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Engineering Notes

Quasi-Periodic Relative Trajectory Generation for Formation Flying Satellites

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I. Introduction

In recent years, many proposed satellite missions have undergone a paradigm shift away from large, expensive spacecraft toward the use of smaller satellites flying in formation. Although formation flying offers a number of advantages, such as robustness of the mission to failure and potential cost savings, the benefits are partially offset by the increased complexity of the guidance, navigation, and control systems and by ΔV-limited mission durations. The development of advantageous relative dynamics between the constituent satellites could help extend mission durations by minimizing fuel usage. In particular, periodic or quasi-periodic relative orbits would reduce the control effort necessary to maintain the formation and significantly prolong the mission's lifetime. Attempts to identify naturally periodic relative orbits can be found throughout the current literature. Vaddi et al. [1] modified initial conditions from the Hill–Clohessy–Wiltshire (HCW) equations to enforce bounded relative motion in the presence of small eccentricity and second-order differential gravity terms. Kasdin and Kolemen [2] solved the Hamilton–Jacobi equation in terms of epicyclic orbital elements to derive bounded, periodic orbits in the presence of higher-order gravity and certain perturbations. Schaub and Alfriend [3] identified the conditions for J_2-invariant relative orbits by matching the secular drift rates of the mean orbital elements of the chief and deputy satellites. In most of these analytical cases, however, the authors established precise periodic motion using approximated dynamical models. When higher fidelity dynamics and realistic orbital parameters are used, these methods break down and exhibit relative orbit drift or a high sensitivity to initial condition errors.

Several numerical approaches to the problem of relative orbit periodicity have also been attempted. Sabatini et al. [4] and Izzo and Sabatini [5] used a refined genetic algorithm to optimize relative trajectories for maximum periodicity in the presence of J_2 perturbations. Quasi-periodic orbits were obtained for two sets of "magic" inclinations (49.11 and 63.43 deg), at which resonance between the formation's in- and out-of-plane motion results in projected circular orbit (PCO) formations with very small orbital drift. Damaren [6] formulated an iterative shooting approach based on the Newton method to close the relative orbit and achieve "almost" periodic initial conditions. This technique requires a low-level state-feedback control loop to track a trajectory based on the HCW equations, but is robust to initial condition errors and exhibits very low drift characteristics in circular orbits. A similar approach was taken by Becerra et al. [7], who applied a nonlinear Hamiltonian model to the Newton method to obtain quasi-periodic relative initial conditions of the deputy in the presence of the J_2 perturbation. A linear quadratic regulator (LQR) controller was used to track reference trajectories that were developed from sinusoidal functions of time and based on a Keplerian orbit.

As yet, a method of numerically designing reference trajectories that match or closely match the natural perturbed relative motion of the deputy satellite in a generic orbit, and that are robust to initial condition errors, has not been developed. Such trajectories, with a properly designed low-authority controller, could be tracked for minimal ΔV requirements, thus enabling formation flying missions to be significantly extended. The intent of this study is to design such trajectories by continuing the development of the methods used to search for almost-periodic orbits, begun in [6], and applying them to the J_2-invariant trajectories in [3]. The resulting relative orbits will approach true periodicity by capitalizing on advantageous initial conditions, numerically designed reference trajectories, and optimal control strategies.

II. Equations of Motion

The orbital propagation of a spacecraft is typically conducted in the geocentric inertial (GCI) reference frame. For two satellites in close formation, $\mathbf{R}_c$ denotes the position of the chief and $\mathbf{R}_d$ the position of the deputy. The motion of these two satellites will evolve in the GCI frame according to

$$\ddot{\mathbf{R}}_c = -(\mu \mathbf{R}_c / R_c^3) + \mathbf{F}(\mathbf{R}_c)_{\text{pert}} \tag{1}$$

$$\ddot{\mathbf{R}}_d = -(\mu \mathbf{R}_d / R_d^3) + \mathbf{F}(\mathbf{R}_d)_{\text{pert}} + \mathbf{u}_i \tag{2}$$

where $R_d = |\mathbf{R}_d|$, μ is Earth's gravitational constant, $\mathbf{u}_i$ is the control force per unit mass applied to the deputy during formation flying maneuvers, and $\mathbf{F}(\mathbf{R})_{\text{pert}}$ is the perturbation force per unit mass acting upon each satellite. In low Earth orbit, the principal perturbing force is the second zonal harmonic of the Earth's gravitational field, J_2. Higher-order terms, J_3–J_6, play a smaller role, but have been included in this study to increase the fidelity of the simulations. The perturbing accelerations of J_2–J_6 are given in Cartesian coordinates in [8].

Formation flying analyses are frequently concerned with the relative motion of the deputy with respect to the chief. This relative motion can be expressed in the Cartesian Hill frame, a local-vertical–local-horizontal reference frame with its origin centered on the chief, its x axis in the orbit radial direction, its z axis in the orbit normal direction, and its y axis completing the right-handed frame. The state of the deputy expressed in this reference frame is $\mathbf{x}(t) = [(\mathbf{r}_d)^T \quad (\mathbf{v}_d)^T]^T = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z}]^T$. The relative position in the Hill frame can be obtained from the inertial position by using the expression $\mathbf{r}_d = [x \quad y \quad z]^T = \mathbf{C}_{\text{hi}}[\mathbf{R}_d - \mathbf{R}_c]$, where $\mathbf{C}_{\text{hi}}$ is the rotation matrix from the GCI to the Hill frame. The relative velocities in the rotating Hill frame are given by $\mathbf{v}_d = [\dot{x} \quad \dot{y} \quad \dot{z}]^T$, the terms of which are fully described in [9]. For a circular chief orbit, the relative dynamics can be approximated by the nonhomogeneous HCW equations:

$$\dot{\mathbf{x}} = \mathbf{A}_h \mathbf{x} + \mathbf{B}_h \mathbf{u}_h \tag{3}$$

where

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$$\mathbf{A}_h = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and where $n = \sqrt{\mu/a}$, a is the semimajor axis of the orbit, and $\mathbf{u}_h = \mathbf{C}_{hi}\mathbf{u}_i$.

III. Shooting Approach to the Newton Method

A derivation of the Newton method used to enforce single orbit periodicity is presented in [6], but is summarized here for continuity. A set of relative initial conditions are sought such that $\mathbf{x}(T) - \mathbf{x}(0)$. Note that this single-orbit-periodicity (or *quasi-periodicity*) requirement does not necessarily produce genuinely periodic motion because nothing is enforcing $\mathbf{x}(2T) = \mathbf{x}(T)$. The iterative solution presented in [6] is given by

$$\mathbf{x}_{k+1}(0) = \mathbf{x}_k(0) + [\boldsymbol{\phi}_k(T,0) - \mathbf{1}]^{-1}(\mathbf{x}_k(0) - \mathbf{x}_k(T)) \quad \text{for } k = 0, 1, 2, \ldots \tag{4}$$

where $\boldsymbol{\phi}_k(T,0)$ is the state transition matrix corresponding to a linearization of the actual dynamics. To expedite the iterative process and reduce computational effort, it is useful to develop an approximation for $\boldsymbol{\phi}_k(T,0)$ that is independent of k. To this end, we can examine the reduced situation (i.e., a circular, Keplerian orbit), for which the linearized HCW equations govern the relative dynamics of the deputy in the Hill frame. The state transition matrix based on the HCW dynamics is given by $\boldsymbol{\phi}_h(T,0) = \exp(\mathbf{A}_h T)$. The analytical form of $\boldsymbol{\phi}_h(T,0)$ is presented by Melton [10], and all six of its eigenvalues are unity. As a result, the inverse, which is required for Eq. (4), does not exist. To make $\boldsymbol{\phi}_k(T,0) - \mathbf{1}$ invertible, we apply a linear state-feedback control law:

$$\mathbf{u}_h(t) = -\mathbf{K}\tilde{\mathbf{x}}(t), \qquad \tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_{\text{ref}}(t) \tag{5}$$

where $\mathbf{K}$ is the controller gain matrix that tracks a reference state, $\mathbf{x}_{\text{ref}}(t)$. An LQR method was used to solve for $\mathbf{K}$ by minimizing the cost function

$$J = \int_0^\infty (\tilde{\mathbf{x}}^T\mathbf{Q}\tilde{\mathbf{x}} + \mathbf{u}_h^T\mathbf{R}\mathbf{u}_h)\,\mathrm{d}t \tag{6}$$

The LQR controller was designed using $\mathbf{Q} = \text{diag}[n^2, n^2, n^2, 1, 1, 1]$ and $\mathbf{R} = \text{diag}[r/n^2, r/n^2, r/n^2]$, where r can be adjusted to affect fuel usage and tracking accuracy. Incorporating the feedback control law into Eq. (3) yields the error dynamics

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A}_h - \mathbf{B}_h\mathbf{K})\tilde{\mathbf{x}} \tag{7}$$

where it is assumed that $\mathbf{x}_{\text{ref}}(t)$ is an appropriate solution of the HCW equations. With the relative dynamics now defined by Eq. (7), the state transition matrix for Eq. (4) becomes $\boldsymbol{\phi}_k(T,0) = \exp[(\mathbf{A}_h - \mathbf{B}_h\mathbf{K})T]$. Therefore, $\boldsymbol{\phi}_k(T,0) - \mathbf{1}$ in Eq. (4) is now invertible. Although the principal motivation for adding a controller is to permit $\boldsymbol{\phi}_k(T,0) - \mathbf{1}$ to be inverted, it will also allow the deputy to actively track a reference trajectory that, in an open loop, would not yield genuinely periodic motion. Nevertheless, to minimize fuel consumption, it is desirable to use low-authority control (corresponding to a high r value). If we can obtain a quasi-periodic reference trajectory that closely matches the natural perturbed motion of the deputy, a low-authority controller will be sufficient to accurately track this trajectory.

IV. Relative Reference Trajectory Generation

To propagate the relative state from $\mathbf{x}_k(0)$ to $\mathbf{x}_k(T)$ using linear state-feedback control, an appropriate reference trajectory must be selected. For a PCO formation, for example, the well-known solutions to the homogenous HCW equations can be used:

$$\mathbf{r}_{\text{ref}} = (\rho/2)[\sin(nt+\alpha) \quad 2\cos(nt+\alpha) \quad 2\sin(nt+\alpha)]^T \tag{8}$$

where ρ is the separation between the satellites, and α is the initial phase angle of the deputy within the PCO formation. These periodic equations approximate the relative state of a spacecraft in a circular orbit in the absence of orbital perturbations and when ρ is much smaller than the orbital radius. In a real orbital environment, however, these assumptions break down and the actual perturbed motion of the deputy will deviate from this idealized trajectory, resulting in high fuel expenditure as the controller seeks to mitigate the tracking error in Eq. (5). Figure 1 illustrates the rate of convergence of Eq. (4) to a set of quasi-periodic initial conditions while tracking the HCW trajectory with different control authorities. In each case, a baseline chief orbit was used, where altitude $= 700$ km, $i = \Omega = 60$ deg, $e = 0$, $\omega = 0$ deg, and $\rho = 400$ m. A choice of the LQR input cost parameter $r = 10^{-2}$ corresponds to a high-control authority, whereas $r = 10^4$ is very weak control. The iteration in Eq. (4) is considered converged when $|\mathbf{r}_k(T) - \mathbf{r}_k(0)| \leq 10^{-6}$ m.

Despite the rapid convergence of the Newton method, the discrepancy between the actual perturbed motion of the deputy and the HCW reference trajectory results in excessive ΔV and low periodicity *after* the first orbit. However, once a solution has been found through the convergence of Eq. (4), a second, more accurate reference trajectory can be generated by fitting a Fourier series to the actual trajectory of the deputy as it tracks the HCW solutions. The generalized Fourier series evaluated on the interval of [0, T] is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi kt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi kt}{T}\right) \tag{9}$$

where a_0, a_n, and b_n are the Fourier coefficients. A second Newton iteration procedure can be performed with the Fourier reference trajectory replacing the HCW trajectory in the feedback control law of Eq. (5) and with the solution of the previous convergence serving as the initial conditions (ICs). However, in the process of iterating Eq. (4) while tracking a new reference trajectory, the deputy will be forced onto yet another relative trajectory. Once more this new trajectory can be fit with a Fourier series, the solution used as the ICs for the next iteration, and the process repeated. The intent is to find the closest agreement possible between the current real trajectory of the deputy and the Fourier series fit to its real trajectory from the *previous* Newton iteration procedure. An exact match would be indicative of a reference trajectory that precisely matches the natural perturbed motion of the deputy, in which case no control effort would be required. In practice, the process is only semiconvergent: after

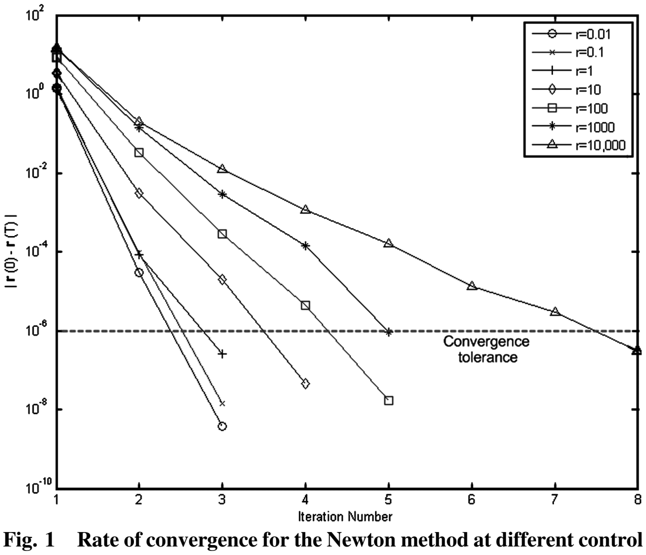

Fig. 1 Rate of convergence for the Newton method at different control authorities.

only two Fourier iterations, the difference between subsequent converged trajectory solutions remains constant.

Once the Newton iteration process has converged to a solution, the resulting relative motion of the deputy often exhibits transient behavior. To avoid fitting the next Fourier series to this initial transient trajectory, the deputy is cycled through five orbits to allow the motion to settle into a steady state, and the Fourier series is fit to the final fifth period. In addition, the periodicity of the solutions to Eq. (4) can be further improved by iterating the initial conditions with the final relative state after five orbits rather than one (i.e., enforce $\mathbf{x}_k(5T) = \mathbf{x}_k(0)$). This avoids the transient behavior of the first orbit and improves the likelihood of finding initial conditions that lead to long-term periodicity.

The selection of the LQR input cost parameter, r, in Eq. (6) strongly impacts both the fuel expenditure and the reference trajectory tracking accuracy. These effects are represented by the metrics ΔV and the relative position periodicity error, E_N, defined as

$$E_N = \sqrt{(x(0) - x(NT))^2 + (y(0) - y(NT))^2 + (z(0) - z(NT))^2} \tag{10}$$

where N is the number of orbits over which the periodicity condition is tested. Because ΔV and E_N are competing metrics, however, it is necessary to optimize the control authority to achieve a balance between fuel use and periodicity. Based on a numerical study, the control authority was set to $r = 8$.

V. Numerical Example: The Quasi-J_2-Invariant Formation

Schaub and Alfriend [3] developed the conditions necessary to achieve periodic relative motion in the presence of J_2 perturbations. The so-called J_2-invariant orbits are formed by using mean orbital element differences and setting the secular drifts of the longitude of the ascending node and the sum of the argument of the perigee and the mean anomaly to zero between neighboring orbits. This results in the two orbits drifting at the same average angular rate and not separating due to the J_2 effect. In practice, the J_2-invariant orbits exhibit some small secular drift because the required osculating–mean–osculating orbital element transformation is only valid to a first-order approximation. Developing J_2-invariant ICs from the baseline chief orbit, we obtain $E_{10} = 0.3951$ m and $E_{50} = 2.875$ m. The primary disadvantage of the J_2-invariant orbits is their sensitivity to initial condition errors. In a real formation flying satellite mission, it is improbable that the *exact* relative starting positions and velocities required for the J_2-invariant orbits can be achieved. Depending on the level of the control authority used to obtain the initial conditions, most deputy satellites will exhibit IC errors on the centimeter or meter level for position and the millimeter or centimeter per second level for velocity. To determine the robustness of the open-loop J_2-invariant trajectories to such errors, position and velocity gradients were determined from the ICs. The initial position was then perturbed 10 cm in the direction of the position gradient and the resulting ICs propagated through 10 orbits. Separately, the initial velocity was perturbed 1 cm/s along the direction of the velocity gradient. The final results for the velocity perturbation were plotted in Fig. 2. The deputy exhibited high secular drift in both cases: the periodicity error for a 0.1 m step along the position gradient is $E_{10} = 37.80$ m; the periodicity error for a 0.01 m/s step along the velocity gradient is $E_{10} = 1274$ m.

The quasi-periodic trajectory design algorithm can be applied to the J_2-invariant orbit baseline. The resulting controlled relative orbits will henceforth be referred to as quasi-J_2-invariant orbits. The same baseline chief orbital elements are used. The initial reference trajectory tracked by the controller during the first Newton iteration sequence is formed by propagating the deputy through a single J_2-invariant orbit with no control and fitting a Fourier series to the actual motion. The control authority was set to $r = 8$, which offers a good balance between E_{10} and ΔV performance. To avoid the small amount of initial transient motion, the Newton method was converged over five orbits. The converged initial conditions are given by

$$\begin{aligned} \mathbf{r}_d(0) &= [1.6893, \quad 398.53, \quad -7.0181]^T \text{ m} \\ \mathbf{v}_d(0) &= [0.21224, \quad 0.00082621, \quad 0.00014407]^T \text{ m/s} \end{aligned} \tag{11}$$

The Fourier reference trajectory fit to the actual motion of the deputy is

$$\mathbf{r}_{\text{ref}}(t) = \begin{bmatrix} 0.13722 + (200.31)\cos((2\pi t/T) - 1.5635) \\ -0.66495 + (400.96)\cos((2\pi t/T) + 0.007302) \\ 0.14596 + (7.1663)\cos((2\pi t/T) - 3.1075) \end{bmatrix} \text{ m} \tag{12}$$

The solution metrics are $\Delta V = 0.007803$ m/s/orbit, $E_{10} = 0.008017$ m, and $E_{50} = 0.08242$ m. Figure 3 illustrates the quasi-J_2-invariant trajectory propagated over 50 orbits. In the close-up view, it is apparent that the actual trajectory of the deputy is distinguishable from the trajectory given in Eq. (12). The + represents the starting point of the deputy, and the O superimposed over it represents the final point.

The robustness test described herein was repeated for the quasi-J_2-invariant orbit. When the position was perturbed 0.1 m along the gradient direction, the metrics were $\Delta V = 0.007910$ m/s/orbit and $E_{10} = 0.008017$ m. When the velocity was perturbed 0.01 m/s along the velocity gradient direction, the metrics were $\Delta V = 0.01032$ m/s/orbit and $E_{10} = 0.008017$ m. The unchanged periodicity error indicates that, although the average fuel requirements increased slightly to compensate for the initial errors, the controller successfully damped out the perturbations by the tenth orbit. Therefore, a weak-authority feedback controller, tracking the quasi-J_2-invariant trajectories and using very little fuel, offers superior robustness to IC errors over the J_2-invariant orbits.

Finally, this method can be compared with the quasi-periodic relative orbits in [5–7]. For long-duration formation keeping, the "quasi-PCO" formations developed at the magic inclinations in [4,5] offer very low average ΔV requirements of approximately 6.33 ×

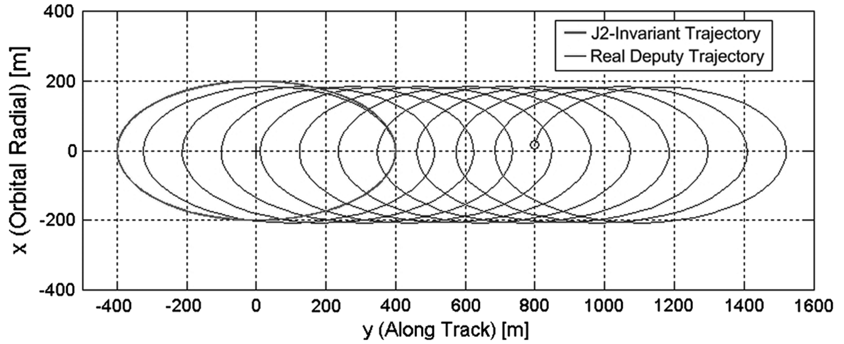

Fig. 2 J_2-invariant orbit with a 0.01 cm/s error on the initial velocity (10 orbits).

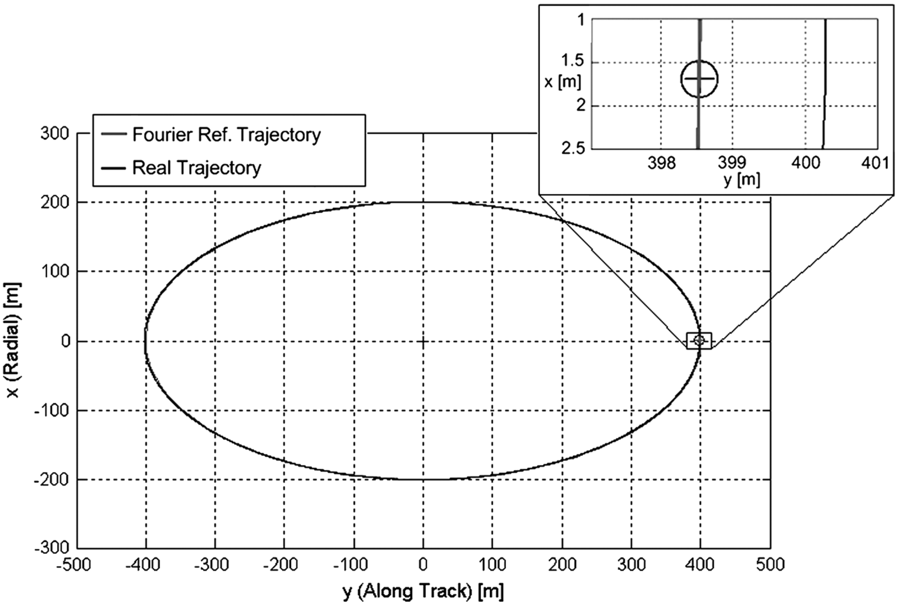

Fig. 3 Relative motion of the quasi-J_2-invariant trajectory (50 orbits).

10^{-4} m/s/orbit for a 1-km-radius PCO. Away from these unique inclinations, however, the fuel requirements rise sharply. The quasi-J_2-invariant method, conversely, results in 0.022 m/s/orbit for a 1 km PCO, but offers consistent performance for a wide range of inclinations. The almost-periodic 400 m PCO in [6] offers a lower average ΔV of 0.00442 m/s/orbit, but is unstable and results in a periodicity error of $E_{10} = 0.626$ m and $E_{50} = 4.98$ m. Finally, the quasi-periodic 10 km PCO of [7] offers a ΔV of 5.99 m/s/orbit, whereas the quasi-J_2-invariant trajectory results in 0.195 m/s/orbit for a similar 10 km relative separation.

VI. Conclusions

A numerical method for generating the initial conditions and reference trajectories for quasi-periodic relative orbits has been presented for formation flying spacecraft under the influence of J_2–J_6 gravitational perturbations. The scheme relies on a shooting approach to the Newton method to find a set of initial conditions that close the relative motion over the number of orbits considered. Reference trajectories are generated iteratively by fitting a Fourier series to the actual perturbed motion of the deputy. The principal advantage of this method is that the deputy tracks a numerically generated reference trajectory that closely approximates its natural perturbed relative motion; consequently, it requires only a low-authority feedback control law, and thus very little fuel, to accurately track this trajectory. This trajectory generation method was successfully applied to the J_2-invariant formation. The resulting quasi-J_2-invariant formations demonstrate a low periodicity error of $E_{50} = 0.08242$ m and $\Delta V = 0.007633$ m/s/orbit. Furthermore, although the classic J_2-invariant orbits are extremely susceptible to initial condition errors, the inclusion of a feedback control law enables a deputy tracking the quasi-J_2-invariant trajectories to rapidly damp out initial perturbations. Because this trajectory generation method is also numerical in nature, it is applicable to simulations with higher-fidelity orbital propagators. The quasi-periodic trajectories found by this method can be viewed as a viable option for long-duration formation flying missions.

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