Almost periodic relative orbits under J2 perturbations

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Abstract: The relative motion of two spacecraft orbiting the Earth in low orbits in nominally circular orbits is studied in the presence of the J_2 perturbation. An iterative scheme is developed which ultimately leads to closure of the relative motion after one orbit. Although not truly periodic, a numerical example indicates that it is very nearly so. The method makes use of a state transition matrix for the relative motion that is developed in the presence of feedback control in order to render the iterative scheme convergent. The example shows that the calculated initial conditions are relatively insensitive to small errors.

Keywords: spacecraft formation flying, almost periodic orbits, J_2 perturbation

1 INTRODUCTION

The problem of spacecraft formation flying is currently attracting much attention. Part of the motivation stems from the attractiveness of using many small satellites to accomplish what one large (expensive) satellite can. An effective formation must exhibit bounded relative motion with periodicity as a special case. This can be accomplished through judicious choice of initial conditions and the use of feedback control. The current article emphasizes the former, but a suitably chosen low authority feedback controller renders the algorithm usable in practice.

The linearized relative motion equations, when one of the satellites is in a circular orbit and perturbations such as oblateness effects are ignored, are known as the Hill–Clohessy–Wiltshire (HCW) equations. Among their useful properties are special classes of periodic solutions whose initial conditions can be used for formation design (references [**1**] and [**2**]). In the presence of eccentricity, higher-order differential gravity terms and oblateness effects, such as the J_2 zonal harmonic of the Earth's gravitational field, the periodic motion predicted by the HCW equations is lost.

Several authors have examined modifications to the HCW initial conditions so that periodicity or something close to it is established for the relative motion in the presence of these effects. Vadali *et al.* [**1**] presented a modification that enforced period matching in the presence of J_2. Vaddi *et al.* [**2**] established modifications to the HCW initial conditions which enforced bounded relative motion in the presence of second-order differential gravity terms and small eccentricity. Inalhan *et al.* [**3**] presented the linearized relative motion equations for elliptic orbits and presented analytical solutions and the initial conditions required for periodicity. In reference [**4**], the extension of the HCW equations to account for second- and third-order differential gravity terms was presented and periodic solutions were established. Guibout and Scheeres [**5**] presented a semianalytical method which could find initial conditions leading to what they called (M, T) stable relative motion (i.e. relative motion remained within distance M for time within $[0, T]$) for problems including J_2 and J_3. Kasdin and Kolemen [**6**] used Hamilton–Jacobi theory and the epicyclic orbital elements to derive bounded, periodic orbits in the presence of certain perturbations.

Schaub and Alfriend [**7**] examined the notion of J_2-invariant orbits and presented modifications of the initial conditions which established equal secular growth for some of the orbital elements. This paper, examines the issue of finding initial conditions that lead to closure of the (non-linear) relative motion, in the presence of the J_2 perturbation, after one period of a reference circular orbit, showing that the ensuing motion is close to periodic but not exactly so in the presence of J_2. An iterative scheme for the initial conditions is presented which yields almost periodic motions that are close to the HCW solutions. The proposed scheme is compared with J_2-invariant initial conditions and shown to be less sensitive to errors in the initial conditions.

Recent work by European researchers in references [**8**] to [**10**] has similarities to the methods presented here. In reference [**8**], a Newton iteration similar to that presented here was used to find periodic initial conditions in the presence of J_2. However, the methods used to enforce invertibility of the key sensitivity matrix are quite different. In particular, the use of feedback control in the present paper is key to enforcing invertiblity of this matrix and ensures that the trajectories remain close to an HCW solution. In reference [**8**], feedback control is developed after the periodic trajectories have been determined. These trajectories are not necessarily close to the initial guess. In reference [**9**], genetic algorithms are used to determine trajectory closure after one period. Genuine periodic orbits are obtained for the special inclinations 49°, 63.4°, 131°, and 116.6°. Otherwise, closure is not obtained after one period. Various control schemes are investigated in reference [**10**], using the results of reference [**9**] as reference trajectories.

2 RELATIVE MOTION EQUATIONS AND ALMOST PERIODIC ORBITS

Consider the motion of two spacecraft in close proximity that are orbiting the Earth at low altitudes. The reference spacecraft is designated as the chief and the other satellite is the deputy (see Fig. 1). It is assumed that the chief is initially at $t = 0$ at the ascending node of an orbit which on average is circular with radius R_o, inclination i, and initial longitude of the ascending node Ω. The mean motion of this average circular orbit is $\omega = \sqrt{\mu/R_o^3}$ and $T = 2\pi/\omega$ is the period where μ is the geocentric gravitational constant.

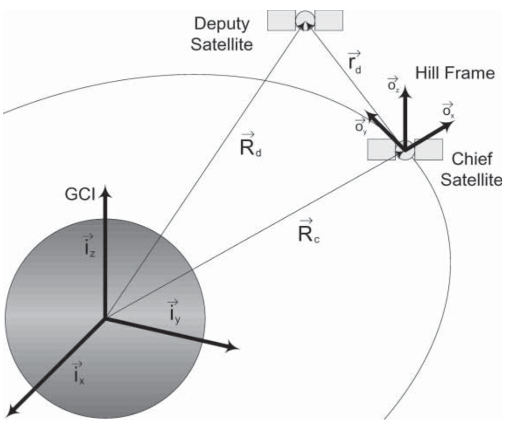

Fig. 1 Orbital geometry

Let $\mathbf{R}_c$ denote the position of the chief and $\mathbf{R}_d$ that of the deputy where $\mathbf{R} = [X \;\; Y \;\; Z]^T$ is the geocentric inertial position of a generic satellite. The motion of the two satellites evolve according to

$$\ddot{\mathbf{R}}_c = -\frac{\mu \mathbf{R}_c}{R_c^3} + \mathbf{F}_{J_2,i}(\mathbf{R}_c) \tag{1}$$

$$\ddot{\mathbf{R}}_d = -\frac{\mu \mathbf{R}_d}{R_d^3} + \mathbf{F}_{J_2,i}(\mathbf{R}_d) + \mathbf{u}_i \tag{2}$$

where a control force (per unit mass) $\mathbf{u}_i$ is acting on the deputy and

$$\mathbf{F}_{J_2,i}(\mathbf{R}) = -\frac{3\mu J_2 R_e^2}{2R^7}\begin{bmatrix} X^3 + XY^2 - 4XZ^2 \\ X^2Y + Y^3 - 4YZ^2 \\ 3X^2Z + 3Y^2Z - 2Z^3 \end{bmatrix} \tag{3}$$

where $\mathbf{F}_{J_2,i}$ is the lowest order term capturing the gravitational force of an oblate mass distribution. The second zonal harmonic coefficient is $J_2 = 1.0\,826\,269 \times 10^{-3}$ and R_e is the Earth's equatorial radius. The motion equations in equations (1) and (2) can be succinctly written in the form

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t) \tag{4}$$

where $\mathbf{X}(t) = \text{col}\{\mathbf{R}_c, \dot{\mathbf{R}}_c, \mathbf{R}_d, \dot{\mathbf{R}}_d\}$ and it has been assumed that $\mathbf{u}_i \equiv \mathbf{u}_i(\mathbf{X}, t)$.

The relative position of the deputy with respect to the chief will be expressed in the local–vertical–local–horizontal (LVLH) or Hill frame which moves with the chief and has its x-axis in the radial direction; z-axis normal to the orbit and y-axis is in the in-track direction and completes the right-handed xyz-coordinate system. The relative position expressed in this frame is given by $\mathbf{r}_d = [x \;\; y \;\; z]^T$ where

$$\mathbf{r}_d = \mathbf{C}_{oi}[\mathbf{R}_d - \mathbf{R}_c], \quad \mathbf{C}_{oi}^T(t) = \begin{bmatrix} \frac{\mathbf{R}_c}{R_c} & \frac{\mathbf{H}_c^\times \mathbf{R}_c}{|\mathbf{H}_c^\times \mathbf{R}_c|} & \frac{\mathbf{H}_c}{H_c} \end{bmatrix} \tag{5}$$

Here, $\mathbf{C}_{oi}$ is the rotation matrix relating the LVLH frame (subscript o) to the geocentric inertial frame (subscript i) and $\mathbf{H}_c = \mathbf{R}_c^\times \dot{\mathbf{R}}_c$ is the chief's angular momentum (per unit mass). The relative velocity with respect to the LVLH frame will be designated $\mathbf{v}_d = \dot{\mathbf{r}}_d$ and the corresponding state vector is given by $\boldsymbol{x}(t) = \text{col}\{\mathbf{r}_d, \mathbf{v}_d\}$. Equation (5) and its derivative can be written in the form

$$\boldsymbol{x}(t) = \mathbf{H}(\mathbf{X}, t) \tag{6}$$

The initial conditions for the relative state vector can be related to those of $\mathbf{X}$ using

$$\mathbf{X}(0) = \begin{bmatrix} \mathbf{R}_c(0) \\ \dot{\mathbf{R}}_c(0) \\ \mathbf{R}_c(0) \\ \dot{\mathbf{R}}_c(0) \end{bmatrix} + \mathbf{B}(0)\boldsymbol{x}(0)$$

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{C}_{\mathrm{oi}}^{\mathrm{T}}(t) & \mathbf{O} \\ \dot{\mathbf{C}}_{\mathrm{oi}}^{\mathrm{T}}(t) & \mathbf{C}_{\mathrm{oi}}^{\mathrm{T}}(t) \end{bmatrix} \tag{7}$$

The search is for a set of initial conditions for $\boldsymbol{x}$ such that $\boldsymbol{x}(T) = \boldsymbol{x}(0)$. Note that this does not lead to periodic relative motion since $\boldsymbol{x}(2T) \neq \boldsymbol{x}(T)$ on account of J_2 effects. However, this will be seen to be a promising avenue for establishing almost periodic relative motion. The corresponding initial conditions are referred to as 'periodic'.

Anticipating an iterative scheme for determining the initial conditions, let $\boldsymbol{x}_k(t), k = 0, 1, 2, \ldots$, denote the relative motion solution emanating from the initial conditions $\boldsymbol{x}_k(0)$ and $\mathbf{X}_k(t)$ the corresponding solution for the inertial state, and linearize about these trajectories assuming small perturbations $\Delta\mathbf{X}_k = \mathbf{X}_{k+1} - \mathbf{X}_k$ and $\Delta\boldsymbol{x}_k = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k$. The linearized equations are given by

$$\Delta\dot{\mathbf{X}}_k = \mathbf{A}_k(t)\Delta\mathbf{X}_k, \quad \mathbf{A}_k(t) = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{X}^{\mathrm{T}}}\right|_{\mathbf{X}=\mathbf{X}_k} \tag{8}$$

$$\Delta\boldsymbol{x}_k(t) = \mathbf{C}_k(t)\Delta\mathbf{X}_k(t), \quad \mathbf{C}_k(t) = \left.\frac{\partial \mathbf{H}}{\partial \mathbf{X}^{\mathrm{T}}}\right|_{\mathbf{X}=\mathbf{X}_k} \tag{9}$$

The solution of equation (8) may be written as

$$\Delta\mathbf{X}_k(t) = \mathbf{\Psi}_k(t, 0)\Delta\mathbf{X}_k(0) \tag{10}$$

where the state transition matrix satisfies

$$\dot{\mathbf{\Psi}}_k(t, t_0) = \mathbf{A}_k(t)\mathbf{\Psi}_k(t, t_0), \quad \mathbf{\Psi}_k(t_0, t_0) = \mathbf{1} \tag{11}$$

and $\Delta\mathbf{X}_k(0) = \mathbf{B}(0)\Delta\boldsymbol{x}_k(0)$. Combining this relation with equations (9) and (10) yields

$$\begin{aligned} &\Delta\boldsymbol{x}_k(T) = \mathbf{\Phi}_k(T, 0)\Delta\boldsymbol{x}_k(0), \\ &\mathbf{\Phi}_k(t, t_0) = \mathbf{C}_k(t)\mathbf{\Psi}_k(t, t_0)\mathbf{B}(t_0) \end{aligned} \tag{12}$$

or

$$\boldsymbol{x}_{k+1}(T) - \boldsymbol{x}_k(T) = \mathbf{\Phi}_k[\boldsymbol{x}_{k+1}(0) - \boldsymbol{x}_k(0)] \tag{13}$$

where $\mathbf{\Phi}_k \equiv \mathbf{\Phi}_k(T, 0)$. Motivated by the shooting method presented in references [**11**] and [**12**], set $\boldsymbol{x}_{k+1}(T) = \boldsymbol{x}_{k+1}(0)$ which ultimately leads to the Newton iteration

$$\begin{aligned} \boldsymbol{x}_{k+1}(0) = \boldsymbol{x}_k(0) + (\mathbf{\Phi}_k - \mathbf{1})^{-1}[\boldsymbol{x}_k(0) - \boldsymbol{x}_k(T)], \\ k = 0, 1, 2, \ldots \end{aligned} \tag{14}$$

where $\boldsymbol{x}_0(0)$ is determined below.

It is advantageous to develop an approximation for $\mathbf{\Phi}_k$. It will in fact be constant (independent of k). To this end, the reduced situation is considered, where J_2 is neglected and the chief's motion is circular. The linearized relative motion equations are the nonhomogeneous form (owing to the control influences on the deputy) of the HCW equations

$$\dot{\boldsymbol{x}} = \mathbf{A}_{\mathrm{h}}\boldsymbol{x} + \mathbf{B}_{\mathrm{h}}\mathbf{u}_{\mathrm{o}} \tag{15}$$

where

$$\mathbf{A}_{\mathrm{h}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_{\mathrm{h}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u}_{\mathrm{o}} = \mathbf{C}_{\mathrm{oi}}\mathbf{u}_{\mathrm{i}}$$

For no control, $\mathbf{u}_{\mathrm{o}} = \mathbf{0}$, and the transition matrix based on equation (15) is given by $\mathbf{\Phi}_k = \exp(\mathbf{A}_{\mathrm{h}}T)$. The analytical form of this matrix is given in reference [**1**] and it is not hard to see that all six of the eigenvalues of $\mathbf{\Phi}_k$ are unity. Hence the inverse required in the iteration of equation (14) does not exist.

In an effort to render the matrix $\mathbf{\Phi}_k - 1$ invertible, a linear feedback controller shall be introduced, which essentially moves the eigenvalues of $\mathbf{A}_{\mathrm{h}}$ off of the imaginary axis and those of $\mathbf{\Phi}_k$ off of the unit circle. This controller will also have the effect of encouraging the converged (almost) periodic trajectories to be close to the initial guess $\boldsymbol{x}_0$. The controller design will be discussed after examining the choice of $\boldsymbol{x}_0$. It is also anticipated that the feedback controller will mitigate against the neglect of other perturbations such as higher order harmonics in the gravity model in the calculation of $\mathbf{\Phi}_k$.

In an effort to identify candidate initial conditions for the iterative scheme and reference trajectories for control action, the homogeneous HCW equations are considered

$$\dot{\boldsymbol{x}}_{\mathrm{h}} = \mathbf{A}_{\mathrm{h}}\boldsymbol{x}_{\mathrm{h}} \tag{16}$$

It is well known that these equations possess periodic solutions, one of which will concern us here. To fix ideas, the projected circular orbit in reference [**1**] is considered

$$\begin{aligned} &x_{\mathrm{h}}(t) = \left(\frac{a}{2}\right)\sin(\omega t), \quad y_{\mathrm{h}}(t) = a\cos(\omega t), \\ &z_{\mathrm{h}}(t) = a\sin(\omega t) \end{aligned} \tag{17}$$

where a is the radius of the circular projection in the yz-plane. Note that it has period T and it is largely for this reason that this has been chosen as the target 'period' for the iteration. An almost periodic solution that approximates the projected circular orbit is desired. To this end, the following control law is adopted

$$\mathbf{u}_o(t) = -\mathbf{K}\tilde{\boldsymbol{x}}(t), \quad \tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_h(t) \tag{18}$$

where $\boldsymbol{x}_h$ corresponds to the solution in equation (17) and $\mathbf{K}$ is a stabilizing state feedback gain to be determined. Subtracting equation (16) from equation (15) yields

$$\dot{\tilde{\boldsymbol{x}}} = \mathbf{A}_h\tilde{\boldsymbol{x}} + \mathbf{B}_h\mathbf{u}_o \tag{19}$$

for the error dynamics. Using linear quadratic regulator (LQR) design, $\mathbf{K}$ is selected to minimize $J = \int_0^\infty (\tilde{\boldsymbol{x}}^T\mathbf{Q}\tilde{\boldsymbol{x}} + \mathbf{u}_o^T\mathbf{R}\mathbf{u}_o)\,dt$. In an effort to minimize fuel, $\mathbf{R}$ should be made fairly large since we desire the smallest possible feedback gain which renders the iteration in equation (14) convergent. Combining equations (18) and (19) yields $\dot{\tilde{\boldsymbol{x}}} = (\mathbf{A}_h - \mathbf{B}_h\mathbf{K})\tilde{\boldsymbol{x}}$ which has the solution $\tilde{\boldsymbol{x}}(T) = \exp[(\mathbf{A}_h - \mathbf{B}_h\mathbf{K})T]\tilde{\boldsymbol{x}}(0)$. Adopting the iterative notation established previously, one obtains $\Delta\tilde{\boldsymbol{x}}_k(t) = \Delta\boldsymbol{x}_k(t)$ since $\boldsymbol{x}_{h,k+1} = \boldsymbol{x}_{h,k}$. Hence

$$\begin{aligned} \Delta\boldsymbol{x}_k(T) &= \boldsymbol{\Phi}_k(T,0)\Delta\boldsymbol{x}_k(0), \\ \boldsymbol{\Phi}_k(T,0) &= \exp[(\mathbf{A}_h - \mathbf{B}_h\mathbf{K})T] \end{aligned} \tag{20}$$

This is the approximation to equation (12) and it can be noted that if $\mathbf{K}$ is stabilizing, then the inverse required in equations (14) exists. It is emphasized that $\boldsymbol{x}_k(T)$ in equation (14) contains non-linearities and J_2 effects since it is determined by integrating equations (1) and (2) and using equation (5) and its derivative. The control in equation (2) is determined from $\mathbf{u}_i = \mathbf{C}_{oi}^T\mathbf{u}_o$ with the understanding that $\boldsymbol{x}$ in equation (18) is determined from equations (1), (2), and (5). Given the numerical nature of the algorithm, considerably more general perturbations than J_2 may be accounted for by including them in the determination of $\boldsymbol{x}_k(T)$. Periodic solutions of the HCW equations other than the projected circular orbit can also be considered.

It should be noted that the shooting method used in equation (14) has also been employed in reference [**8**] but in lieu of the Cartesian state, a Hamiltonian formulation was used. In that case the required inverse did not exist as well. This was remedied by replacing one of the equations by a statement of conservation of energy and employing a pseudoinverse. Clearly, the development in equations (8) to (14) can be applied to more general formulations than the Cartesian description used in equations (1) to (7).

3 NUMERICAL EXAMPLE

It is assumed that the initial chief orbit has $R_o = 7078$ km and $i = \Omega = 60°$ which yields an orbital period of $T = 1.65$ h. The initial conditions for the iteration, $\boldsymbol{x}_0(0)$ are selected according to equation (17) with $a = 400$ m. The differential equations in equation (4) are propagated using a fixed-stepsize (0.1 s) fourth-order Runge–Kutta scheme. The LQR feedback controller is designed using $\mathbf{Q} = \text{diag}\{\omega^2, \omega^2, \omega^2, 1, 1, 1\}$ and $\mathbf{R} = \text{diag}\{r/\omega^2, r/\omega^2, r/\omega^2\}$ with $r = 10^4$. To understand the significance of this choice, note that $r = 10^{-2}$ results in a feedback law that renders the tracking errors in $\{|\tilde{x}_0(t)|, |\tilde{y}_0(t)|, |\tilde{z}_0(t)|\}$ (i.e. the response to the HCW initial conditions) to be just less than one meter. The choice $r = 10^4$ corresponds to very low authority control.

The evolution of the iterative scheme is illustrated in Fig. 2 where $\{|x_k(T) - x_k(0)|, |y_k(T) - y_k(0)|, |z_k(T) - z_k(0)|\}$ are shown as a function of iteration number k. The behaviour of the corresponding rates is shown in Fig. 3. Convergence is very rapid and after seven iterations, the relative position closes to within 10^{-7} m and the relative velocity to within 10^{-9} m s^{-1}. Submetre errors are obtained after one iteration and subcentimetre errors are obtained after three iterations.

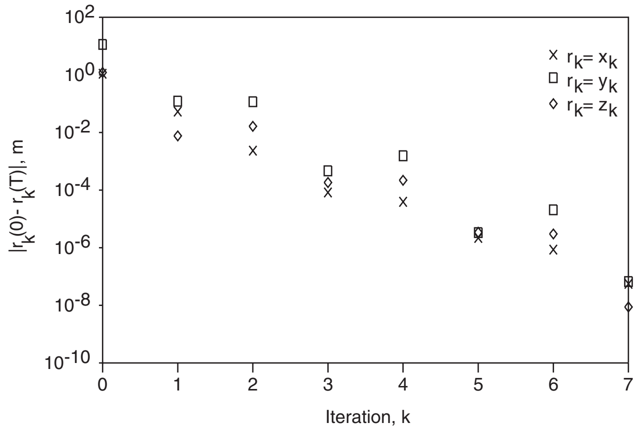

Fig. 2 Relative position error versus iteration number

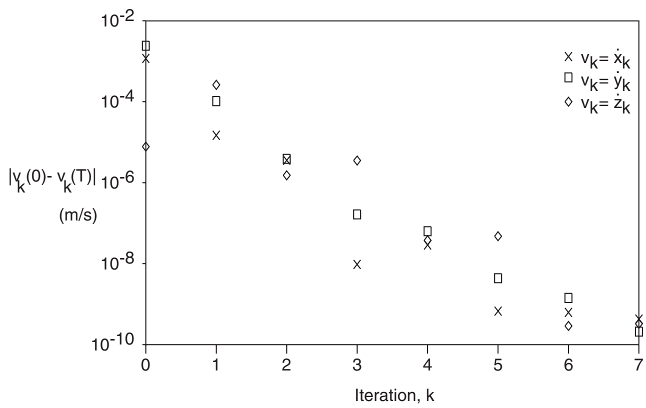

Fig. 3 Relative velocity error versus iteration number

The converged initial conditions are given by

$$\mathbf{r}_d(0) = [5.4459 \ \ 375.22 \ \ 27.712]^T \ \mathrm{m}$$
$$\mathbf{v}_d(0) = [0.206\,37 \ \ -0.011\,943 \ \ 0.417\,89]^T \ \mathrm{m\,s^{-1}}$$

The converged relative trajectory $\tilde{\boldsymbol{x}}_7(t)$ is shown in Fig. 4 which illustrates the discrepancy from the projected circular orbit. The shape of the converged relative orbit is shown in Fig. 5 in the yz-plane and compared with the projected circular orbit.

As noted in the previous section, the solution is not periodic. The converged solution has been propagated for ten orbits in Fig. 6 and 50 orbits in Fig. 7. The former illustrates the nearly periodic nature of the solution in the short run but the latter begins to show the breakdown of the periodicity. It is tempting to view the feedback controller as an artifice which renders the iterative scheme functional and a means to obtain appropriate initial conditions. If the feedback controller is dropped from the final simulation using the converged initial conditions, the result is Fig. 8 (20 orbits). Clearly, the periodicity is not as good as the case with feedback control but is a large improvement on the use of the HCW initial conditions $\boldsymbol{x}_0(0)$.

One may wonder if anything may be gained upon using a better approximation to the state transition matrix. The linearized dynamics (in the presence of J_2 with a reference circular orbit) presented by Ross [**13**] are a system of linear time-varying ODE's. They can be integrated over one period with different initial conditions to produce the state transition matrix analog of the matrix exponential used here. Its use presented no advantages over the simple case presented here. It is

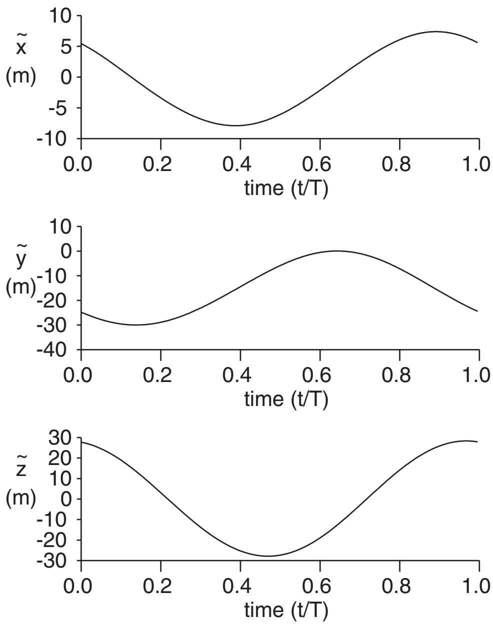

Fig. 4 Converged 'periodic' solution relative to HCW solution

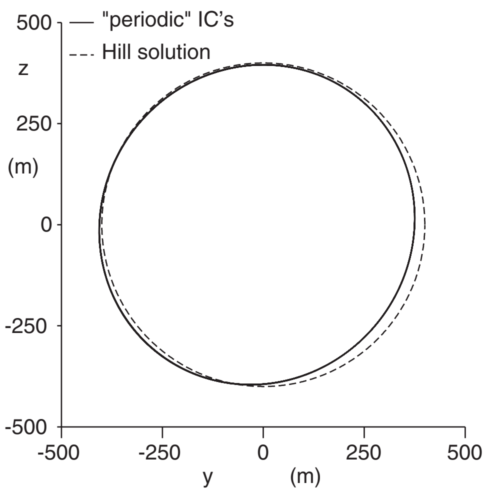

Fig. 5 Converged 'periodic' solution in yz-plane

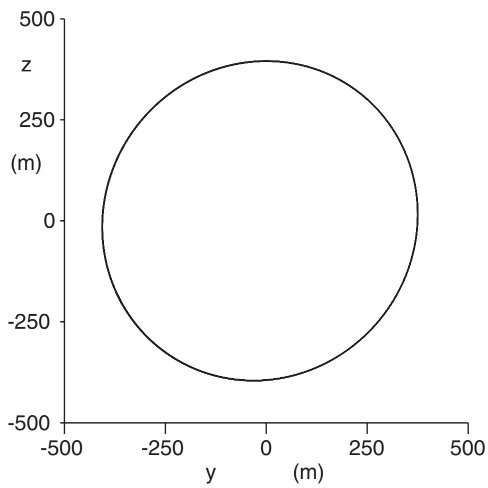

Fig. 6 'Periodic' ICs with control (10 orbits)

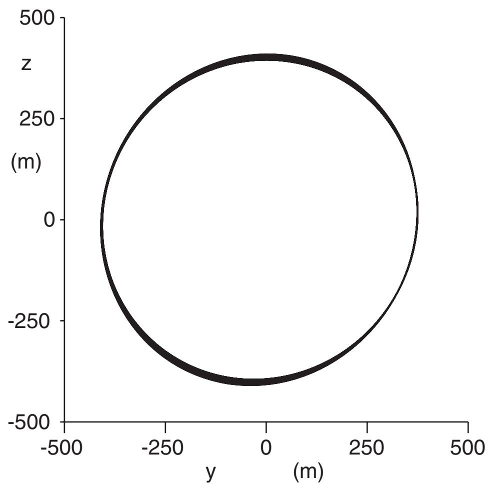

Fig. 7 'Periodic' ICs with control (50 orbits)

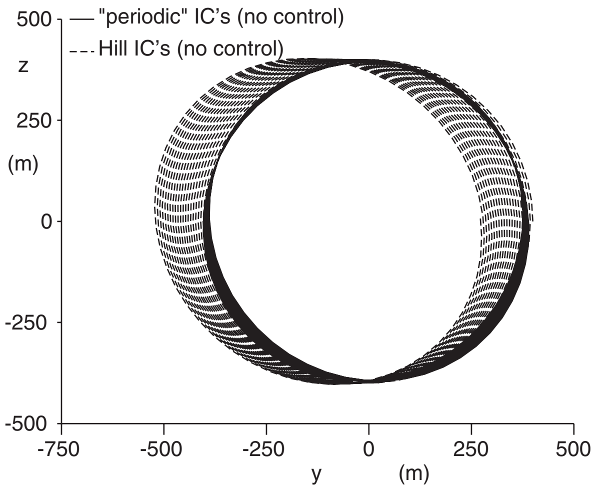

Fig. 8 'Periodic' ICs without control (20 orbits)

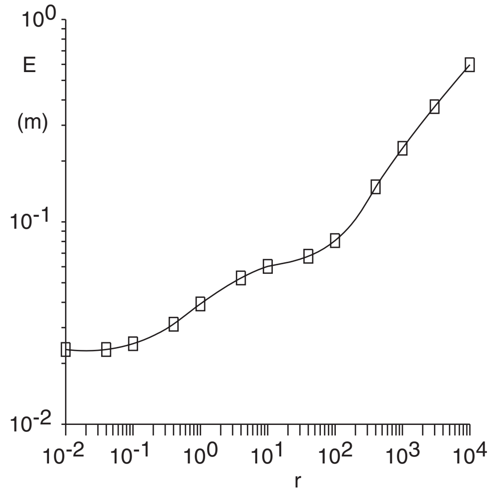

Fig. 10 Periodicity error measure (10 orbits) versus LQR control weighting r

expected that the linear time-varying equations presented by Melton [**14**] along with the corresponding state transition matrix will be useful in treating almost periodic motion between elliptic orbits.

In an effort to gauge the impact of other parameters on the periodicity resulting from the calculated initial conditions, the following periodicity error measure is defined as

$$E = \sqrt{\begin{matrix}[x(0) - x(10T)]^2 + [y(0) - y(10T)]^2 \\ +[z(0) - z(10T)]^2\end{matrix}} \tag{21}$$

which measures the periodicity error over ten orbits. Keeping all of the parameters the same as above, the inclination is varied from 0 to 90° and the resulting error measure is shown in Fig. 9. In all cases, the error is less than 2 m with the best results obtained for polar orbits ($E = 38$ cm). Note that we are unable to obtain the truly periodic ($E = 0$) results of reference [**9**] for

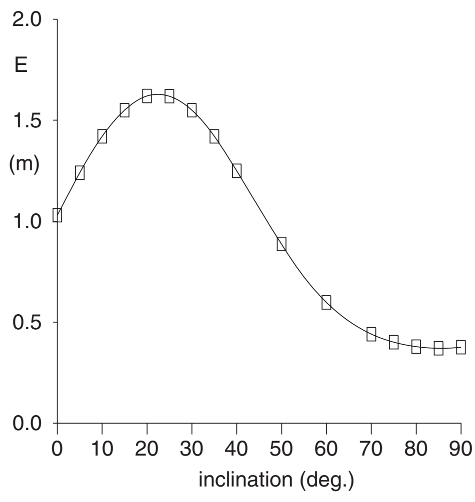

Fig. 9 Periodicity error measure (10 orbits) versus inclination

inclinations of 49 and 63.4° since this technique must use a feedback controller, which encourages the resulting trajectories to remain close to the HCW solution. There is no such restriction on the genetic algorithm approach of reference [**9**] which searches for a periodic solution over a very general range of trajectory parameters. On the other hand, that technique does not generate trajectory closure over even one orbit away from the special inclinations.

In Fig. 10, the periodicity error measure is shown as a function of control weighting parameter r for an inclination of 60°. As r is reduced, the periodicity measure improves and the resulting trajectories approach the HCW solution. The price paid for this is increased fuel consumption. On the other hand, as r is increased, the periodicity error increases but one has the possibility of approximating the behaviour without any control effort at all (beyond that required to establish the initial conditions).

It is instructive to compare the approach here with the use of initial conditions obtained using the J_2-invariant technique of reference [**7**]. In order to implement the latter, it is started by choosing the Cartesian position and velocity of the chief and deputy in an identical fashion to above. These are then converted to osculating orbital elements which are then converted to mean orbital elements using the procedure outlined in reference [**15**]. The J_2-invariant modifications are then applied to the deputy's mean elements i_d and a_d

$$i_d = i_c - \frac{4\delta\eta}{(\eta_c \tan i_c)}, \quad \eta_c = \sqrt{1 - e_c^2}$$

$$a_d = a_c + \frac{2Da_c^2\delta\eta}{R_e}, \quad \delta\eta = \eta_d - \eta_c$$

$$D = \frac{J_2}{4L_c^4\eta_c^5}(4 + 3\eta_c)(1 + 5\cos^2 i_c), \quad L_c = \sqrt{\frac{a_c}{R_e}}$$

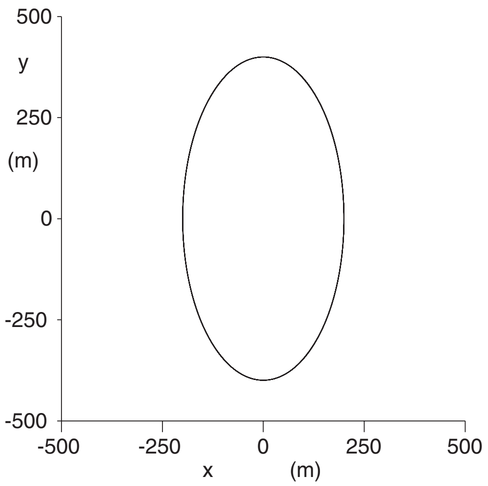

Fig. 11 J_2-invariant ICs (10 orbits)

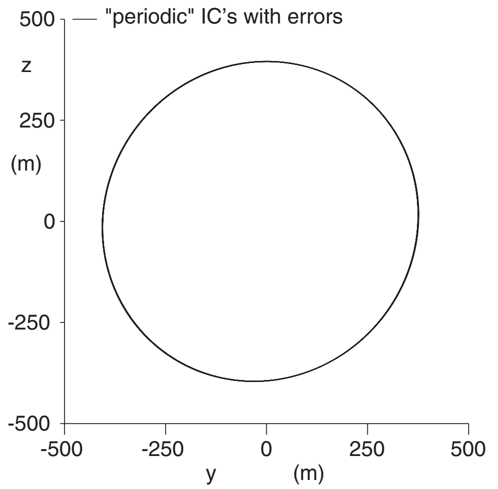

Fig. 13 'Periodic' ICs with errors (10 orbits)

where the subscripts c and d refer to mean elements of the chief and deputy, respectively. After these modifications are made, the mean elements are converted to osculating elements from which the Cartesian positions and velocities can be obtained.

The ensuing relative motion for ten orbits is depicted in Fig. 11. Departures from periodicity are not discernible. It should also be noted that a depiction of the motion has been made in the xy-plane since the J_2-invariant conditions effectively suppress the z motion and the motion is essentially an ellipse in the xy-plane. In an effort to assess the sensitivity to errors in the initial conditions, the deputy's initial conditions are perturbed as follows: $\delta\mathbf{R}_\mathrm{d}(0) = [0.2 \;\; -0.3 \;\; 0.5]^\mathrm{T}$ m. The ensuing relative motion is depicted in Fig. 12 which clearly shows that periodicity is lost and a secular motion in the in-track direction appears. When the same perturbation is made to the deputy's initial conditions obtained using the approach of the paper, the results analogous to Fig. 6 are shown in Fig. 13. Clearly, the approach is much less sensitive to errors in the initial conditions than the J_2-invariant approach. Another advantage is that the orbit obtained from the converged initial conditions is not radically different than that used for initialization, i.e. the projected circular orbit, whereas this is not the case for the J_2-invariant approach.

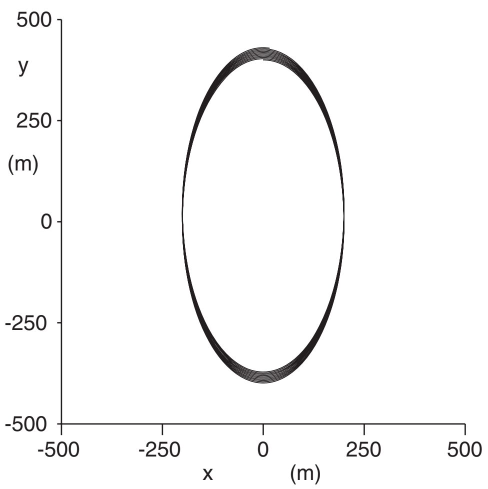

Fig. 12 J_2-invariant ICs with errors (10 orbits)

4 CONCLUSION

An iterative scheme for generating initial conditions yielding closed relative orbits has been presented for spacecraft in nearly circular orbits under the influence of the J_2 gravitational perturbation. The scheme uses an approximation to the relevant state transition matrix based on the HCW equations with an additional small feedback controller. The relative orbits obtained from the initial conditions were shown to be nearly periodic and this persisted when the feedback controller was removed. Since the technique is numerical in character, it is anticipated that more general problems (i.e. higher order harmonics in the gravitational field and eccentric orbits) may be dealt with using the proposed method. It was also demonstrated for the example used in the paper that the method is less sensitive to errors in the initial conditions than the J_2-invariant approach.

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