbf{1} - \boldsymbol{\varepsilon}_d^{\times} & -\boldsymbol{\varepsilon}_d \\ \boldsymbol{\varepsilon}_d^T & \eta_d \end{bmatrix}\begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}$$

Note that if $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_d$ and $\eta = \eta_d$, then

$$\boldsymbol{\varepsilon}_e = \mathbf{0}, \quad \eta_e = 1$$

The motion equations governing the system are

$$\begin{aligned} \mathbf{I}\dot{\boldsymbol{\omega}} &+ \boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \mathbf{u} \\ \dot{\boldsymbol{\varepsilon}} &= -\tfrac{1}{2}\boldsymbol{\omega}^{\times}\boldsymbol{\varepsilon} + \tfrac{1}{2}\eta\boldsymbol{\omega} \\ \dot{\eta} &= -\tfrac{1}{2}\boldsymbol{\omega}^{T}\boldsymbol{\varepsilon} \end{aligned}$$

We propose the following controller:

$$\mathbf{u}(t) = -\mathbf{K}_d\boldsymbol{\omega}(t) - k\boldsymbol{\varepsilon}_e(t), \quad \mathbf{K}_d = \mathbf{K}_d^T,\ k > 0 \tag{10}$$

Claim: The equilibrium $\boldsymbol{\omega} = \mathbf{0}$, $\varepsilon_e = \mathbf{0}$, and $\eta_e = 1$ is globally asymptotically stable.

Proof: For simplicity consider $\boldsymbol{\varepsilon}_d = \mathbf{0}$, $\eta_d = 1$, *i.e.*, $\phi_d = 0$. Therefore, $\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon}$ and $\eta_e = \eta$. For a Lyapunov function, we adopt

$$V(t) = \tfrac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega} + k[\boldsymbol{\varepsilon}_e^T\boldsymbol{\varepsilon}_e + (\eta_e - 1)^2]$$

which is strictly positive unless $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\varepsilon}_e = \mathbf{0}$, $\eta_e = 1$. Therefore

$$\begin{aligned} \dot{V} &= \boldsymbol{\omega}^T\mathbf{I}\dot{\boldsymbol{\omega}} + 2k[\boldsymbol{\varepsilon}^T\dot{\boldsymbol{\varepsilon}} + (\eta - 1)\dot{\eta}] \\ &= \boldsymbol{\omega}^T(-\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{u}) + 2k[\boldsymbol{\varepsilon}^T(-\tfrac{1}{2}\boldsymbol{\omega}^{\times}\boldsymbol{\varepsilon} + \tfrac{1}{2}\eta\boldsymbol{\omega}) + (\eta - 1)(-\tfrac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\varepsilon})] \\ &= -\boldsymbol{\omega}^T(\mathbf{K}_d\boldsymbol{\omega} + k\boldsymbol{\varepsilon}) + k[\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}^{\times}\boldsymbol{\omega} + \eta\boldsymbol{\varepsilon}^T\boldsymbol{\omega} - \eta\boldsymbol{\varepsilon}^T\boldsymbol{\omega} + \boldsymbol{\omega}^T\boldsymbol{\varepsilon}] \\ &= -\boldsymbol{\omega}^T\mathbf{K}_d\boldsymbol{\omega} \leq 0 \end{aligned}$$

Applying LaSalle's theorem:

$$\dot{V} = 0 \Rightarrow \boldsymbol{\omega} = \mathbf{0} \Rightarrow \dot{\boldsymbol{\omega}} = \mathbf{0}$$

Hence,

$$\mathbf{u} = -\mathbf{K}_d\boldsymbol{\omega} - k\boldsymbol{\varepsilon}_e = \mathbf{0}$$

Therefore $\boldsymbol{\varepsilon}_e = \mathbf{0}$ which implies that $\eta_e^2 = 1$. Therefore, $\eta_e = \pm 1$ which corresponds to the same attitude since $\cos(\phi/2) = \pm 1$ implies that $\phi = 0$ or $\phi = 2\pi$.