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Transient Free-Surface Hydrodynamics Using Rational Approximation of the Green's Function

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Rational approximations in the frequency domain are developed for the source function of linear free-surface hydrodynamics using the recently uncovered fourth-order ordinary differential equation (ODE) satisfied by the time-domain source function. The radiation problem for a floating body in deep water is formulated using a source plus wave-free potential expansion for the fluid. The inherent rational dependence on frequency of the wave-free potentials as well as the source approximation are used to develop a system of constant-coefficient ODE's for the radiation impedance which can be used to develop the motion of the body in a simple manner. The technique is applied to the heaving motion of a floating sphere with good results. The application to more general body geometries is explored by formulating the frequency-domain problem using the variational principle of Chen and Mei and exploiting its polynomial dependence on frequency.

Introduction

THIS PAPER considers the small motions of a body that floats on the surface of an infinitely deep ocean of infinite extent in the presence of small-amplitude surface waves. The fluid medium is incompressible, of constant density, inviscid, and irrotational. The ultimate goal is the determination of a finite-dimensional, linear time-invariant (LTI) representation for the mapping from (transient) wave motion to body motion. This can form a basis for simulation and is a necessary prelude to control system design. This work considers only the radiation problem, and the body and fluid motion are assumed to be driven by appropriate initial conditions.

John (1950) studied the problem in the frequency domain in terms of steady-state time-harmonic solutions and Cummins (1962) and Wehausen (1967,1971) noted the connection to time-domain transient solutions. The latter author noted that inverse Fourier transformation of time-harmonic solutions was but one possibility and showed that the problem could be formulated directly in the time domain using a time-varying transient Green's function. These functions are known from the work of Finkelstein (1957) and have been used by Yeung (1982), Newman (1985), Beck & Liapis (1987), and Pot & Jami (1991) who studied the radiation problem for various cylinders and spheres. Beck & Magee (1990) presented transient calculations for a realistic ship section and discussed the computation of the Green's function.

Ursell (1964) and Maskell & Ursell (1970) studied the transient motion of a two-dimensional cylinder and the properties of the added mass and damping coefficients were used to infer asymptotic properties of the temporal solution. The full solution was obtained numerically using inverse Fourier transformation.

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The key ideas used here are the relationship between the transient and time-harmonic problems furnished by the Fourier transformation and the recognition that linear, constant-coefficient ordinary differential equations (ODE's) in the time domain correspond to rational dependence on the frequency variable in the frequency domain. This motivates the use of a far-field expansion using wave-free potentials and source terms located at the origin of the coordinate system. Such expansions were pioneered by Ursell (1949) in two dimensions. The wave-free potentials exhibit polynomial dependence on wave number K, hence frequency in the deep water case.

The recent work of Clément (1998) uncovered a linear fourth-order time-varying ODE whose solution yields the time-domain source function. The simple form of this ODE is used here to generate sufficiently accurate rational approximations to the source function in the frequency domain. With this in hand, multiplication by a source function (or its spatial derivatives) in the frequency domain yields convolution operators in the time domain whose action can be obtained as the "output" of a system of linear constant-coefficient ODE's corresponding to the rational approximation.

The effectiveness of the expansion is gaged by applying it to the entire radiation field exterior to a heaving hemisphere. This follows directly from the work of Barakat (1962) and Hulme (1982) who used such an expansion to solve the problem on a frequency-by-frequency basis. We use it to obtain a set of first-order ODE's that are forced by the heave velocity and an appropriate output yields the hydrodynamic pressure force. These must be coupled to the body motion equation to develop the complete statement of the transient hydrodynamics of a floating body. With this single set of equations, it is demonstrated that the added mass and damping coefficients can be reliably derived over a large range of frequencies as well as yielding the transient motion of the body.

The extension of the approach to more general body geometries is handled by formulating the frequency domain problem using the variational principle of Chen & Mei (1974) (Mei

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1989). This is a flexible approach wherein an inner solution—inside a hemisphere enclosing the wetted portion of the body—can be expanded using finite elements to handle complex geometries. Furthermore, contributions to the motion equations from this region exhibit polynomial dependence on K given the form of the variational principle. For the outer region, we propose the use of the wave-free potentials plus source terms approximated according to the scheme of the paper. This outer solution satisfies all of the requirements of the field problem and is neatly matched along with its normal derivative to the inner solution along the hemispherical boundary. The resulting equations in the time domain for the expansion coefficients will be of the required form. Nestegard & Sclavounos (1984) used such an expansion in two dimensions in the frequency domain to model the far field.

Yu & Falnes (1995) were motivated by the same goal of a state-space representation for the radiation map from body velocity to hydrodynamic force. Their technique required the solution of the boundary value problem at several frequencies to obtain the added mass and damping coefficients. The corresponding impulse response was obtained by (numerical) inverse Fourier transformation and "fitted" with an LTI system. In previous work (Damaren 1997,1999), we used a similar approach but the radiation impedance was fitted in the frequency domain with specially chosen rational analytic functions.

The advantages of the approach used here are myriad. The transformation from frequency to time domain is performed analytically. The computational effort is roughly that of the frequency domain problem for a single frequency. The convolution operators characteristic of approaches using the time-domain Green's function are avoided as are the accompanying computer memory requirements, Green's function evaluations, and the time-stepping nature of the solution. Given the standard first-order form of the combined body and "radiation" equations, the solution of the initial value problem is obtainable in terms of a matrix exponential. Since the radiation potentials and their spatial derivatives are readily available on the body, it is anticipated that the diffraction forces due to an incident wave field can be obtained using the transient form of the Haskind relations (Wehausen 1967, Mei 1989). Most important of all, is the embedding of the problem within the framework of linear system theory. This opens the door to exploiting many standard tools for model order reduction and the formulation of control problems for dealing with stationkeeping and vibration suppression.

Transient hydrodynamics for a floating body

The small displacements of a floating body $\mathcal{B}$ will be described by $\mathbf{w}(\mathbf{r},t) = [w_1\ w_2\ w_3]^T$ where $\mathbf{r} = [x\ y\ z]^T$. The z-axis is vertically upwards and the origin of the coordinate system lies in the undisturbed free surface. The external forces acting on $\mathcal{B}$ are given by $\mathbf{f}(\mathbf{r},t)$. A spatial discretization of the form

$$\mathbf{w}(\mathbf{r},t) = \sum_{\alpha=1}^{N} \mathbf{\Gamma}_\alpha(\mathbf{r}) u_\alpha(t) \tag{1}$$

is employed and included in the $\mathbf{\Gamma}_\alpha$ are the six rigid-body motions. The corresponding (generalized) forces are given by

$$f_\alpha(t) \triangleq \int_{\mathcal{B}} \mathbf{\Gamma}_\alpha^T \mathbf{f}\, dV \tag{2}$$

Let $\mathbf{n}(\mathbf{r})$ denote the components of the outward normal to the wetted portion of $\mathcal{B}$, S, and define $n_\alpha(\mathbf{r}) \triangleq \mathbf{n}^T \mathbf{\Gamma}_\alpha$.

The motion of the fluid in $V = \{z \le 0 \backslash \mathcal{B}\}$ is governed by the velocity potential $\Phi(\mathbf{r},t)$ which satisfies

$$\begin{aligned} \nabla^2 \Phi &= 0, \qquad \mathbf{r} \in V; \\ \frac{\partial^2 \Phi}{\partial t^2} &= -g \frac{\partial \Phi}{\partial z}, \qquad \mathbf{r} \in F; \\ \lim_{z \to -\infty} \frac{\partial \Phi}{\partial z} &= 0 \end{aligned} \tag{3}$$

where $F = \{z = 0 \backslash \mathcal{B}\}$ denotes the free surface and g is the acceleration due to gravity. On the surface of the body,

$$\frac{\partial \Phi}{\partial n} = V_n(\mathbf{r},t) \triangleq \mathbf{n}^T \dot{\mathbf{w}} = \sum_{\alpha=1}^{N} n_\alpha(\mathbf{r}) \dot{u}_\alpha, \qquad \mathbf{r} \in S \tag{4}$$

In addition, Φ is bounded everywhere in the fluid as are its first derivatives as $r \triangleq \sqrt{x^2+y^2} \to \infty$. As we are only concerned with the free radiation problem here, the problem statement is completed by specifying appropriate initial conditions. We take $\Phi(\mathbf{r},0) = 0$ in V, $\partial\Phi(\mathbf{r},0)/\partial t = 0$ on F, and assume that $u_\alpha(0)$ and $\dot{u}_\alpha(0)$ are given.

Using Bernoulli's equation, the (linearized) component of $f_\alpha(t)$ in (2) stemming from fluid forces is

$$f_\alpha(t) = \int_S \left(\rho \frac{\partial \Phi}{\partial t} + \rho g z\right) n_\alpha\, dS = f_{R\alpha}(t) - \sum_{\beta=1}^{N} K_{s,\alpha\beta} u_\beta(t) \tag{5}$$

Here, ρ is the fluid density and $K_{s,\alpha\beta}$ are the hydrostatic restoring coefficients; see Newman (1977) for the rigid-body case. The radiation forces satisfy

$$f_{R\alpha}(t) = \rho \int_S \left[\frac{\partial \Phi}{\partial t} n_\alpha\right] dS \tag{6}$$

The problem is to now obtain the subsequent motion of the body for $t \ge 0$ given equations (3)–(6) augmented by the body motion equations and the initial conditions.

It is helpful to consider the frequency domain formulation of the transient problem and the relationship between them. The Laplace transform of Φ is defined by

$$\mathcal{L}\{\Phi(\mathbf{r},t)\} = \tilde{\Phi}(\mathbf{r},s) = \int_0^\infty e^{-st} \Phi(\mathbf{r},t)\, dt, \qquad Re\{s\} > 0$$

a notation used throughout. The corresponding Fourier transform will also be used and is designated $\mathcal{F}\{\Phi(\mathbf{r},t)\} = \tilde{\Phi}(\mathbf{r},j\omega)$. In the absence of arguments, say $\tilde{\Phi}$, the Fourier transform is intended. Transformation of the boundary value problem in (5) leads to

$$\begin{aligned} \nabla^2 \tilde{\Phi} &= 0, \qquad \mathbf{r} \in V; \\ \frac{\partial \tilde{\Phi}}{\partial z} &= K\tilde{\Phi}, \qquad \mathbf{r} \in F; \\ \lim_{z \to -\infty} \frac{\partial \tilde{\Phi}}{\partial z} &= 0 \end{aligned} \tag{7}$$

where $K = \omega^2/g$ and the boundary condition on the body can be written as

$$\frac{\partial \tilde{\Phi}}{\partial n} = \tilde{V}_n = \sum_{\alpha=1}^{N} n_\alpha(\mathbf{r}) \tilde{u}_\alpha \tag{8}$$

This statement of the problem is formally equivalent to assuming time-harmonic dependence of the form $\Phi(\mathbf{r},t) = \tilde{\Phi}(\mathbf{r},j\omega) \times e^{j\omega t}$. In this light, $\tilde{\Phi}$ must also satisfy an outgoing radiation condition as $r \to \infty$.

The potential can be obtained as a solution to a Fredholm integral equation of the form

$$
\begin{aligned}
& -2\pi\widetilde{\Phi}(\mathbf{r}, j\omega) + \int_S \frac{\partial \widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; j\omega)}{\partial n_\xi} \widetilde{\Phi}(\boldsymbol{\xi}) dS_\xi \\
&= \int_S \widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; j\omega) \frac{\partial \widetilde{\Phi}(\boldsymbol{\xi}, j\omega)}{\partial n_\xi} dS_\xi
\end{aligned} \tag{9}
$$

where $\widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; j\omega)$ is the Green's function which satisfies Laplace's equation and the free surface, bottom, and radiation conditions (Thorne 1953). Letting $\boldsymbol{\xi} = [\xi \;\; \eta \;\; \zeta]^T$:

$$
\begin{aligned}
\widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; j\omega) &\triangleq \frac{1}{R_\xi} + \int_0^\infty \frac{\kappa + K}{\kappa - K} e^{\kappa(z+\zeta)} J_0(\kappa r_\xi)\, d\kappa \\
&= \frac{1}{R_\xi} - \frac{1}{R_1} + 2\int_0^\infty \frac{\kappa}{\kappa - K} e^{\kappa(z+\zeta)} J_0(\kappa r_\xi)\, d\kappa
\end{aligned} \tag{10}
$$

where $r_\xi^2 = (x-\xi)^2 + (y-\eta)^2$, $R_\xi^2 = r_\xi^2 + (z-\zeta)^2$, and $R_1^2 = r_\xi^2 + (z+\zeta)^2$. The integration in (10) passes above the pole at $\kappa = K$ in order to satisfy the radiation condition.

Taking the inverse Fourier transform of (9) yields the transient form of the boundary value problem:

$$
\begin{aligned}
& -2\pi\Phi(\mathbf{r}, t) + \int_0^t \int_S \frac{\partial \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t-\tau)}{\partial n_\xi} \Phi(\boldsymbol{\xi}, \tau)\, dS_\xi\, d\tau \\
&= \int_0^t \int_S \mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t-\tau) \frac{\partial \Phi(\boldsymbol{\xi}, \tau)}{\partial n_\xi}\, dS_\xi\, d\tau
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t) &= \left[\frac{1}{R_\xi} - \frac{1}{R_1}\right] \delta(t) \\
&+ 2\int_0^\infty e^{\kappa(z+\zeta)} J_0(\kappa r_\xi) \sqrt{g\kappa} \sin(t\sqrt{g\kappa})\, d\kappa\, H(t)
\end{aligned} \tag{12}
$$

Here, $\delta(t)$ is the Dirac delta function and $H(t)$ is the Heaviside step function. It is readily verified that the Fourier transform of this expression is given by (10), although $\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t-\tau)$ is more commonly introduced as the time-varying potential due to an impulsive source at $t = \tau$ and $\mathbf{r} = \boldsymbol{\xi}$. Given the linear nature of the problem, the surface S may be taken as that of the fixed equilibrium position in (9) and (11).

In the absence of incident waves, Φ can be further decomposed as

$$
\begin{aligned}
\Phi &= \sum_{\alpha=1}^N \phi_\alpha * \dot{u}_\alpha; \\
\frac{\partial \phi_\alpha}{\partial n} &= n_\alpha \delta(t), \qquad \mathbf{r} \in S
\end{aligned} \tag{13}
$$

where $(*)$ denotes temporal convolution. It is readily verified that $\widetilde{\phi}_\alpha$ satisfies (9) with $\partial\widetilde{\phi}_\alpha/\partial n = n_\alpha$ on S. Correspondingly, $\phi_\alpha(\mathbf{r}, t)$ satisfies (11) which has the advantage of being independent of the body motion. Further decompositions into memoryless and memory portions are possible along the lines of Cummins (1962) and Wehausen (1967) as used by Beck & Liapis (1987).

The radiation forces in (6) upon substitution of (13) satisfy

$$
f_{R\alpha} = \rho \sum_{\beta=1}^N \int_S \left[\frac{\partial}{\partial t}\left(\phi_\beta * \dot{u}_\beta\right) n_\alpha\right] dS \tag{14}
$$

and taking the Fourier transform gives

$$
\begin{aligned}
\widetilde{f}_{R\alpha} &= -\sum_{\alpha=1}^N \widetilde{H}_{\alpha\beta}(j\omega)\widetilde{u}_\beta, \\
\widetilde{H}_{\alpha\beta}(j\omega) &\triangleq -j\omega\rho \int_S \left(\widetilde{\phi}_\beta n_\alpha\right) dS
\end{aligned} \tag{15}
$$

where $\widetilde{H}_{\alpha\beta}$ are the matrix elements of what shall be termed the radiation impedance. Their real and imaginary parts may be related to the added damping and mass coefficients, respectively.

The goal here is a formulation in the frequency domain which yields an approximation to $\widetilde{H}_{\alpha\beta}$ exhibiting rational dependence on frequency. This allows the radiation map to be realized as a linear system of constant-coefficient differential equations forced by the body velocities $\dot{u}_\beta(t)$ with the radiation forces $f_{R\alpha}(t)$ corresponding to appropriate outputs. In previous papers (Damaren 1997,1999), this was accomplished by fitting values $\widetilde{H}_{\alpha\beta}(j\omega)$ with rational functions of s exhibiting key properties, namely analyticity in the right half-plane coupled with nonnegativity of the added damping coefficient. The drawback of that approach is the need to solve the boundary value problem in (9) at many frequencies for each mode n_α.

In the present work, we seek to *analytically* construct a rational description of $\widetilde{H}_{\alpha\beta}(j\omega)$. A necessary step en route is the development of a similar representation for the Green's function $\widetilde{\mathcal{G}}$. This will permit temporal convolutions such as those occurring in (11) to be replaced with the solution of a system of nonhomogeneous *constant-coefficient* ODE's. A potential source of confusion arises since the rational approximation of $\widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; s)$ is obtained in the next section using a *time-varying* ODE satisfied by (the memory portion of) $\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t)$.

Rational approximation of the Green's function

The Green's function in equation (12) can be decomposed into memoryless and memory portions by writing

$$
\mathcal{G}(\mathbf{r}, \boldsymbol{\xi}; t) = \left[\frac{1}{R_\xi} - \frac{1}{R_1}\right] \delta(t) + G(\mathbf{r}, \boldsymbol{\xi}; t)
$$

where $G(\mathbf{r}, \boldsymbol{\xi}; t)$ denotes the integral in (12). Correspondingly, $\widetilde{G}(\mathbf{r}, \boldsymbol{\xi}; j\omega)$ is the integral in (10). Note that if $\boldsymbol{\xi} = \mathbf{0}$, a special case to be considered in detail later, $G_0(\mathbf{r}, t) \triangleq G(\mathbf{r}, \mathbf{0}; t) = \mathcal{G}(\mathbf{r}, \mathbf{0}; t)$ since $R_\xi = R_1$. Clément (1998) has shown that $G(\mathbf{r}, \boldsymbol{\xi}; t)$ and its spatial derivatives satisfy linear time-varying fourth-order equations of the following form:

$$
A_4 \frac{\partial^4 F}{\partial \hat{t}^4} + A_3 \hat{t} \frac{\partial^3 F}{\partial \hat{t}^3} + (A_2 + \bar{A}_2 \hat{t}^2) \frac{\partial^2 F}{\partial \hat{t}^2} + A_1 \hat{t} \frac{\partial F}{\partial \hat{t}} + A_0 F = 0 \tag{16}
$$

with

$$
F(0) = \frac{\partial^2 F}{\partial \hat{t}^2}(0) = 0, \qquad \frac{\partial F}{\partial \hat{t}}(0) = B_1, \qquad \frac{\partial^3 F}{\partial \hat{t}^3}(0) = B_3 \tag{17}
$$

For $F(\hat{t}) = \widehat{G}(\mathbf{r}, \boldsymbol{\xi}; \hat{t}) \triangleq \sqrt{R_1^3/g}\, G(\mathbf{r}, \boldsymbol{\xi}; \hat{t}\sqrt{R_1/g})$ (the spatial dependence is momentarily dropped for convenience), the "constants" A_i, B_i are given by

$$
\begin{aligned}
& A_4 = 1, \qquad A_3 = \mu, \qquad A_2 = 4\mu, \qquad \bar{A}_2 = \frac{1}{4}, \\
& A_1 = \frac{7}{4}, \qquad A_0 = \frac{9}{4}, \qquad B_1 = 2\mu, \qquad B_3 = 2 - 6\mu^2
\end{aligned}
$$

where $\mu = -(z+\zeta)/R_1$ and R_1 was defined after equation (10). Note that the nondimensionalized function $\widehat{G}$ depends (spatially) only on μ. The corresponding expressions for the

A_i, B_i when $F = R_1\partial\widehat{G}/\partial r$ or $F = R_1\partial\widehat{G}/\partial z$, can be found in Clément (1998).

Taking the Laplace transform of (16) while using the initial conditions in (17) gives

$$\gamma_2 s^2 \widetilde{F}''(s) + (\beta_3 s^3 + \beta_1 s)\widetilde{F}'(s) + (\alpha_4 s^4 + \alpha_2 s^2 + \alpha_0)\widetilde{F}(s) = \delta_2 s^2 + \delta_0 \tag{18}$$

where

$$\begin{aligned}
\gamma_2 &= \bar{A}_2, \\
\beta_1 &= 4\bar{A}_2 - A_1, \\
\beta_3 &= -A_3, \\
\alpha_0 &= A_0 - A_1 + 2\bar{A}_2, \\
\alpha_2 &= A_2 - 3A_3, \\
\alpha_4 &= A_4, \\
\delta_0 &= (A_2 - A_3)B_1 + A_4 B_3, \\
\delta_2 &= A_4 B_1
\end{aligned}$$

We seek a rational approximation of $\widetilde{F}(s)$ satisfying (18) while noting that $\widetilde{F}(s)$ is analytic in $Re\{s\} > 0$. It is tempting to seek a power series solution in s but this converges in disk-like regions which is inconsistent with the behavior of $\widetilde{F}$ in the half-plane. This can be remedied by using the bilinear transformation $s = (1-z)/(1+z)$ and its inverse $z = (1-s)/(1+s)$ which isomorphically maps the open right half of the s-plane onto the open unit disk $|z| < 1$. (Hopefully no confusion will arise between the complex variable z and the spatial depth variable which has been implicitly absorbed into R_1 and μ.) Writing $\widehat{F}(z) = \widetilde{F}[(1-z)/(1+z)]$, and transforming (18) yields

$$\begin{aligned}
&\gamma_2(z-1)^2(z+1)^6\widehat{F}''(z) \\
&\quad + 2[\gamma_2(z-1)^2(z+1)^5 + \beta_3(z-1)^3(z+1)^3 \\
&\qquad + \beta_1(z-1)(z+1)^5]\widehat{F}'(z) \\
&\quad + 4[\alpha_4(z-1)^4 + \alpha_2(z-1)^2(z+1)^2 + \alpha_0(z+1)^4]\widehat{F}(z) \\
&= 4(z+1)^2[\delta_2(z-1)^2 + \delta_0(z+1)^2]
\end{aligned} \tag{19}$$

where $(\cdot)'$ now refers to differentiation with respect to z.

Since $\widehat{F}(z)$ is a bounded analytic function in the unit disk, it permits a representation in terms of the uniformly convergent power series

$$\widehat{F}(z) = \lim_{N\to\infty} \sum_{i=0}^{N} h_i z^i \tag{20}$$

where the $\{h_i\}$ are known as the Hankel coefficients. Substituting this into (19) and matching powers of z yields an infinite system of linear equations for the $\{h_i\}$ which can be truncated at $N = 2n$. A rational approximation to $\widetilde{F}(s)$ is then obtained by letting $z = (1-s)/(1+s)$ in (20).

Alternatively, state-space manipulations are possible. Letting

$$\widehat{F}(z) = h_0 + L(z^{-1}), \qquad L(z) \triangleq \sum_{i=1}^{2n} h_i z^{-1} \tag{21}$$

Kailath (1980) shows that

$$L(z) = \frac{\sum_{i=1}^{n} b_i z^{n-i}}{z^n + \sum_{i=1}^{n} a_i z^{n-i}} \tag{22}$$

where $\mathbf{a}^T \triangleq [a_1 \cdots a_n]$, $\mathbf{b}^T \triangleq [b_1 \cdots b_n]$ are obtained from $\mathbf{L}\mathbf{a} = -\mathbf{h}$, $\mathbf{b} = \mathbf{T}\bar{\mathbf{a}}$, and

$$\mathbf{L} = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n-1} \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ h_n & h_{n-1} & \cdots & h_1 \end{bmatrix},$$

$$\mathbf{h}^T = [\,h_{n+1} \quad h_{n+2} \quad \cdots \quad h_{2n}\,],$$

$$\bar{\mathbf{a}}^T = [\,1 \quad a_1 \quad \cdots \quad a_{n-1}\,] \tag{23}$$

A state-space representation of $h_0 + L(z)$ can be written in the companion form

$$h_0 + L(z) = \mathbf{c}_d^T(z\mathbf{1} - \mathbf{A}_d)^{-1}\mathbf{b}_d + d_d \tag{24}$$

where

$$\mathbf{A}_d = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\begin{aligned}
\mathbf{c}_d^T &= [b_1 \cdots b_n], \\
\mathbf{b}_d^T &= [1\ 0 \cdots 0], \\
d_d &= h_0
\end{aligned}$$

The s-domain Green's function $\widetilde{F}(s)$ can then be obtained as

$$\widetilde{F}(s) = \widehat{F}\left(\frac{1-s}{1+s}\right) = h_0 + L\left(\frac{1+s}{1-s}\right) = \mathbf{c}^T(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b} + d \tag{25}$$

where

$$\begin{aligned}
\mathbf{A} &= (\mathbf{A}_d - \mathbf{1})(\mathbf{A}_d + \mathbf{1})^{-1}, \\
\mathbf{b} &= \mathbf{b}_d, \\
\mathbf{c}^T &= \mathbf{c}_d^T(\mathbf{1} + \mathbf{A}_d)^{-1}(\mathbf{1} - \mathbf{A}), \\
d &= d_d - \mathbf{c}_d^T(\mathbf{1} + \mathbf{A}_d)^{-1}\mathbf{b}_d
\end{aligned}$$

Since, $\widetilde{F}(s)$ is strictly proper, i.e., $\lim_{s\to\infty} \widetilde{F}(s) = 0$, d should vanish which provides a check on the numerical procedure. At this point the (nondimensionalized) memoryless portion of $\widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; s)$ in (10) could be incorporated by setting $d = R_1(R_\xi^{-1} - R_1^{-1})$; $\widetilde{F}(j\omega)$ can then be identified with $R_1\widetilde{\mathcal{G}}(\mathbf{r}, \boldsymbol{\xi}; j\omega)$ in (10).

The major advantage of (25) lies in its time-domain realization. Consider $\widetilde{y}(s) = \widetilde{F}(s)\widetilde{q}(s)$, which mimics the form of the spatial integrand on the right-hand side of (9). Then,

$$\begin{aligned}
y(\hat{t}) &= \int_0^{\hat{t}} F(\hat{t} - \tau)q(\tau)\,d\tau \\
&= \mathbf{c}^T\mathbf{x}(\hat{t}) + dq(\hat{t}), \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}q(\hat{t})
\end{aligned} \tag{26}$$

Hence the convolution, which mirrors that in the spatial integrand of the right-hand side of (11), can be evaluated for fixed $(\mathbf{r}, \boldsymbol{\xi})$ by solving (26). Note that $F(\hat{t})$ is the impulse response, i.e., $y(\hat{t})$ when $q(\hat{t}) = \delta(\hat{t})$ which can also be obtained by solving $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{b}$ leading to $y(\hat{t}) = \mathbf{c}^T\exp(\mathbf{A}t)\mathbf{b} + d\delta(\hat{t})$. It is worthy of note that Clément (1998) has suggested representing the mapping from q to y in (26) using nonhomogeneous ODE's with time-varying coefficients. Such an approach must necessarily yield a time-varying convolution operator which is at odds with the known time-invariant nature of the convolution as required by its origins as multiplication in the frequency domain. This latter fact must remain in spite of $F(\hat{t})$ being the exact solution of a time-varying ODE.

Source plus wave-free potential expansion for $\widetilde{\Phi}$

The above shows how temporal convolutions involving the Green's function can be approximated by a system of first-order constant-coefficient ODE's. In principle, this can be applied to the solution of (11) but this requires many such systems corresponding to several points on the body. As an alternative, we suggest the use of an expansion for $\widetilde{\Phi}$ consisting of a discrete source(s) at the origin and a series of wave-free potentials (Barakat 1962). A spherical coordinate system will be used from here on with

$$x = R\sqrt{1-\mu^2}\cos\phi$$
$$y = R\sqrt{1-\mu^2}\sin\phi$$
$$z = -R\mu, \qquad \mu = \cos\theta \tag{27}$$

This definition of μ is consistent with that used previously when $\boldsymbol{\xi} = \mathbf{0}$.

In the reduced case of a heaving body exhibiting vertical axisymmetry, Hulme (1982) notes that $\widetilde{\Phi}$ can be expanded as

$$\widetilde{\Phi}(\mathbf{r}, j\omega) = a^2\widetilde{G}_0(\mathbf{r}, j\omega)\widetilde{q}_0(j\omega) + \sum_{n=1}^{\infty}\widetilde{g}_n(\mathbf{r}, j\omega)\widetilde{q}_n(j\omega) \tag{28}$$

where

$$\widetilde{g}_n = \left[\frac{K}{2n}\frac{1}{R^{2n}}P_{2n-1}(\mu) + \frac{1}{R^{2n+1}}P_{2n}(\mu)\right]a^{2n+2} \tag{29}$$

are axisymmetric wave-free potentials, $P_n(\mu)$ are Legendre polynomials, $\widetilde{G}_0(\mathbf{r}, j\omega) \triangleq \widetilde{G}(\mathbf{r}, \mathbf{0}; j\omega)$ is the Green's function for a source at the origin, $\widetilde{q}_n(j\omega)$, $n = 0, 1, 2, \cdots$, are unknown "source strengths," and a is a reference dimension. The time-domain equivalent of (28) is

$$\Phi(\mathbf{r}, t) = a^2\int_0^t G_0(\mathbf{r}, t-\tau)\, q_0(\tau)\, d\tau + \text{wave-free potential terms} \tag{30}$$

which has the advantage of a single convolution for given $\mathbf{r}$ and the wave-free terms involve linear combinations of $q_n(t)$ and $\ddot{q}_n(t)$ given the affine dependence of the $\widetilde{g}_n$ on $K = -(j\omega)^2/g$.

The source strengths in (28) [or (30)] must be selected to yield the boundary condition in (8) [or (4)] and (30) must be substituted into (6) to yield the hydrodynamic forces. In the next section, these operations are shown to give rise to more general expressions of the form

$$\mathcal{L}^{-1}\left\{\int_S w(\mathbf{r})\widetilde{G}_0(\mathbf{r}, s)\widetilde{q}_0(s)\, dS\right\}$$
$$= \int_0^t \int_S w(\mathbf{r})G_0(\mathbf{r}, t-\tau)\, dS\, q_0(\tau)\, d\tau \tag{31}$$

for given spatial weightings $w(\mathbf{r})$. What is needed is an LTI system whose impulse response is $\int_S w(\mathbf{r})G_0(\mathbf{r}, t)\, dS$. Its Laplace transform is $\int_S w(\mathbf{r})\widetilde{G}_0(\mathbf{r}, s)\, dS$. In terms of z, (20) may be used to write

$$\widehat{F}(\mathbf{r}, z) = a\widetilde{G}_0[\mathbf{r}, (1-z)/(1+z)] = \lim_{N\to\infty}\sum_{i=0}^{N} h_i(\mathbf{r})z^i \tag{32}$$

Given the uniformly convergent nature of the series, this gives

$$\int_S w(\mathbf{r})\widehat{F}(\mathbf{r}, z)\, dS = \lim_{N\to\infty}\sum_{i=0}^{N} H_i z^i, \qquad H_i = \int_S w(\mathbf{r})h_i(\mathbf{r})\, dS \tag{33}$$

It is proposed to evaluate the spatial integration by Gauss-Legendre quadrature. Therefore, the Hankel parameters $\{h_i\}$ are evaluated for a discrete set of points on S, $\mathbf{r} = \mathbf{r}_j$, and the integration performed as a weighted sum of the h_i evaluated at these points thus yielding the effective parameters $\{H_i\}$. These can then be subjected to the operations in equations (21)–(25), resulting in effective $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ matrices; the convolution in (31) can then be obtained (in nondimensionalized form) as the solution of (26).

The ability to do spatially weighted convolutions in this manner is another reason for working in the z-plane. The linear nature of the dependence on the h_i in (32) is heavily exploited when spatially superimposing them in (33). This would not be possible if one worked directly with a rational expression in the s-plane. We shall also require $\widetilde{F}(s)$ corresponding to $F(\hat{t}) = a\partial\widehat{G}_0/\partial R$ where $\partial\widehat{G}_0/\partial R = \sqrt{1-\mu^2}\partial\widehat{G}_0/\partial r - \mu\partial\widehat{G}_0/\partial z$. The $\{h_i\}$ can be obtained for $a\partial\widehat{G}_0/\partial r$ and $a\partial\widehat{G}_0/\partial z$ and the linear combination constructed. Spatial integrations can be dealt with in the manner given above.

Numerical examples

Here, we take $\boldsymbol{\xi} = \mathbf{0}$ and $\mathbf{r}$ is selected according to (27) with $R = R_1 = a$ for various values of $\mu = \cos\theta$. The behavior of $\widetilde{F}(j\omega) = \mathbf{c}^T(j\omega\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}$ generated by (25) was computed and the real and imaginary parts compared with those of $a\widetilde{G}_0(\mathbf{r}, j\omega)$. The latter were determined using the algorithms presented by Newman (1984) for evaluating (10). The impulse response corresponding to (26) was also calculated and compared with $\widehat{G}_0(\mathbf{r}, \hat{t})$, which was obtained by integrating (16) using the initial conditions in (17). In both cases, a fourth-order Runge-Kutta technique was used with a step-size of $\Delta\hat{t} = 0.05$. The convolution in (26) was tested for $q(\hat{t}) = \sin\hat{t}$. This was obtained using the same Runge-Kutta integration and compared

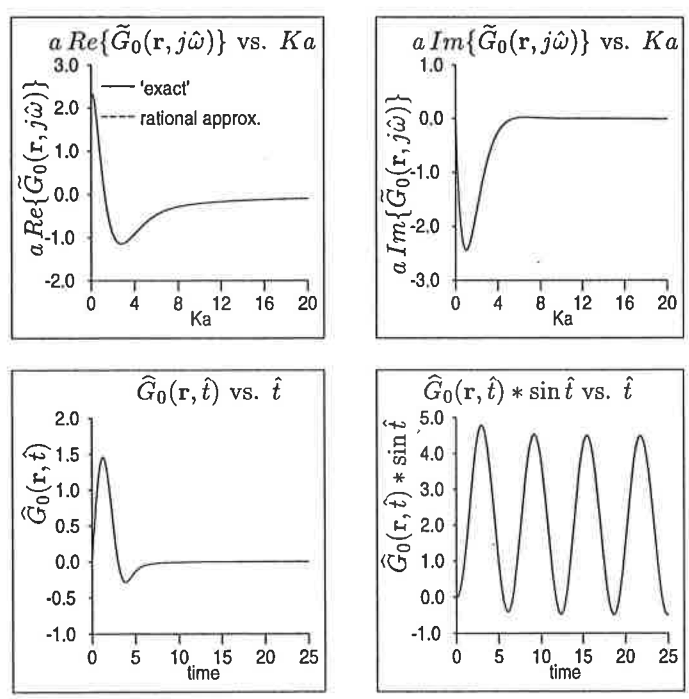

Fig. 1 Rational approximation of $\widetilde{G}_0$ vs. exact solution ($\mu = 0.9$)

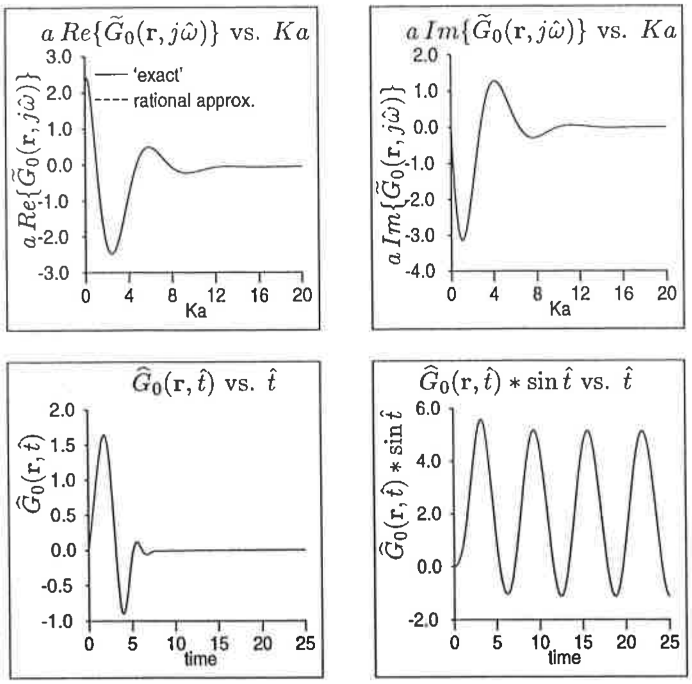

Fig. 2 Rational approximation of $\widetilde{G}_0$ vs. exact solution ($\mu = 0.5$)

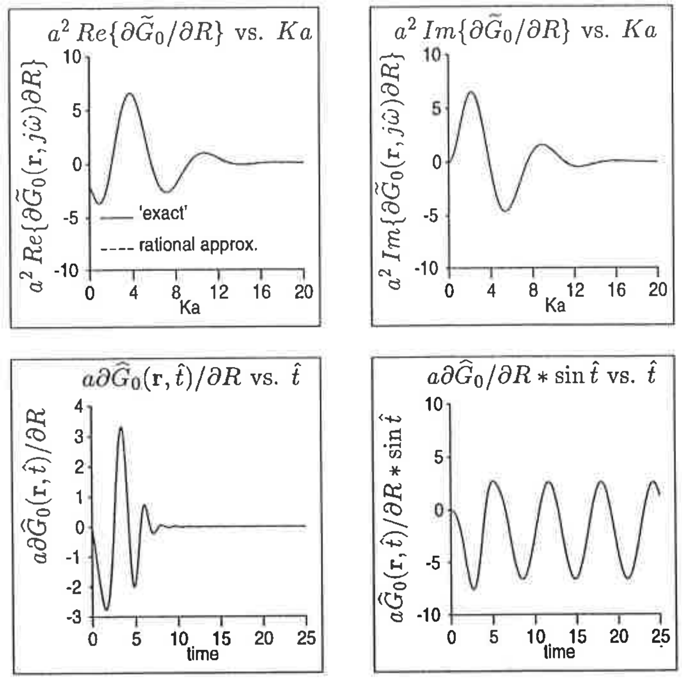

Fig. 4 Rational approximation of $\partial\widetilde{G}_0/\partial R$ vs. exact solution ($\mu = 0.5$)

with the result of convolving the "exact" solution of (16) with $q(\hat{t})$ using the trapezoidal rule with the same step-size.

The results of these calculations are presented in Figs. 1–3 for $\mu = 0.9$, $\mu = 0.5$, and $\mu = 0.1$, respectively. It is clear that excellent agreement is furnished by the rational approximation ($n = 20$) and corresponding ODE's for $\mu = 0.9$ and $\mu = 0.5$. As the free surface is approached ($\mu \to 0$), the accuracy decreases for a given value of n. However, a good fit to $\widetilde{G}_0$ is obtained in the range $0 \leq Ka \leq 10$ for $\mu = 0.1$ and the convolution at unit frequency is quite accurate. Notice the large error near $Ka = 14$ which can be attributed to an s-plane pole that is close to the imaginary axis. This pole corresponds to the frequency of the steady-state oscillation in the graph of the impulse response. Removing the pole created noticeable errors at low frequency. Similar behavior to that above for the Green's function was observed for approximations of $a^2\partial\widetilde{G}_0/\partial R$ and the results for $\mu = 0.5$ ($n = 14$) are shown in Fig. 4.

In Fig. 5, the dependence of the errors $|\widetilde{F}(j\omega) - a\widetilde{G}_0|$ and $|\widetilde{F}(j\omega) - a^2\partial\widetilde{G}_0/\partial R|$ on n are shown as a function of frequency for the corresponding cases in Figs. 1–4. The error is monotonically decreasing with n at each frequency until about $n = 15$. Thereafter, an improvement in accuracy at high frequencies appears to occur at the expense of accuracy in the low frequency region. This can be attributed to the fact that as $kr \to \infty$,

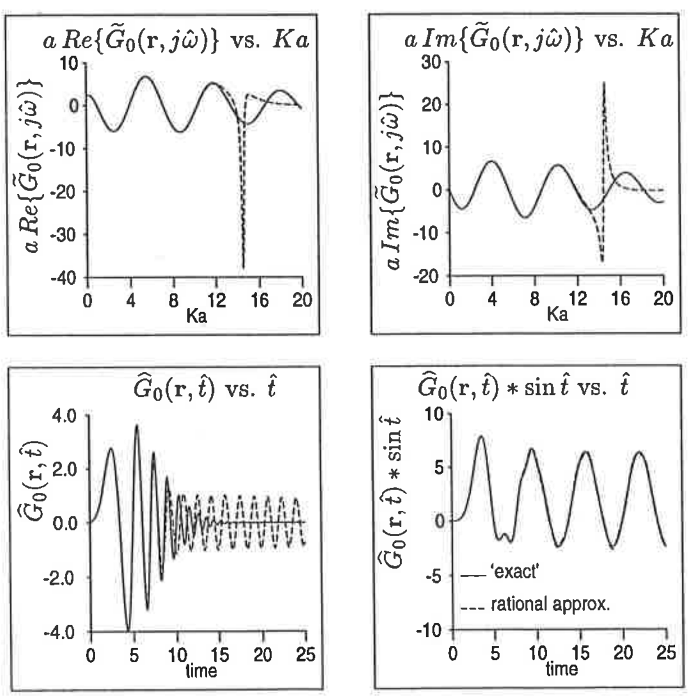

Fig. 3 Rational approximation of $\widetilde{G}_0$ vs. exact solution ($\mu = 0.1$)

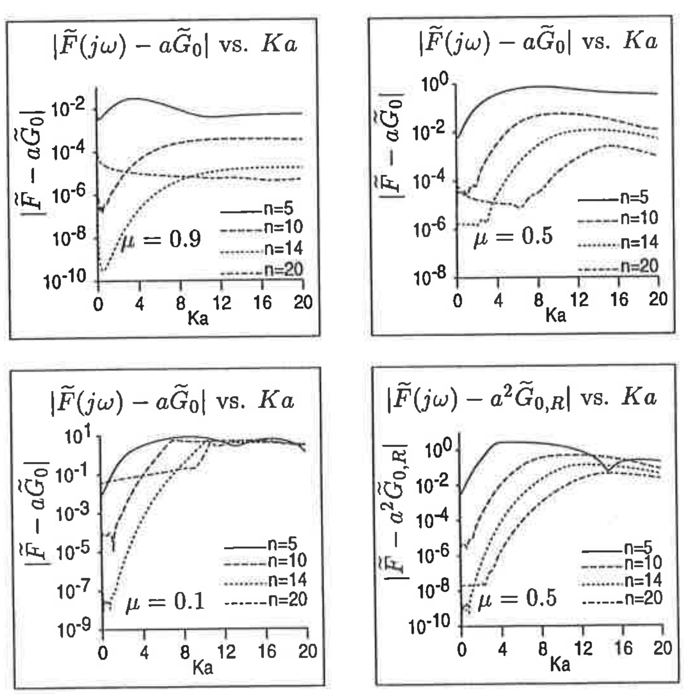

Fig. 5 Error in rational approximation of $\widetilde{G}_0$ for varying order

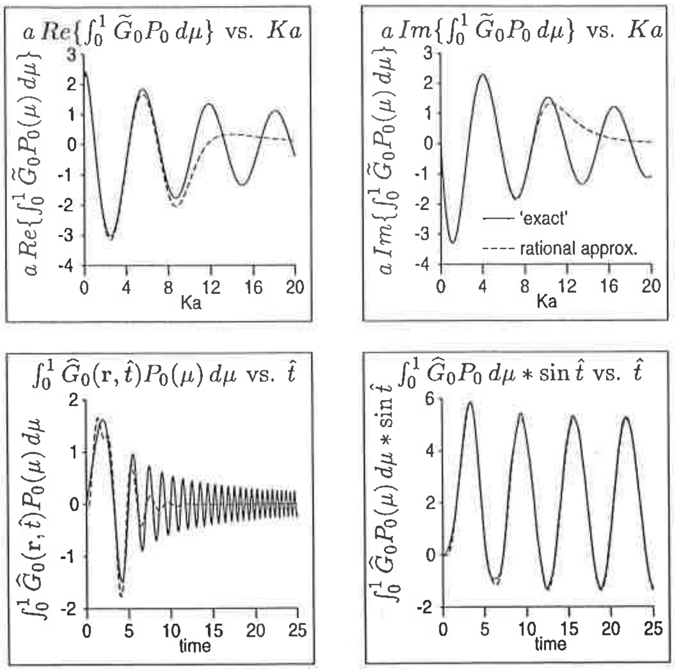

Fig. 6 Spatially weighted rational approximation vs. exact solution ($w = P_0$)

$a\widetilde{G}_0(\mathbf{r}, j\omega) \sim -2\pi jka \exp(kz)\sqrt{2/(\pi kr)} \exp[-j(kr-\pi/4)]$ with $z = -a\mu$ and $r = a\sqrt{1-\mu^2}$, which stems from the asymptotic approximation of the Hankel function. This is evidently difficult to approximate with rational functions especially as $\mu \to 0$. The exact form of $a\widehat{G}_0(\mathbf{r}, jw)$ when $\mu = 0$ is discussed by Clément (1998) and Beck & Magee (1990). The graph given here for $\mu = 0.1$ and $n = 20$ represents the case where the offending pole noted above has been removed.

The graphs analogous to Fig. 1 are shown in Fig. 6 for the weighted convolution in (31) when $w(\mathbf{r}) = P_0(\mu) = 1$

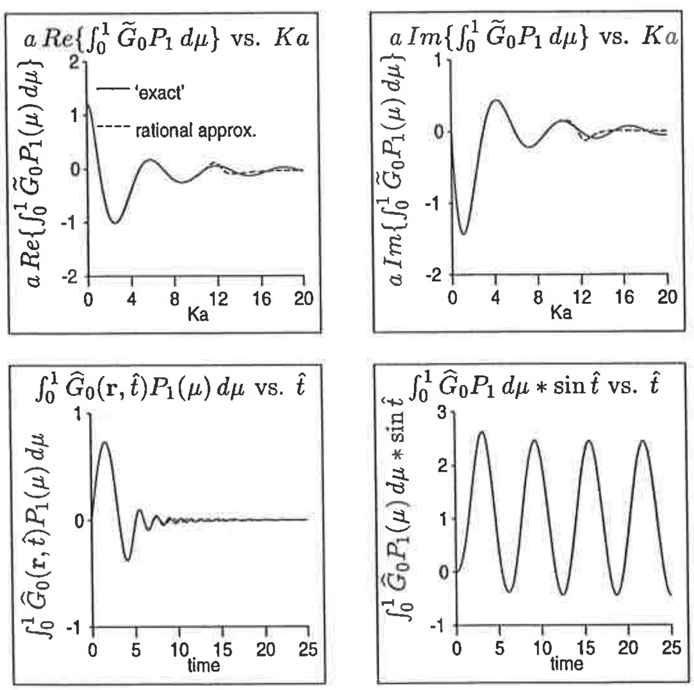

Fig. 7 Spatially weighted rational approximation vs. exact solution ($w = P_1$)

($dS = d\mu$). The approximate system was obtained using (33) with a 48-point Gauss-Legendre rule and truncated so that $n = 15$. The exact solutions were obtained by applying the same quadrature scheme to the "exact" time-domain solutions obtained at the required spatial locations. Similar results are shown in Fig. 7 for $w(\mathbf{r}) = P_1(\mu) = \mu$. As expected, the smaller weighting near the free surface improves the accuracy.

It should be noted that $\widetilde{G}_0(\mathbf{r}, s)$ is analytic in $Re\{s\} > 0$ given the causal nature of the corresponding time function. Naturally, its approximation $\widetilde{F}(s)$ as given by (25) should also have this property. This is equivalent to the requirement that the eigenvalues of $\mathbf{A}$ have negative real parts. Experience has shown that the algorithm of the previous section generates some eigenvalues with positive real parts but they correspond to modes of (25) that have very poor observability and/or controllability (Kailath 1980). Hence, they do not affect the overall input-output relationship between q and y in (26) and can be discarded. This is done through eigendecomposition of $\mathbf{A}$ followed by removal of the offending eigenvalues and eigenvectors prior to reconstructing $\mathbf{A}$. The values of the approximation order (n) given above reflect this removal.

Example: The heaving hemisphere

As a direct application of these results, consider the free heaving motion, $u(t)$, of a half-submerged sphere of radius a. The motion equation consistent with (5) is

$$\frac{2}{3}\rho\pi a^3 \frac{d^2u}{dt^2} = -\rho g\pi a^2 u(t) + f_R(t) \tag{34}$$

Letting $V(t) = du/dt$, the essential goal is the determination of a system of equations relating $V(t)$ to $-f_R(t)$ that are analogous in form to those in (26) relating q to y.

The frequency domain problem in the guise of steady-state harmonic motions has been formulated by Hulme (1982). The Fourier transform of the velocity potential satisfies (7) and a radiation condition as $r \to \infty$ and can be expressed according to (28). It is helpful to realize that $\mathcal{F}^{-1}\{a\widetilde{G}_0(\mathbf{r}, j\omega)\widetilde{q}_0(j\omega)\} = \widehat{G}_0(\mathbf{r}, \hat{t}) * q_0(\hat{t})$ where $\mathcal{F}^{-1}$ is the (dimensional) inverse Fourier transform and $\hat{t} = t\sqrt{g/a}$. The following convolution operators will be required:

$$z_1 \triangleq \int_0^1 P_1(\mu)\widehat{G}_0\Big|_{R=a} d\mu * q_0(\hat{t}), \tag{35}$$

$$y_0 \triangleq \int_0^1 a\,\frac{\partial \widehat{G}_0}{\partial R}\Big|_{R=a} d\mu * q_0(\hat{t}) \tag{36}$$

$$y_n \triangleq \int_0^1 aP_{2n-1}(\mu)\,\frac{\partial \widehat{G}_0}{\partial R}\Big|_{R=a} d\mu * q_0(\hat{t}), \qquad n = 1, 2, 3, \cdots \tag{37}$$

which are realized using the technique of the previous section as

$$z_1 = \bar{\mathbf{c}}_1^T \chi_1, \quad \dot{\chi}_1 = \bar{\mathbf{A}}_1\chi_1 + \bar{\mathbf{b}}_1 q_0 \tag{38}$$

$$y_0 = \mathbf{c}_0^T \mathbf{x}_0, \quad \dot{\mathbf{x}}_0 = \mathbf{A}_0\mathbf{x}_0 + \mathbf{b}_0 q_0 \tag{39}$$

$$y_n = \mathbf{c}_n^T \mathbf{x}_n, \quad \dot{\mathbf{x}}_n = \mathbf{A}_n\mathbf{x}_n + \mathbf{b}_n q_0, \qquad n = 1, 2, 3, \cdots \tag{40}$$

where $\dot{(\ )}$ will refer to the nondimensional time derivative $d(\ )/d\hat{t}$. The last of these is concisely written as

$$\mathbf{y}_a = \mathbf{C}_a\mathbf{x}_a, \quad \dot{\mathbf{x}}_a = \mathbf{A}_a\mathbf{x}_a + \mathbf{b}_a q_0 \tag{41}$$

where $\mathbf{y}_a = \text{col}\{y_n\}$, $\mathbf{x}_a = \text{col}\{\mathbf{x}_n\}$, $\mathbf{C}_a = \text{diag}\{\mathbf{c}_n^T\}$, $\mathbf{A}_a = \text{diag}\{\mathbf{A}_n\}$, and $\mathbf{b}_a = \text{col}\{\mathbf{b}_n\}$.

On the body, the boundary equation in (8) is

$$\left.\frac{\partial\widetilde{\Phi}}{\partial R}\right|_{R=a} = -\widetilde{V}(j\omega)P_1(\cos\theta) \tag{42}$$

Substitution of (28) into the body condition (42) and integration with respect to μ from 0 to 1 gives

$$-\frac{1}{2}\widetilde{V}(j\omega) = \int_0^1 a^2\frac{\partial\widetilde{G}_0}{\partial R}d\mu\,\widetilde{q}_0(j\omega) - \sum_{n=1}^{N_p}(Ka)I_{0,2n-1}\widetilde{q}_n(j\omega) \tag{43}$$

where the series (28) has been truncated at N_p terms and

$$I_{nm} \triangleq \int_0^1 P_n(\mu)P_m(\mu)\,d\mu$$

Analytical expressions for these integrals are given in (Hulme 1982).

Transforming (43) into the time domain while noting (36) and (39) gives

$$-\frac{1}{2}V(\hat{t}) = \mathbf{c}_0^T\mathbf{x}_0 + \mathbf{d}_0^T\ddot{\mathbf{q}}(\hat{t}) \tag{44}$$

where $\mathbf{q} = \text{col}\{q_n\}$, $\mathbf{d}_0^T = \text{row}\{I_{0,2n-1}\}$. Now, multiply the result of substituting (28) into (42) by $P_{2n-1}(\mu)$ and integrate from 0 to 1:

$$-\widetilde{V}I_{2n-1,1} = \int_0^1 a^2P_{2n-1}(\mu)\frac{\partial\widetilde{G}_0}{\partial R}d\mu\,\widetilde{q}_0 - \sum_{m=1}^{N_p}\widetilde{q}_m[(Ka)I_{2n-1,2m-1} + (2m+1)I_{2n-1,2m}]$$

Transforming to the time domain, we have

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{C}_a\mathbf{x}_a = \mathbf{b}_pV(t) \tag{45}$$

where $M_{nm} = I_{2n-1,2m-1}$, $K_{nm} = -(2m+1)I_{2n-1,2m}$, $b_{pn} = -I_{2n-1,1}$, and (41) has been used to describe (37) for $n = 1\cdots N_p$.

It is desirable to write (45) in conjunction with (38)–(40) in an equivalent first-order form. Defining $\mathbf{X} = \text{col}\{\dot{\mathbf{q}}, \mathbf{q}, \mathbf{x}_0, \chi_1, \mathbf{x}_a\}$,

$$\dot{\mathbf{X}} = \mathcal{A}_0\mathbf{X} + \mathcal{B}_0q_0 + \mathcal{B}_1V \tag{46}$$

where

$$\mathcal{B}_0 = \text{col}\{\mathbf{0}, \mathbf{0}, \mathbf{b}_0, \bar{\mathbf{b}}_1, \mathbf{b}_a\}$$
$$\mathcal{B}_1 = \text{col}\{\mathbf{M}^{-1}\mathbf{b}_p, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}$$

and the entries in $\mathcal{A}_0$ are readily inferred from the original ODE's.

The challenge is to now eliminate the unknown source strength q_0 using (44) and (45). Differentiating (45) with respect to time, solving for $\mathbf{q}^{(3)}$, while using (41) for $\dot{\mathbf{x}}_a$, and substituting into the derivative of (44), while using (39) for $\dot{\mathbf{x}}_0$, and then solving for q_0 gives

$$q_0(\hat{t}) = \mathbf{Q}_0\mathbf{X}(\hat{t}) + Q_1\dot{V}(\hat{t}) \tag{47}$$

where

$$\mathbf{Q}_0 \triangleq \nu_0^{-1}\text{row}\{\mathbf{d}_0^T\mathbf{M}^{-1}\mathbf{K}, \mathbf{0}, -\mathbf{c}_0^T\mathbf{A}_0, \mathbf{0}, \mathbf{d}_0^T\mathbf{M}^{-1}\mathbf{C}_a\mathbf{A}_a\}$$
$$Q_1 \triangleq \nu_0^{-1}[-\frac{1}{2} - \mathbf{d}_0^T\mathbf{M}^{-1}\mathbf{b}_p]$$
$$\nu_0 \triangleq \mathbf{c}_0^T\mathbf{b}_0 - \mathbf{d}_0^T\mathbf{M}^{-1}\mathbf{C}_a\mathbf{b}_a$$

This can be used to eliminate q_0 in (46):

$$\dot{\widehat{\mathbf{X}}} = \mathcal{A}\widehat{\mathbf{X}} + \bar{\mathcal{B}}_0\dot{\widehat{V}} + \mathcal{B}_1\widehat{V}, \qquad \mathcal{A} \triangleq \mathcal{A}_0 + \mathcal{B}_0\mathbf{Q}_0, \qquad \bar{\mathcal{B}}_0 \triangleq \mathcal{B}_0Q_1 \tag{48}$$

where $\widehat{\mathbf{X}} = \mathbf{X}/\sqrt{ag}$, $\widehat{V} = V/\sqrt{ag}$.

The hydrodynamic pressure acting on the body is $p = -\rho(\partial\Phi/\partial t)$ so that the total upward radiation force is

$$f_R(t) = -2\pi a^2\rho\int_0^1\frac{\partial\Phi(a,\mu)}{\partial t}P_1(\mu)\,d\mu$$
$$= -2\pi a^2\rho\sqrt{g/a}\int_0^1\dot{\Phi}P_1(\mu)\,d\mu$$

Substituting the inverse Fourier transform of (28) while using (35) and (38) gives

$$\underbrace{-(2\pi\rho g a^3/3)^{-1}f_R(\hat{t})}_{}$$
$$= \frac{3}{\sqrt{ag}}\frac{d}{d\hat{t}}\left[z_1(\hat{t}) + \sum_{n=1}^{N_p}\left(-\frac{1}{2n}I_{2n-1,1}\ddot{q}_n + I_{2n,1}q_n(\hat{t})\right)\right]$$
$$-\widehat{f}_R = \frac{3}{\sqrt{ag}}\left[\bar{\mathbf{c}}_1^T\dot{\chi}_1 + \mathbf{e}_2^T\mathbf{q}^{(3)} + \mathbf{e}_3^T\dot{\mathbf{q}}\right]$$

where $\mathbf{e}_2^T = \text{row}\{-I_{2n-1,1}/(2n)\}$, $\mathbf{e}_3^T = \text{row}\{I_{2n,1}\}$. Now substitute $\dot{\chi}_1$ from (38) and $\mathbf{q}^{(3)}$ from the derivative of (45), while using (39) for $\dot{\mathbf{x}}_a$, to get:

$$-\widehat{f}_R = \frac{3}{\sqrt{ag}}[\mathcal{C}_0\mathbf{X} + \mathcal{D}_0q_0 + \mathcal{D}_1\dot{V}]$$
$$= \underbrace{3(\mathcal{C}_0 + \mathcal{D}_0\mathbf{Q}_0)}_{\mathcal{C}}\widehat{\mathbf{X}} + \underbrace{3(\mathcal{D}_0Q_1 + \mathcal{D}_1)}_{\mathcal{M}_\infty}\dot{\widehat{V}} \tag{49}$$

where

$$\mathcal{C}_0 = \text{row}\{\mathbf{e}_3^T - \mathbf{e}_2^T\mathbf{M}^{-1}\mathbf{K}, \mathbf{0}, \mathbf{0}, \bar{\mathbf{c}}_1^T\bar{\mathbf{A}}_1, -\mathbf{e}_2^T\mathbf{M}^{-1}\mathbf{C}_a\mathbf{A}_a\}$$
$$\mathcal{D}_0 = (\bar{\mathbf{c}}_1^T\mathbf{b}_1 - \mathbf{e}_2^T\mathbf{M}^{-1}\mathbf{C}_a\mathbf{b}_a), \mathcal{D}_1 = \mathbf{e}_2^T\mathbf{M}^{-1}\mathbf{b}_p$$

The map from $\widehat{V}$ to $-\widehat{f}_R$ is given by (48) and (49).

The occurrence of $\dot{\widehat{V}}$ in (48) can be eliminated. Taking Laplace transforms gives

$$-\widehat{f}_R(s) = \mathcal{C}(s\mathbf{1} - \mathcal{A})^{-1}(\mathcal{B}_1 + s\bar{\mathcal{B}}_0)\widehat{V}(s) + \mathcal{M}_\infty\dot{\widehat{V}}(s)$$
$$= \underbrace{[\mathcal{C}(s\mathbf{1} - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}_\infty + s\mathcal{M}_\infty]}\widehat{V}(s)$$
$$\triangleq \qquad \widetilde{H}(s)\widehat{V}(s) \tag{50}$$

where $\mathcal{B} \triangleq \mathcal{B}_1 + \mathcal{A}\bar{\mathcal{B}}_0$ and $\mathcal{D}_\infty = \mathcal{C}\bar{\mathcal{B}}_0$. In the time domain this is

$$-\widehat{f}_R(\hat{t}) = \mathcal{C}\widehat{\mathbf{X}}(\hat{t}) + \mathcal{D}_\infty\widehat{V}(\hat{t}) + \mathcal{M}_\infty\dot{\widehat{V}}(\hat{t}) \tag{51}$$
$$\dot{\widehat{\mathbf{X}}} = \mathcal{A}\widehat{\mathbf{X}} + \mathcal{B}\widehat{V} \tag{52}$$

which is the desired realization of the radiation impedance $\widetilde{H}(s)$ orginally defined in (15). It is related to the standard frequency-domain added damping and mass coefficients by

$$\mathcal{D}(\hat{\omega}) = Re\{\widetilde{H}(j\hat{\omega})\}/\hat{\omega},$$
$$\mathcal{M}(\hat{\omega}) = Im\{\widetilde{H}(j\hat{\omega})\}/\hat{\omega} \qquad (\hat{\omega} = \sqrt{Ka}) \tag{53}$$

In this light, $\mathcal{D}_\infty/\hat{\omega}$ and $\mathcal{M}_\infty$ are equivalent to the high frequency behavior of these coefficents; for the hemisphere $\mathcal{M}_\infty = 0.5$, $\mathcal{D}_\infty = 0$ which provides a check on the entire procedure.

The transient motion of the body can then be obtained by combining (51) with the motion equation (34), i.e., in nondimensional form

$$\dot{\widehat{u}} = \widehat{V}, \qquad \dot{\widehat{V}} = -\frac{3}{2}\widehat{u} + \widehat{f}_R \tag{54}$$

where $\widehat{u} = u/a$. By augmenting the state vector $\widehat{\mathbf{X}}$ with $\widehat{V}$ and $\widehat{u}$, one arrives at a first-order system of equations whose

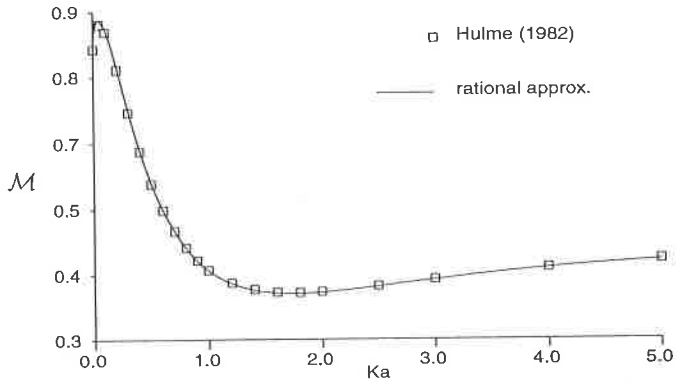

Fig. 8 Added-mass coefficients for a heaving hemisphere

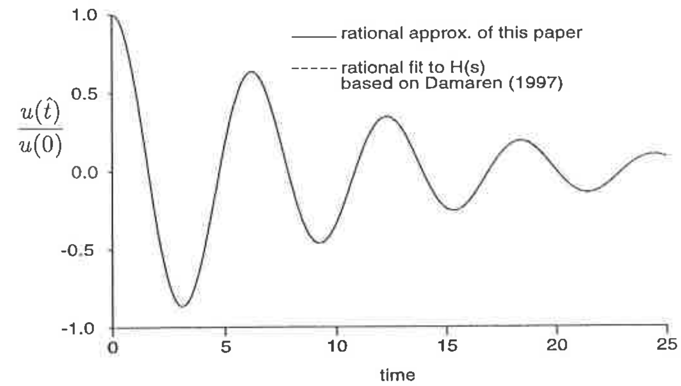

Fig. 10 Free decay of a heaving hemisphere (initial displacement)

solution can be expressed in terms of a matrix exponential given the initial conditions $\widehat{u}(0)$, $\widehat{V}(0)$, and $\widehat{\mathbf{X}}(0) = \mathbf{0}$ for an initially quiescent sea.

Numerical Results

Consider the use of ten wave-free potentials, 48-point Gauss-Legendre quadrature in forming the coefficients $\{H_i\}$ required to generate the convolution operators in equations (38)–(40), and a 12th order approximation to these operators. This yields a state vector $\widehat{\mathbf{X}}$ in (52) of dimension 109 after removing unstable eigenvalues and eigenvectors in $\mathcal{A}$. Like those in the previous section, these correspond to modes with poor controllability and/or observability. The resulting values of M_∞ and $\mathcal{D}_\infty$ were 0.5 (to numerical precision) and 4×10^{-17} respectively. The resulting added mass and damping coefficients obtained from (50) and (53) are given in Figs. 8 and 9. They are compared with Hulme's results which were originally obtained by using (28) on a frequency-by-frequency basis whereas ours are obtained at all frequencies from the time-domain formulation. The agreement is quite satisfactory.

Some comments are in order on the nature of the (necessarily rational) approximation to $\widetilde{H}(s)$ obtained in (50). Since $\widetilde{H}(s)$ is analytic for $Re\{s\} > 0$ and $Re\{\widetilde{H}(j\omega)\} \geq 0$, $\widetilde{H}(s)$ is a positive real (PR) function (Zemanian 1965) which is known to be equivalent to the passivity of an LTI system, i.e., in the present case

$$-\int_0^T f_R(t)\dot{u}(t)\,dt \geq 0, \qquad \forall T \geq 0$$

which states that radiation is a dissipative process. The radiation impedance can be further written as

$$\widetilde{H}(s) = \widetilde{H}_r(s) + s\mathcal{M}_\infty$$

where $\widetilde{H}_r(s)$ continues to enjoy the PR property but it is strictly proper.

Standard properties of rational PR functions can be found in (Tao & Ioannou, 1988). It is interesting to note that all strictly proper rational PR functions have necessarily relative degree one so that rational approximations to the added damping and mass coefficients satisfy $\mathcal{D} \propto \omega^{-3}$ and $[\mathcal{M} - \mathcal{M}_\infty] \propto \omega^{-2}$ as $\omega \to \infty$. This is at odds with known asymptotics for these coefficients but the discrepancy can be pushed to arbitrarily high frequency by increasing the order of the approximation. Although the added mass behavior is correct, Hulme has shown that $\mathcal{D} \propto \omega^{-8}$ for the hemisphere.

The low-frequency asymptotics are similarly mismatched. The approximation satisfies $\mathcal{D} \propto \omega$ and

$$\mathcal{M} - \mathcal{M}_\infty = \lim_{\omega \to 0} Im[\widetilde{H}_r(j\omega)/\omega] + \mathcal{O}(\omega^2)$$

whereas the exact coefficients satisfy

$$\mathcal{M} = \mathcal{M}(0) - \frac{3}{4}(Ka)\ln Ka + \mathcal{O}(Ka)$$
$$\mathcal{D} = \frac{3}{4}\pi Ka + \mathcal{O}(Ka)^2 \qquad (55)$$

Hence, neither the damping nor the added mass is of the correct form.

The prediction for the free heave motions resulting from $\widehat{u}(0) = 1$, $\widehat{V}(0) = 0$ and $\widehat{u}(0) = 0$, $\widehat{V}(0) = 1$ are shown in Figs. 10 and 11, respectively. Good agreement with the theoretical (and experimental) results of Beck & Liapis (1987) is in evidence. As noted by Kotik & Lurye (1968), the asymptotic time dependence is of the form $6/t^4$. Our results necessarily exhibit a decaying exponential envelope but the discrepancy can be pushed to large enough values of time where it is not significant.

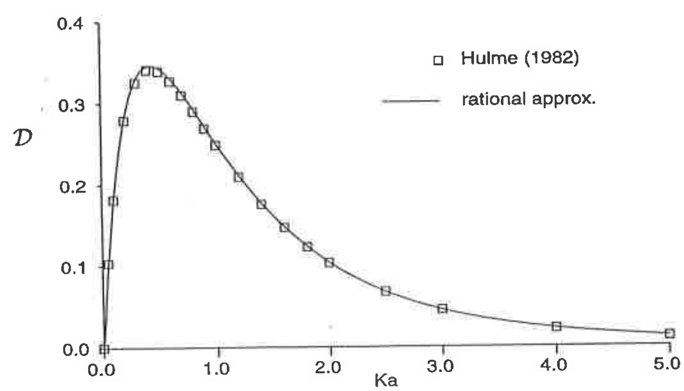

Fig. 9 Added-damping coefficients for a heaving hemisphere

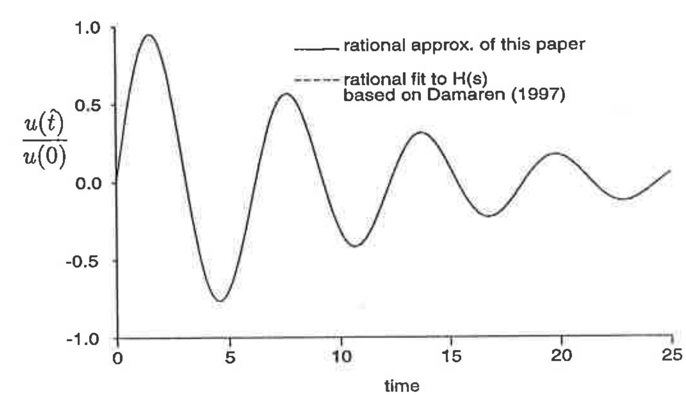

Fig. 11 Free decay of a heaving hemisphere (initial velocity)

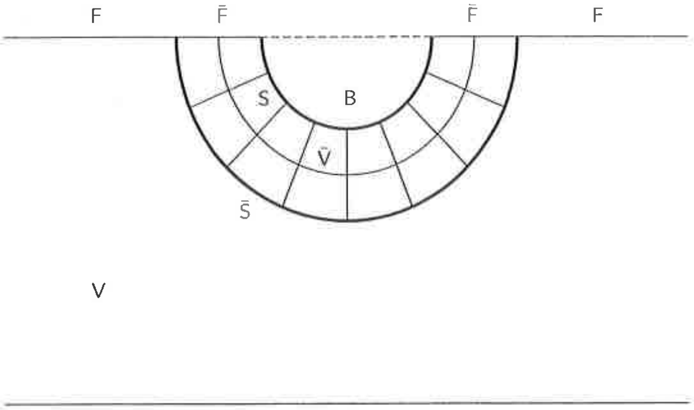

Fig. 12 Geometry for variational problem

Also shown in Figs. 10 and 11 are the solutions of (54) with (50) obtained by fitting a rational function $\widetilde{H}(s)$ with $s = j\omega$ to the accurate added mass and damping coefficients provided by Hulme (Damaren 1997,1999). The agreement between the two transient solutions is quite good. An accurate fit in the range $0 \leq Ka \leq 10$ was obtained using a 10th-order system. This is considerably less than that obtained above (namely, 109) but the fit is capable of supplying only $f_R(t)$ whereas the former can be used to obtain the entire radiation field on the body, $\Phi(\mathbf{r}, t)$, $\mathbf{r} \in S$. It is expected that this will be quite useful in addressing transient diffraction problems using the transient form of the Haskind relations.

General formulation using a variational principle

Guided by the experience of the last section, it is proposed that general radiation problems be handled by formulation in the frequency domain followed by (analytical) inversion to the time domain while capitalizing on rational frequency dependence. This is the motivation for the use of the source plus wave-free potential expansion. The latter exhibit polynomial frequency dependence and the techniques of this paper provide a method for dealing with the source terms.

For ease of exposition, consider the heaving motion of a general axisymmetric floating body (Fig. 12). If the body is enclosed by a hemisphere of radius a centered on the line of symmetry, whose surface is denoted by $\bar{S}$, then the potential function outside $\bar{S}$ (call this region V) can be described by (28). Note that it satisfies the free surface, bottom, and radiation conditions exactly as well as being harmonic in the fluid; furthermore, it is eminently suited to the hemispherical geometry.

In the interior of the hemisphere, $\bar{V}$, we propose to represent $\Phi(\mathbf{r}, t)|_{\bar{V}} = \Psi$ by finite elements or more generally by

$$\widetilde{\Psi}(\mathbf{r}, j\omega) = \sum_{i=1}^{N_1} \psi_i(\mathbf{r}) \widetilde{\eta}_i(j\omega) \tag{56}$$

which is simply $\Psi = \sum_i \psi_i(\mathbf{r})\eta_i(t)$ in the time domain. Notice that the shape functions do not exhibit frequency dependence. What is required is a technique for generating the motion equations for $\widetilde{\Psi}$, hence the $\widetilde{\eta}_i$, which is consistent with the field problem, (7) and (8), and accomplishes matching of $\widetilde{\Psi}$ and its normal derivative with that of the source plus wave-free potential expansion on $\bar{S}$. An elegant solution to this problem has been furnished by Chen & Mei (1974) who showed that stationary values of the functional

$$J(\widetilde{\Psi}, \widetilde{\Phi}) = \frac{1}{2}\int_{\bar{V}} \nabla\widetilde{\Psi} \cdot \nabla\widetilde{\Psi}\, dV - \frac{1}{2}K \int_{\bar{F}} \widetilde{\Psi}^2\, dS - \int_S \widetilde{\Psi}\widetilde{V}_n\, dS + \int_{\bar{S}} \left(\frac{1}{2}\widetilde{\Phi} - \widetilde{\Psi}\right) \frac{\partial \widetilde{\Phi}}{\partial n}\, dS \tag{57}$$

yield functions $\widetilde{\Phi}$ and $\widetilde{\Psi}$ that satisfy the field problem in conjunction with the required matching. Here, $\bar{F}$ is the portion of F inside $\bar{S}$. The advantage of this approach for our problem is the polynomial dependence on K exhibited by the functional in the interior region $\bar{V}$. It also allows more complicated body geometries to be handled via finite element discretization of the interior region while maintaining the simplicity of the exterior solution.

The next step is substitution of the expansions (56) and (28) into (57) followed by minimization with respect to $\widetilde{\eta}_i$, $\widetilde{q}_n$, and $\widetilde{q}_0$ which yields the required frequency domain equations. In particular, the first three terms in (57) become

$$J_1 = \frac{1}{2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_1} \left[\Xi_{ij}^{(0)} - (Ka)\Xi_{ij}^{(2)}\right] \widetilde{\eta}_i \widetilde{\eta}_j - \sum_{i=1}^{N_1} \Upsilon_i \widetilde{\eta}_i \dot{\widetilde{u}}_\alpha \tag{58}$$

where

$$\Xi_{ij}^{(0)} = \int_{\bar{V}} \nabla\psi_i \cdot \nabla\psi_j\, dV$$

$$\Xi_{ij}^{(2)} = a^{-1} \int_{\bar{F}} \psi_i \psi_j\, dS$$

$$\Upsilon_i = \int_S \psi_i n_\alpha\, dS$$

and we have used $\widetilde{V}_n = n_\alpha \dot{\widetilde{u}}_\alpha$ from (4). Minimization of J_1 with respect to $\widetilde{\eta}_i$ would give rise in the time domain to an ODE similar in form to (45) with $\mathbf{q}$ replaced with $\boldsymbol{\eta} = \mathrm{col}\{\eta_i\}$ and $\mathbf{x}_a = \mathbf{0}$.

The last term in (57) yields expressions of the form

$$-\int_{\bar{S}} \psi_i \frac{\partial \widetilde{g}_n}{\partial R}\, dS, \quad \frac{1}{2}\int_{\bar{S}} \widetilde{g}_n \frac{\partial \widetilde{g}_m}{\partial R}\, dS, \quad -a^2 \int_{\bar{S}} \psi_i \frac{\partial \widetilde{G}_0}{\partial R}\, dS \tag{59}$$

$$\frac{1}{2}a^2 \int_{\bar{S}} \widetilde{g}_n \frac{\partial \widetilde{G}_0}{\partial R}\, dS, \quad \frac{1}{2}a^2 \int_{\bar{S}} \widetilde{G}_0 \frac{\partial \widetilde{g}_n}{\partial R}\, dS, \quad \frac{1}{2}a^4 \int_{\bar{S}} \widetilde{G}_0 \frac{\partial \widetilde{G}_0}{\partial R}\, dS \tag{60}$$

which will be multiplied by pairs selected from $\widetilde{\eta}_i$, $\widetilde{q}_n$, or $\widetilde{q}_0$ and summation over the indices n, m, and i has been omitted. Given the axisymmetry, $dS = 2\pi a^2\, d\mu$ with $\mu \in [0, 1]$. It is helpful to rewrite the wave-free potentials in a form that makes explicit the dependence on K:

$$\widetilde{g}_n(\mathbf{r}, j\omega) = g_n^{(0)}(\mathbf{r}) - (Ka) g_n^{(2)}(\mathbf{r}) \tag{61}$$

where the definitions of $g_n^{(0)}$ and $g_n^{(2)}$ are clear from (29).

The contributions to the functional from the first two terms in (59) are

$$J_2 = \sum_{i=1}^{N_1}\sum_{n=1}^{N_p} \left[\Pi_{i,n}^{(0)} - (Ka)\Pi_{i,n}^{(2)}\right] \widetilde{\eta}_i \widetilde{q}_n + \frac{1}{2}\sum_{n=1}^{N_p}\sum_{m=1}^{N_p} \left[\Delta_{nm}^{(0)} - 2(Ka)\Delta_{nm}^{(2)} + (Ka)^2\Delta_{nm}^{(4)}\right] \widetilde{q}_n \widetilde{q}_m \tag{62}$$

where

$$\Pi_{i,n}^{(\alpha)} = -\int_{\bar{S}} \psi_i \frac{\partial g_n^{(\alpha)}}{\partial R}\, dS,$$

$$\Delta_{nm}^{(\alpha+\beta)} = \frac{1}{2}\int_{\bar{S}} \left(g_n^{(\alpha)} \frac{\partial g_m^{(\beta)}}{\partial R} + g_n^{(\beta)} \frac{\partial g_m^{(\alpha)}}{\partial R}\right) dS$$

for $\alpha, \beta = 0, 2$.

In order to treat the source terms in (59) and (60), it is assumed that $\widetilde{G}_0$ and its normal derivative are described on $\bar{S}$ analogous to (32):

$$a\widetilde{G}_0(\mathbf{r}, j\omega) = \sum_j h_{G,j}(\mathbf{r})z^j,$$

$$a^2\frac{\partial \widetilde{G}_0}{\partial R}(\mathbf{r}, j\omega) = \sum_j h_{R,j}(\mathbf{r})z^j, \qquad z = \frac{1-j\omega}{1+j\omega} \tag{63}$$

where we have set $s = j\omega$ in the definition of z to be consistent with the use of the Fourier transform. In practice, the coefficients $h_{G,j}$ and $h_{R,j}$ will be calculated at discrete points $\mathbf{r} \in \bar{S}$. Using $w(\mathbf{r}) = -\psi_i$, the third term in (59) can be treated using the technique of equations (32) and (33) to give

$$J_3 = \sum_{i=1}^{N_1}\left[\sum_j H^{(1)}_{Ri,j}z^j\right]\widetilde{\eta}_i\widetilde{q}_0,$$

$$H^{(1)}_{Ri,j} = -\int_{\bar{S}} \psi_i h_{R,j}\, dS \tag{64}$$

The ensuing series in square brackets, after performing the operations in equations (21)–(25) and setting $s = j\omega$, will be denoted by $\widetilde{F}^{(1)}_{Ri}(j\omega)$, a notational pattern to be used throughout.

The first two terms in (60), upon substitution of (61) and (63), give

$$J_4 = \sum_{n=1}^{N_p}\left[\sum_j H^{(0)}_{Rn,j}z^j - (Ka)H^{(2)}_{Rn,j}z^j + H^{(0)}_{Gn,j}z^j - (Ka)H^{(2)}_{Gn,j}z^j\right]\widetilde{q}_n\widetilde{q}_0 \tag{65}$$

where

$$H^{(\alpha)}_{Rn,j} = \frac{1}{2}\int_{\bar{S}} g^{(\alpha)}_n h_{R,j}\, dS,$$

$$H^{(\alpha)}_{Gn,j} = \frac{1}{2}a\int_{\bar{S}} \frac{\partial g^{(\alpha)}_n}{\partial R} h_{G,j}\, dS, \qquad \alpha = 0, 2$$

Collecting terms in powers of Ka leads naturally to the definitions $H^{(\alpha)}_{RGn,j} = H^{(\alpha)}_{Rn,j} + H^{(\alpha)}_{Gn,j}$ with corresponding series $\widetilde{F}^{(\alpha)}_{RG,n}$. The last term in (60) yields

$$J_5 = \frac{1}{2}a\left[\sum_j H_{G,j}z^j\right]\widetilde{q}_0^2 = \frac{1}{2}a\widetilde{F}_G(j\omega)\widetilde{q}_0^2 \tag{66}$$

where the coefficients in the series are obtained by multiplying the two series in (63) and integrating the effective coefficients over $\bar{S}$.

Forming $J(\widetilde{\eta}, \widetilde{\mathbf{q}}, \widetilde{q}_0) = J_1 + J_2 + J_3 + J_4 + J_5$ and setting $\partial J/\partial[\widetilde{\eta}^T\ \widetilde{\mathbf{q}}^T\ \widetilde{q}_0^T]^T = \mathbf{0}$ leads to the system of equations

$$\begin{bmatrix} \mathbf{\Xi}^{(0)} - (Ka)\mathbf{\Xi}^{(2)} & \mathbf{\Pi}^{(0)} - (Ka)\mathbf{\Pi}^{(2)} & \widetilde{\mathbf{F}}^{(1)}_R \\ \mathbf{\Pi}^{(0)T} - (Ka)\mathbf{\Pi}^{(2)T} & \mathbf{\Delta}^{(0)} - 2(Ka)\mathbf{\Delta}^{(2)} + (Ka)^2\mathbf{\Delta}^{(4)} & \widetilde{\mathbf{F}}^{(0)}_{RG} - (Ka)\widetilde{\mathbf{F}}^{(2)}_{RG} \\ \widetilde{\mathbf{F}}^{(1)T}_R & \widetilde{\mathbf{F}}^{(0)T}_{RG} - (Ka)\widetilde{\mathbf{F}}^{(2)T}_{RG} & a\widetilde{F}_G \end{bmatrix}\begin{bmatrix}\widetilde{\eta} \\ \widetilde{\mathbf{q}} \\ \widetilde{q}_0\end{bmatrix} = [\mathbf{\Upsilon}^T\ \ \mathbf{0}\ \ 0]^T\widetilde{u} \tag{67}$$

An obvious notation has been used where

$$\mathbf{\Xi}^{(0)} = \mathrm{matrix}\{\Xi^{(0)}_{ij}\}$$

$$\widetilde{\mathbf{F}}^{(1)}_R = \mathrm{col}\{\widetilde{F}^{(1)}_{R,i}\},\ \text{etc.}$$

The coefficient matrix exhibits rational dependence on frequency given the polynomial dependence on Ka and the construction procedure used for each entry in the $\widetilde{\mathbf{F}}(j\omega)$ matrices, i.e., each entry can be described as in (25) with $s = j\omega$.

Reusing the notation of the last section, assume that the time domain realization of $\widetilde{\mathbf{F}}^{(1)}_R\widetilde{q}_0$ in the first row of (67) follows the pattern of equations (40) and (41). Then, inverse Fourier transformation of this row gives

$$\mathbf{\Xi}^{(2)}\ddot{\eta}(t) + \mathbf{\Xi}^{(0)}\eta(t) + \mathbf{\Pi}^{(2)}\ddot{\mathbf{q}}(t) + \mathbf{\Pi}^{(0)}\mathbf{q}(t) + \mathbf{C}_a\mathbf{x}_a(t) = \mathbf{\Upsilon}\dot{u}_\alpha(t) \tag{68}$$

where $\mathbf{x}_a(t)$ satisfies (41). Similar operations can be performed on the second and third rows to furnish the remaining ODE's. The resulting differential equations will involve the unknown source strength $q_0(t)$ which must be eliminated. The hydrodynamic pressure follows from substituting $\Phi(\mathbf{r}, t) = \Psi$ from the inverse Fourier transform of (56) into $p = -\rho\partial\Phi/\partial t$. Projection on the spatial description of the normal component of each body motion yields the generalized hydrodynamic forces:

$$f_R(t) = \rho\mathbf{\Upsilon}^T\frac{d}{dt}\eta(t) \tag{69}$$

The resulting radiation impedance will be described by equations identical in form to those in (49) and (50) for appropriate state vector $\mathbf{X}$.

Further challenges are posed by the fact that typically $\mathbf{\Xi}^{(2)}$ above is not invertible (this is evident from its definition after equation (58) where the integral defining its entries is only over the free surface $\bar{F}$). This can be remedied by eliminating some of the η_i from the equations using the constraint formed by (68) within the nullspace of $\mathbf{\Xi}^{(2)}$. The detailed implementation of this is left for a future exposition. We note that more general body geometries and motions can be handled by augmenting the expansion for $\widetilde{\Phi}$ to include additional multipoles at the origin and a more complete set of wave-free potentials. However, the basic features outlined here are left unchanged.

Concluding remarks

A systematic method for obtaining rational approximations to the source function of linear free-surface hydrodynamics has been presented. The approximations are analytical in character, being obtained from the fourth-order ODE satisfied by the Green's function, as opposed to interpolation of numerical evaluations of the source function. By working in the unit disk (z-plane), the approximations took the form of a linear combination of readily determined coefficients from which spatial integrals of the source function could be determined which had the same form.

It has been demonstrated how the rational approximations coupled with polynomial frequency dependence of the wave-free potentials give rise to a system of a constant-coefficient ODE's which describe the radiation impedance. By applying this "multipole" expansion to the problem of the heaving hemisphere, good results were obtained for relevant quantities in the time and frequency domains.

We have also shown how the approach can be extended to more general body geometries using the variational principle of Chen & Mei. This forms the basis for future work along with the application to two-dimensional problems. The technique is especially suitable in this case given the reduced spatial dimensionality and the simple nature of the far-field expansion: one source, one dipole, plus even and odd wave-free potentials. In two dimensions, an ODE analogous to the fourth-order one uncovered by Clément needs to be established. It is expected that diffraction forces arising in the general problem of a floating body in an incident wave field can be described using the transient Haskind relations, given the availability of the transient radiation field.

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