AER1503H
Spacecraft Dynamics and Control II

A course presented at the
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1 Stability Theory

1.1 General Idea of a System

A system is a mathematical representation of the relationship between a collection of inputs ("the causes") and outputs ("the effects"). Typically the inputs and outputs are functions of time and the relationship between them is described by differential equations and static mappings. The most general representation we shall require is given by:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{x}(0) = \mathbf{x}_0 &\quad (1) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, t) &\quad (2) \end{aligned}$$

where

$$\begin{aligned} t &\in \Re^+ = [0, \infty] \\ \mathbf{x} &: \Re^+ \to \Re^n = \text{state vector} \\ \mathbf{u} &: \Re^+ \to \Re^m = \text{control or input vector} \\ \mathbf{y} &: \Re^+ \to \Re^p = \text{output vector} \\ \mathbf{f} &: \Re^n \times \Re^m \times \Re^+ \to \Re^n \\ \mathbf{h} &: \Re^n \times \Re^m \times \Re^+ \to \Re^p \end{aligned}$$

Example

$$\begin{aligned} \dot{x}(t) &= -2x(t) + u(t), \quad x(0) = x_0 \\ y(t) &= 4x(t) + 5u(t) \end{aligned}$$

The solution of this system is given by

$$\begin{aligned} x(t) &= \exp(-2t)x_0 + \int_0^t \exp[-2(t-\tau)]u(\tau)\, d\tau \\ y(t) &= 4x(t) + 5u(t) \\ &= 4\exp(-2t)x_0 + 4\int_0^t \exp[-2(t-\tau)]u(\tau)\, d\tau + 5u(t) \end{aligned}$$

It is clear that the output y is effected by the input u and the initial conditions x_0. There are two distinct notions of stability depending on which of these is selected as the primary "cause" of the "effect".

In order to extend these ideas, we need to establish some ideas of "function spaces," *i.e.*, the collection of objects which contain $\mathbf{u}$ and $\mathbf{y}$.

Example: Euler's Equation

Euler's equation governing the evolution of a rigid body's angular velocity (of a body-fixed centre of mass frame) $\mathbf{y}(t) = \boldsymbol{\omega}(t) = [\omega_1\ \omega_2\ \omega_3]^T$ under the action of an external torque $\mathbf{u}(t) = [u_1\ u_2\ u_3]^T$ is given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \mathbf{u} \tag{3}$$

where $\mathbf{I} = \mathbf{I}^T > \mathbf{O}$ is the 3×3 moment of inertia matrix with respect to the centre of mass frame and

$$\boldsymbol{\omega}^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

This can be rewritten as

$$\dot{\boldsymbol{\omega}} = -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{u}$$

which is of the form of Eq. (1) with state vector $\mathbf{x} = \boldsymbol{\omega}$. The output equation $\mathbf{y}(t) = \boldsymbol{\omega}(t)$ is of the form in Eq. (2).

1.2 Input-Output (I/O) Stability

We define the 2-norm of a (vector) function of time by

$$||\mathbf{u}||_2 = \sqrt{\int_0^\infty \mathbf{u}^T(t)\mathbf{u}(t)\,dt} \tag{4}$$

It is readily verified that this satisfies all of the properties of a norm including the triangle inequality:

$$\begin{aligned} ||\mathbf{u}||_2 &\geq 0 \\ ||\mathbf{u}||_2 &= 0 \Leftrightarrow \mathbf{u} = \mathbf{0} \\ ||c\mathbf{u}||_2 &= |c| \cdot ||\mathbf{u}||_2,\ c \in \Re \\ ||\mathbf{u}_1 + \mathbf{u}_2||_2 &\leq ||\mathbf{u}_1||_2 + ||\mathbf{u}_2||_2 \end{aligned}$$

The L_2-space is defined by

$$L_2 = \{\mathbf{u}(t) \mid ||\mathbf{u}||_2 < \infty\}$$

i.e., $\mathbf{u}$ is square-integrable. For example,

$$u(t) = \begin{cases} \exp(at) \in L_2, & a < 0 \\ \exp(at) \notin L_2, & a > 0 \\ 1 \notin L_2 \end{cases}$$

In general, if $\mathbf{u} \in L_2$ this does not imply that $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$. An example is given by $u(t) = \exp(-t^4 \sin^2 t)$. However, if $\mathbf{u} \in L_2$ and $\dot{\mathbf{u}} \in L_2$, then $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$.

The truncation of a function is defined as

$$\mathbf{u}_T(t) = \begin{cases} \mathbf{u}(t), & t \leq T \\ \mathbf{0}, & t > T \end{cases}$$

The extended L_2-space is defined by

$$L_{2e} = \{\mathbf{u}(t) \mid \mathbf{u}_T(t) \in L_2,\ 0 < T < \infty\}$$

i.e.,

$$\mathbf{u}(t) = \begin{bmatrix} 1 \\ \exp(2t) \\ t \end{bmatrix} \in L_{2e}$$

but none of its elements belongs to L_2.

A system is a mapping (or operator) $\boldsymbol{\mathcal{G}} : L_{2e} \to L_{2e}$ and we write $\mathbf{y} = \boldsymbol{\mathcal{G}}\mathbf{u}$ indicating that the system can be thought of as an operator $\boldsymbol{\mathcal{G}}$ which maps the function $\mathbf{u}$ into a function $\mathbf{y}$. In most cases the act of $\boldsymbol{\mathcal{G}}$ operating on $\mathbf{u}$ to produce $\mathbf{y}$ corresponds to solving Eq. (1) for $\mathbf{x}(t)$ (given $\mathbf{u}$ and $\mathbf{x}(0)$) and then substituting $\mathbf{x}(t)$, $\mathbf{u}(t)$, and t into Eq. (2) to produce $\mathbf{y}(t)$. The concise notation $\mathbf{y} = \boldsymbol{\mathcal{G}}\mathbf{u}$ corresponds to these operations.

The operator notation is particularly appropriate when the initial conditions are neglected. In this case, we draw:

$$\mathbf{u} \longrightarrow \boxed{\boldsymbol{\mathcal{G}}} \longrightarrow \mathbf{y}$$

$\boldsymbol{\mathcal{G}}$ is L_2-*stable* if $\mathbf{u} \in L_2 \Rightarrow \mathbf{y} \in L_2$, i.e., "finite energy in implies finite energy out." The L_2-*gain* of a system is related to the peak energy amplification by

$$||\boldsymbol{\mathcal{G}}|| = \sup_{\mathbf{0} \neq \mathbf{u} \in L_2} \frac{||\boldsymbol{\mathcal{G}}\mathbf{u}||_2}{||\mathbf{u}||_2} = \sup_{\mathbf{0} \neq \mathbf{u} \in L_2} \frac{||\mathbf{y}||_2}{||\mathbf{u}||_2}$$

If $||\boldsymbol{\mathcal{G}}|| < \infty$, then $\boldsymbol{\mathcal{G}}$ is L_2-stable since

$$||\mathbf{y}||_2 = ||\boldsymbol{\mathcal{G}}\mathbf{u}||_2 \leq ||\boldsymbol{\mathcal{G}}|| \cdot ||\mathbf{u}||_2$$

Hence, $||\mathbf{u}||_2 < \infty \Rightarrow ||\mathbf{y}||_2 < \infty$ if $||\boldsymbol{\mathcal{G}}|| < \infty$.

Also, by considering an input $\mathbf{u}_T$, we have

$$||\mathbf{y}_T||_2 \leq ||\mathbf{y}||_2 \leq ||\boldsymbol{\mathcal{G}}|| \cdot ||\mathbf{u}_T||_2$$

Another useful property that we shall need is the Cauchy-Schwarz inequality:

$$\int_0^T \mathbf{u}_1^T \mathbf{u}_2 \, dt \leq ||\mathbf{u}_{1T}||_2 \cdot ||\mathbf{u}_{2T}||_2, \quad 0 \leq T \leq \infty$$

Example of L_2-gain

Consider the single-input/single-output (SISO), linear time-invariant (LTI) system:

$$y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau = \boldsymbol{\mathcal{G}} u$$

where $g(t)$ is the impulse response, *i.e.*, $y(t) = g(t)$ when $u(t) = \delta(t)$. Taking Laplace transforms, *i.e.*,

$$\hat{g}(s) = \int_0^\infty e^{-st} g(t)\, dt$$

we have

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s)$$

where $\hat{g}(s)$ is the transfer function.

This system, represented by the operator $\boldsymbol{\mathcal{G}}$, is L_2-stable if

$$\hat{g}(s) \in \mathcal{H}_\infty = \{\hat{g}(s) \;\mid\; \hat{g}(s) \text{ is analytic and bounded in } \Re e\{s\} > 0\}$$

The space of complex-valued functions $\mathcal{H}_\infty$ contains all of the stable transfer functions.

If $\hat{g}(s) \in \mathcal{H}_\infty$, we define the $\mathcal{H}_\infty$ norm by

$$||\hat{g}(s)||_\infty = \sup_{\Re e\{s\}>0} |\hat{g}(s)| = \sup_{\omega\in\mathcal{R}} |\hat{g}(j\omega)|$$

If $\hat{g}(s) = n(s)/d(s)$ is real rational, *i.e.*, $n(s)$ and $d(s)$ are polynomials with real coefficients, then $\hat{g}(s) \in \mathcal{H}_\infty$ if $\deg n(s) \leq \deg d(s)$ and $d(s)$ has roots in the open LHP, $\Re e\{s\} < 0$.

For $s = j\omega$, we have

$$\hat{y}(j\omega) = \hat{g}(j\omega)\hat{u}(j\omega)$$

where

$$\hat{u}(j\omega) = \int_0^\infty u(t)e^{-j\omega t}\,dt$$

is the Fourier transform. Parseval's theorem relates the time-domain L_2-norm to a calculation in the frequency domain:

$$\int_0^\infty u^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{u}(j\omega)|^2\,d\omega \tag{5}$$

Therefore

$$\begin{aligned}
||y||_2^2 &= \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{y}(j\omega)|^2\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{g}(j\omega)|^2|\hat{u}(j\omega)|^2\,d\omega \\
&\leq \sup_{\omega\in\mathcal{R}} |\hat{g}(j\omega)|^2 \cdot \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{u}(j\omega)|^2\,d\omega \\
&\leq ||\hat{g}(s)||_\infty^2 \cdot ||u||_2^2
\end{aligned}$$

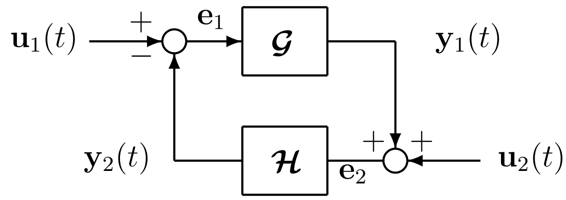
Therefore,

$$\frac{||y||_2}{||u||_2} \leq ||\hat{g}(s)||_\infty$$

It can be shown that this bound can be reached arbitrarily closely for certain choices of $u \in L_2$. Hence for SISO LTI systems

$$||\boldsymbol{\mathcal{G}}|| = \sup_{0\neq u\in L_2} \frac{||y||_2}{||u||_2} = ||\hat{g}(s)||_\infty$$

1.3 Stability of Feedback Systems



The feedback system shown below can be summarized by

$$\begin{aligned}\mathbf{y}_1 &= \mathcal{G}\mathbf{e}_1\\ \mathbf{y}_2 &= \mathcal{H}\mathbf{e}_2\\ \mathbf{e}_1 &= \mathbf{u}_1 - \mathbf{y}_2\\ \mathbf{e}_2 &= \mathbf{u}_2 + \mathbf{y}_1\end{aligned}$$

Assume that

$$\begin{aligned}\mathcal{G} &: L_{2e} \to L_{2e}\\ \mathcal{H} &: L_{2e} \to L_{2e}\\ \mathbf{u}_1, \mathbf{u}_2 \in L_{2e} &\Rightarrow \mathbf{e}_1, \mathbf{e}_2 \in L_{2e}\end{aligned}$$

We say that the feedback system is L_2-stable if

$$\text{(a) } \mathbf{u}_1, \mathbf{u}_2 \in L_2 \;\Rightarrow\; \mathbf{y}_1, \mathbf{y}_2 \in L_2$$

or

$$\text{(b) } \mathbf{u}_1, \mathbf{u}_2 \in L_2 \;\Rightarrow\; \mathbf{e}_1, \mathbf{e}_2 \in L_2$$

Claim: (a) $\Leftrightarrow$ (b)

Proof: [(a) $\Rightarrow$ (b)] Assume that (a) holds and $\mathbf{u}_1, \mathbf{u}_2 \in L_2$ and $\mathbf{y}_1, \mathbf{y}_2 \in L_2$. Therefore, using the triangle inequality

$$\begin{aligned}\mathbf{e}_1 = \mathbf{u}_1 - \mathbf{y}_2 &\Rightarrow ||\mathbf{e}_1||_2 \leq ||\mathbf{u}_1||_2 + ||\mathbf{y}_2||_2 < \infty\\ \mathbf{e}_2 = \mathbf{u}_2 + \mathbf{y}_1 &\Rightarrow ||\mathbf{e}_2||_2 \leq ||\mathbf{u}_2||_2 + ||\mathbf{y}_1||_2 < \infty\end{aligned}$$

[(b) $\Rightarrow$ (a)] Assume that (b) holds and $\mathbf{u}_1, \mathbf{u}_2 \in L_2$ and $\mathbf{e}_1, \mathbf{e}_2 \in L_2$. Therefore,

$$\begin{aligned}\mathbf{y}_1 = \mathbf{e}_2 - \mathbf{u}_2 &\Rightarrow ||\mathbf{y}_1||_2 \leq ||\mathbf{e}_2||_2 + ||\mathbf{u}_2||_2 < \infty\\ \mathbf{y}_2 = \mathbf{u}_1 - \mathbf{e}_1 &\Rightarrow ||\mathbf{y}_2||_2 \leq ||\mathbf{u}_1||_2 + ||\mathbf{e}_1||_2 < \infty\end{aligned}$$

$\square$

Small Gain Theorem: Consider the feedback system and let $\gamma_1 = ||\boldsymbol{\mathcal{G}}||$, $\gamma_2 = ||\boldsymbol{\mathcal{H}}||$. If $\gamma_1\gamma_2 < 1$, then the feedback system is L_2-stable.

Proof: Consider

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{u}_1 - \mathbf{y}_2 = \mathbf{u}_1 - \boldsymbol{\mathcal{H}}\mathbf{e}_2 \\ \mathbf{e}_2 &= \mathbf{u}_2 + \mathbf{y}_1 = \mathbf{u}_2 + \boldsymbol{\mathcal{G}}\mathbf{e}_1 \end{aligned}$$

Taking truncations, we have

$$\begin{aligned} ||\mathbf{e}_{1T}||_2 &\leq ||\mathbf{u}_{1T}||_2 + ||\mathbf{y}_{2T}||_2 \leq ||\mathbf{u}_{1T}||_2 + \gamma_2||\mathbf{e}_{2T}||_2 \\ ||\mathbf{e}_{2T}||_2 &\leq ||\mathbf{u}_{2T}||_2 + ||\mathbf{y}_{1T}||_2 \leq ||\mathbf{u}_{2T}||_2 + \gamma_1||\mathbf{e}_{1T}||_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow ||\mathbf{e}_{1T}||_2 &\leq ||\mathbf{u}_{1T}||_2 + \gamma_2||\mathbf{u}_{2T}||_2 + \gamma_1\gamma_2||\mathbf{e}_{1T}||_2 \\ ||\mathbf{e}_{2T}||_2 &\leq ||\mathbf{u}_{2T}||_2 + \gamma_1||\mathbf{u}_{1T}||_2 + \gamma_1\gamma_2||\mathbf{e}_{2T}||_2 \end{aligned}$$

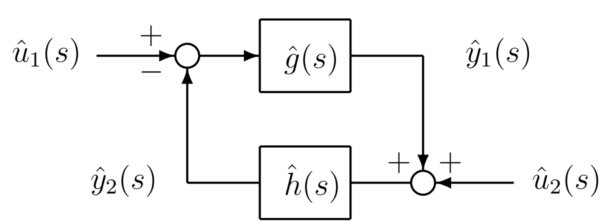
$$\begin{aligned} \Rightarrow (1-\gamma_1\gamma_2)||\mathbf{e}_{1T}||_2 &\leq ||\mathbf{u}_{1T}||_2 + \gamma_2||\mathbf{u}_{2T}||_2 \\ (1-\gamma_1\gamma_2)||\mathbf{e}_{2T}||_2 &\leq ||\mathbf{u}_{2T}||_2 + \gamma_1||\mathbf{u}_{1T}||_2 \end{aligned}$$

Now assume that $\mathbf{u}_1 \in L_2$, $\mathbf{u}_2 \in L_2$, and let $T \to \infty$. Therefore

$$\begin{aligned} ||\mathbf{e}_1||_2 &\leq (1-\gamma_1\gamma_2)^{-1}(||\mathbf{u}_1||_2 + \gamma_2||\mathbf{u}_2||_2) \\ ||\mathbf{e}_2||_2 &\leq (1-\gamma_1\gamma_2)^{-1}(||\mathbf{u}_2||_2 + \gamma_1||\mathbf{u}_1||_2) \end{aligned}$$

Hence, $\mathbf{e}_1 \in L_2$, $\mathbf{e}_2 \in L_2$, and the system is L_2-stable. □

Example. Consider the feedback interconnection of two single-input/single-output, linear time-invariant systems and assume that $\hat{g}(s), \hat{h}(s) \in \mathcal{H}_\infty$.



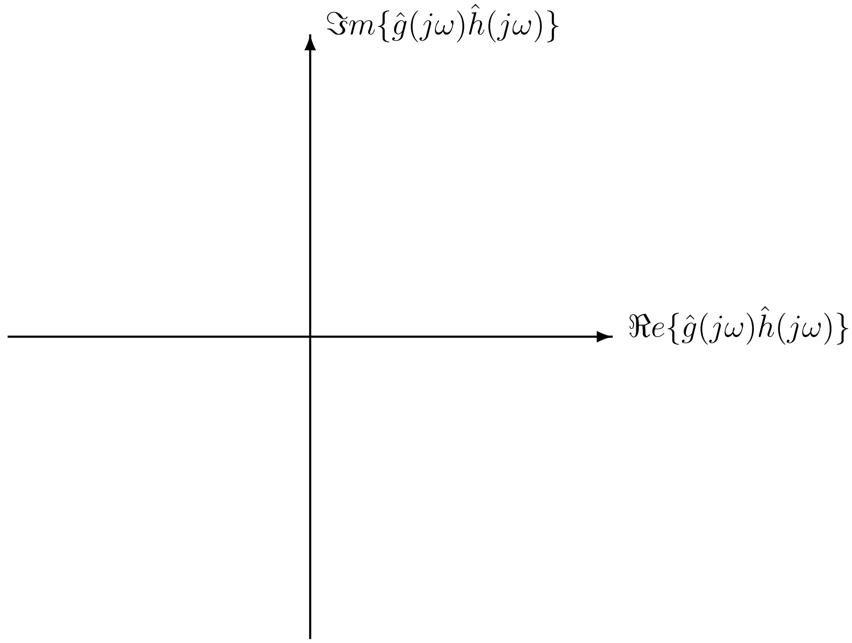
Let

$$\begin{aligned} \gamma_1 &= \sup_{\omega\in\Re} |\hat{g}(j\omega)| \\ \gamma_2 &= \sup_{\omega\in\Re} |\hat{h}(j\omega)| \end{aligned}$$

If $\gamma_1\gamma_2 < 1$, then

$$\sup_{\omega\in\Re} |\hat{g}(j\omega)\hat{h}(j\omega)| \leq \sup_{\omega\in\Re} |\hat{g}(j\omega)| \cdot \sup_{\omega\in\Re} |\hat{h}(j\omega)| < 1$$

Hence, $|\hat{g}(j\omega)\hat{h}(j\omega)| < 1$, $\forall\omega \in \Re$.



Therefore the Nyquist plot cannot possibly encircle the -1 point. This is termed *gain stabilization*.

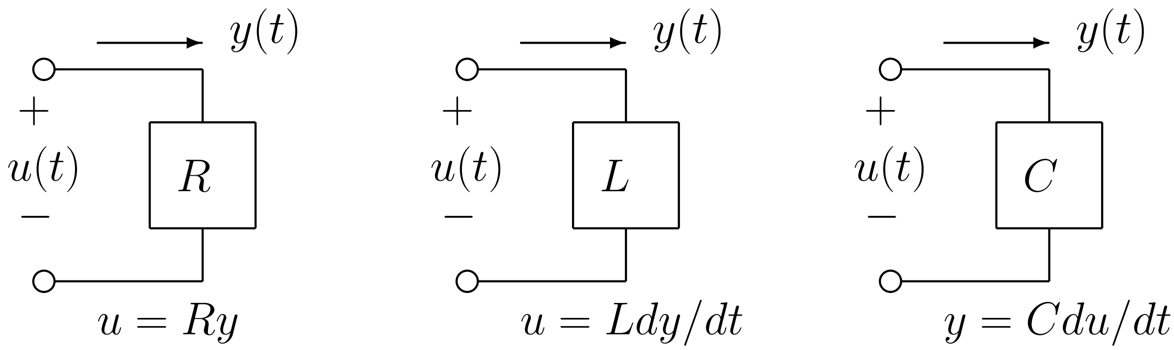
1.4 Passivity

Consider a square system $\mathbf{y} = \boldsymbol{\mathcal{G}}\mathbf{u}$ with $\mathbf{u} \in \Re^m$, $\mathbf{y} \in \Re^m$, and assume that $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. $\boldsymbol{\mathcal{G}}$ is passive if

$$\int_0^T \mathbf{u}^T \boldsymbol{\mathcal{G}}\mathbf{u}\, dt = \int_0^T \mathbf{y}^T(t)\mathbf{u}(t)\, dt \geq 0, \quad \forall \mathbf{u} \in L_{2e}, \forall T \geq 0 \tag{6}$$

In the case where $\boldsymbol{\mathcal{G}}$ corresponds to the system in Eqs. (1) and (2), it is assumed that the initial conditions satisfy $\mathbf{x}(0) = \mathbf{x}_0 = \mathbf{0}$ when performing the calculation.

Passive Circuit Motivation:



For the resistor (R), we have

$$\int_0^T y(t)u(t)\,dt = \int_0^T \frac{1}{R}u^2(t)\,dt \geq 0 \tag{7}$$

For the inductor (L) with $y(0) = 0$, we have

$$\int_0^T y(t)u(t)\,dt = \int_0^T y(t)L\frac{dy(t)}{dt}\,dt = \tfrac{1}{2}Ly^2(T) \geq 0 \tag{8}$$

For the capacitor (C) with $u(0) = 0$, we have

$$\int_0^T y(t)u(t)\,dt = \int_0^T u(t)C\frac{du(t)}{dt}\,dt = \tfrac{1}{2}Cu^2(T) \geq 0 \tag{9}$$

Hence, these three components correspond to passive systems. They correspond to the mathematical relations of constant gain, temporal integration, and temporal differentiation between input and output:

$$\begin{aligned}
y(t) &= Ku(t) \;\; (K = R^{-1} > 0) \\
y(t) &= K\int_0^t u(\tau)\,d\tau \;\; (K = L^{-1} > 0) \\
y(t) &= K\frac{du(t)}{dt} \;\; (K = C > 0)
\end{aligned}$$

It is trivial to show that the parallel connection of two passive systems (corresponding to summing their outputs) is also passive. On this basis, the classic proportional-integral-derivative (PID) controller is also passive:

$$y(t) = K_p u(t) + K_i \int_0^t u(\tau)\,d\tau + K_d\frac{du(t)}{dt}$$

with $K_p > 0, K_i > 0, K_d > 0$.

A square system $\mathbf{y} = \boldsymbol{\mathcal{H}}\mathbf{u}$ with $\boldsymbol{\mathcal{H}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is *strictly passive* if there exists $\epsilon > 0$ such that

$$\int_0^T \mathbf{u}^T \boldsymbol{\mathcal{H}}\mathbf{u}\, dt = \int_0^T \mathbf{y}^T(t)\mathbf{u}(t)\, dt \geq \epsilon \int_0^T \mathbf{u}^T(t)\mathbf{u}(t)\, dt, \quad \forall \mathbf{u} \in L_{2e}, \forall T \geq 0 \quad (10)$$

From (7), the resistor corresponds to a strictly passive system with $\epsilon = R^{-1}$. It is easy to show that the parallel connection of a strictly passive and a passive one is also strictly passive. Hence, the PID controller is strictly passive and so is a PI one ($K_d = 0$) and a PD one ($K_i = 0$).

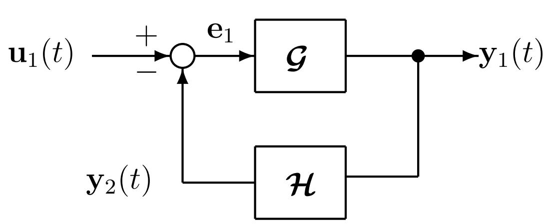
Example. The system $\mathbf{y}(t) = \mathbf{K}\mathbf{u}(t)$ with $\mathbf{K} = \mathbf{K}^T > \mathbf{O}$ is strictly passive.

Proof. Exhibit the eigendecomposition of $\mathbf{K}$ as $\mathbf{K} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^T$, $\boldsymbol{\Lambda} = \text{diag}\{\lambda_i\}$ with $\lambda_i > 0$. Defining $\hat{\mathbf{u}} = \mathbf{E}^T\mathbf{u}$, we have $\hat{\mathbf{u}}^T\hat{\mathbf{u}} = \mathbf{u}^T\mathbf{u}$ since $\mathbf{E}^T = \mathbf{E}^{-1}$ and

$$\begin{aligned}
\mathbf{y}^T\mathbf{u} &= \mathbf{u}^T\mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^T\mathbf{u} \\
&= \hat{\mathbf{u}}^T\boldsymbol{\Lambda}\hat{\mathbf{u}} \\
&= \sum_{i=1}^n \hat{u}_i^2\lambda_i \\
&\geq \sum_{i=1}^n \epsilon\hat{u}_i^2, \quad \epsilon = \min_i \lambda_i > 0 \\
&\geq \epsilon\sum_{i=1}^n u_i^2 = \epsilon\mathbf{u}^T\mathbf{u}
\end{aligned}$$

After integrating both side of this inequality with respect to time, (10) is satisfied.

Passivity Theorem: Consider the following feedback system:



If $\boldsymbol{\mathcal{G}}$ is passive and $\boldsymbol{\mathcal{H}}$ is strictly passive, then $\mathbf{u}_1 \in L_2 \;\Rightarrow\; \mathbf{y}_1 \in L_2$.

Proof. Using the passivity of $\boldsymbol{\mathcal{G}}$, we have

$$\begin{aligned}\int_0^T \mathbf{y}_1^T \mathbf{e}_1 \, dt &= \int_0^T \mathbf{y}_1^T (\mathbf{u}_1 - \mathbf{y}_2) \, dt \geq 0 \qquad (11)\\ \Rightarrow \int_0^T \mathbf{y}_1^T \mathbf{u}_1 \, dt &- \int_0^T \mathbf{y}_1^T \boldsymbol{\mathcal{H}} \mathbf{y}_1 \, dt \geq 0\\ \Rightarrow \int_0^T \mathbf{y}_1^T \mathbf{u}_1 \, dt &\geq \epsilon \int_0^T \mathbf{y}_1^T \mathbf{y}_1 \, dt \qquad (12)\end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\int_0^T \mathbf{y}_1^T \mathbf{u}_1 \, dt \leq \left[\int_0^T \mathbf{y}_1^T \mathbf{y}_1 \, dt\right]^{1/2} \left[\int_0^T \mathbf{u}_1^T \mathbf{u}_1 \, dt\right]^{1/2} \qquad (13)$$

Combining (12) and (13) gives

$$\begin{aligned}\epsilon \int_0^T \mathbf{y}_1^T \mathbf{y}_1 \, dt &\leq \left[\int_0^T \mathbf{y}_1^T \mathbf{y}_1 \, dt\right]^{1/2} \left[\int_0^T \mathbf{u}_1^T \mathbf{u}_1 \, dt\right]^{1/2}\\ \Rightarrow \left[\int_0^T \mathbf{y}_1^T \mathbf{y}_1 \, dt\right]^{1/2} &\leq \epsilon^{-1} \left[\int_0^T \mathbf{u}_1^T \mathbf{u}_1 \, dt\right]^{1/2}\end{aligned}$$

Assuming that $\mathbf{u}_1 \in L_2$ and letting $T \to \infty$ gives

$$||\mathbf{y}_1||_2 \leq \epsilon^{-1} ||\mathbf{u}_1||_2$$

Therefore, $\mathbf{u}_1 \in L_2 \Rightarrow \mathbf{y}_1 \in L_2$. □

Consider (11) in the above proof, and assume that $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{H}}$ are passive. Then

$$\int_0^T \mathbf{y}_1^T \mathbf{e}_1 \, dt = \int_0^T \mathbf{y}_1^T (\mathbf{u}_1 - \mathbf{y}_2) \, dt$$

Hence,

$$\begin{aligned}\int_0^T \mathbf{y}_1^T \mathbf{u}_1 \, dt &= \int_0^T \mathbf{y}_1^T \mathbf{e}_1 \, dt + \int_0^T \mathbf{y}_1^T \mathbf{y}_2 \, dt\\ &= \int_0^T \mathbf{e}_1^T \boldsymbol{\mathcal{G}} \mathbf{e}_1 + \int_0^T \mathbf{y}_1^T \boldsymbol{\mathcal{H}} \mathbf{y}_1 \, dt\\ &\geq 0\end{aligned}$$

Hence, the negative feedback interconnection of two passive systems is also passive.

Example: Fully Actuated Mechanical System

The motion equations are given as

$$\mathbf{y} = \dot{\mathbf{q}}, \quad \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{u}(t), \quad \dot{\mathbf{q}}(0) = \mathbf{q}(0) = \mathbf{0}$$

where $\mathbf{M} = \mathbf{M}^T > \mathbf{O}$ and $\mathbf{K} = \mathbf{K}^T \geq \mathbf{O}$.

Claim: The system mapping $\mathbf{u}$ to $\mathbf{y}$ is passive.

Proof: Consider the total energy (Hamiltonian),

$$H(t) = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \geq 0$$

Therefore

$$\dot{H} = \dot{\mathbf{q}}^T[\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q}] = \mathbf{y}^T\mathbf{u}$$

Integrating both sides from $t = 0$ to $t = T$ gives

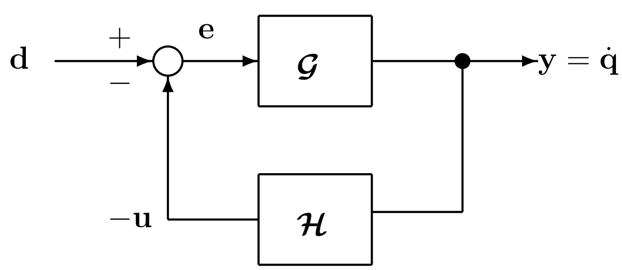
$$\int_0^T \mathbf{y}^T\mathbf{u}\,dt = \int_0^T \dot{H}\,dt = H(T) - H(0) = H(T) \geq 0$$

Stabilization

Now consider there to be two external inputs, controls $\mathbf{u}$ and disturbances $\mathbf{d}$:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{e}(t) = \mathbf{u}(t) + \mathbf{d}(t), \quad \mathbf{y} = \dot{\mathbf{q}}$$

Assuming a negative feedback controller $\mathbf{u} = -\boldsymbol{\mathcal{H}}\mathbf{y}$, the situation can be represented by the following block diagram:



If we select $\boldsymbol{\mathcal{H}}$ to be a strictly passive system such as

$$\mathbf{u} = -\mathbf{K}_d\dot{\mathbf{q}}, \quad \mathbf{K}_d = \mathbf{K}_d^T > \mathbf{O}$$

then $\mathbf{d} \in L_2 \;\Rightarrow\; \dot{\mathbf{q}} \in L_2$ by the passivity theorem.

Example: Euler's Equation

Consider a rigid body with external torque $\mathbf{u}$ about the mass centre and output $\boldsymbol{\omega}$. The equation of motion is

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} = \mathbf{u}(t)$$

where $\mathbf{I} = \mathbf{I}^T > \mathbf{O}$. Consider the kinetic energy, $H(t) = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega} \geq 0$. Its time derivative satisfies

$$\begin{aligned}\dot{H} &= \boldsymbol{\omega}^T\mathbf{I}\dot{\boldsymbol{\omega}} \\ &= \boldsymbol{\omega}^T(-\boldsymbol{\omega}^\times\mathbf{I}\boldsymbol{\omega} + \mathbf{u}) \\ &= \boldsymbol{\omega}^T\mathbf{u}\end{aligned}$$

Integrating both sides gives

$$\int_0^T \boldsymbol{\omega}^T\mathbf{u}\,dt = H(T) - H(0) = H(T) \geq 0 \quad (H(0) = 0)$$

Hence the mapping from $\mathbf{u}$ to $\boldsymbol{\omega}$ is passive.

<u>**Example**: SISO LTI systems</u>

Consider the SISO LTI system in the frequency domain

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s)$$

Assume that $\hat{g}(s) \in \mathcal{H}_\infty$ and let $\epsilon = \inf_{\omega\in\Re} \Re e\{\hat{g}(j\omega)\}$. A more general form of Parseval's relation is

$$\int_0^\infty y(t)u(t)\,dt = \frac{1}{2\pi}\Re e\left\{\int_{-\infty}^\infty \hat{y}^*(j\omega)\hat{u}(j\omega)\,d\omega\right\}$$

Applying this we have

$$\begin{aligned}\int_0^T y(t)u(t)\,dt &= \int_0^\infty y_T(t)u_T(t)\,dt \\ &= \frac{1}{2\pi}\Re e\left\{\int_{-\infty}^\infty \widehat{y_T}(j\omega)\widehat{u_T}^*(j\omega)\,d\omega\right\} \\ &= \frac{1}{2\pi}\Re e\left\{\int_{-\infty}^\infty \hat{g}(j\omega)\widehat{u_T}(j\omega)\widehat{u_T}^*(j\omega)\,d\omega\right\} \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \Re e\left\{\hat{g}(j\omega)\right\}\widehat{u_T}(j\omega)\widehat{u_T}^*(j\omega)\,d\omega \\ &\geq \epsilon\cdot\frac{1}{2\pi}\int_{-\infty}^\infty \widehat{u_T}(j\omega)\widehat{u_T}^*(j\omega)\,d\omega \\ &= \epsilon||\mathbf{u}_T||_2^2\end{aligned}$$

Hence, if $\epsilon \geq 0$, the system is passive. We say that $\hat{g}(s)$ is positive real. If $\epsilon > 0$, the system is strictly passive and $\hat{g}(s)$ is strictly positive real.

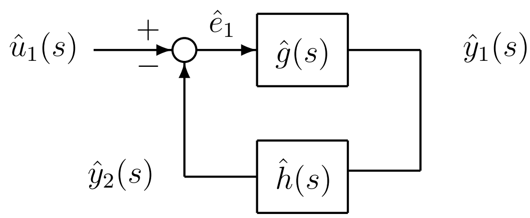
For $\epsilon > 0$,

$$-\frac{\pi}{2} < \arg\hat{g}(j\omega) < \frac{\pi}{2}$$

For $\epsilon = 0$,

$$-\frac{\pi}{2} \leq \arg \hat{g}(j\omega) \leq \frac{\pi}{2}$$

Now consider the feedback system:



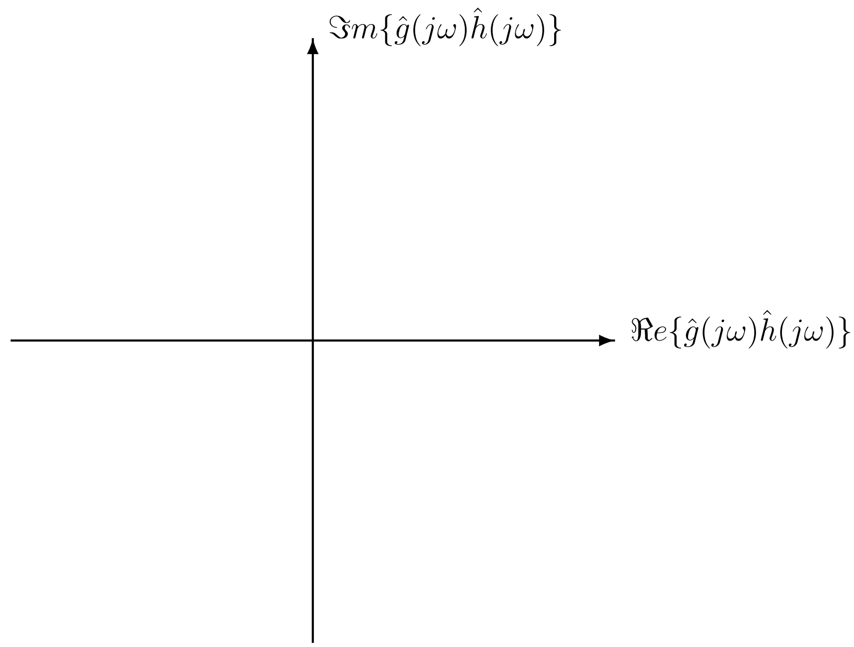
Assume that $\hat{g}(s), \hat{h}(s) \in \mathcal{H}_\infty$ and

$$\Re e\{\hat{g}(j\omega)\} \geq 0, \quad \Re e\{\hat{h}(j\omega)\} \geq \epsilon > 0$$

Since, $\arg\{\hat{g}(j\omega)\hat{h}(j\omega)\} = \arg\{\hat{g}(j\omega)\} + \arg\{\hat{h}(j\omega)\}$, we have

$$-\pi < \arg\{\hat{g}(j\omega)\hat{h}(j\omega)\} < \pi$$

Therefore, the passivity theorem guarantees that the -1 point is not encircled by the Nyquist plot. This is termed *phase stabilization*.



$\Im m\{\hat{g}(j\omega)\hat{h}(j\omega)\}$
$\Re e\{\hat{g}(j\omega)\hat{h}(j\omega)\}$