Pergamon

S0016-0032(96)00008-7

J. Franklin Inst. Vol. 333(B), No. 2, pp. 245–256, 1996

Published by Elsevier Science Ltd
Printed in Great Britain
0016-0032/96 $15.00+0.00

Analysis and Synthesis of Strictly Positive Real Transfer Functions

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(Received 27 December 1994; revised 28 November 1995; accepted 26 December 1995)

ABSTRACT: *The concept of strictly positive real (SPR) transfer functions is examined. It is shown that commonly used frequency domain conditions for SPR do not satisfy some of the most basic elements of the definition and properties of this class of functions. For a given Hurwitz polynomial* a, *a degree* n, *we find the set of all possible polynomials* b *that make the ratio* b/a *SPR, and (i) proper, and (ii) improper. Further, we show that the set of all possible* bs *can be parametrized in terms of, respectively,* n + 1 *and* n + 2 *numbers that satisfy a simple constraint.* Copyright © 1996 Published by Elsevier Science Ltd

I. Introduction

In science and engineering, a theory is usually developed expanding the notion of a few simple concepts and definitions which are relevant to a particular problem. In systems and control, one of the most fundamental concepts encountered over the past 40 years is that of *strictly positive real* (SPR) transfer functions. The concept has its origins in the notion of *positive real* (PR), introduced in circuit theory. Positive realness in turn, was the fundamental tool used in the development of the theory of synthesis of passive networks. See for example (**1, 2**) and the references therein.

In control theory, SPR transfer functions played a fundamental role in the solution of the Lur'e problem (**3**) and more recently in the stability of adaptive systems, and the control of passive plants, such as flexible manipulators and large space structures with colocated sensors and actuators. In the latter case, it is possible to show that *any* (possible nonlinear) passive plant is always stabilized by *any* SPR compensator. Notice the formidable robustness implications of this statement. The result implies that, assuming colocation, an SPR compensator will ensure closed loop stability, no matter how large the uncertainty in the plant parameters.

Even with all of the research conducted during the last 40 years, SPR is possibly the concept that has originated more errors and confusion in the history of circuit and systems theory. Indeed, more often than not, the definition of SPR is incorrectly stated in technical papers. Errors usually arise from the fact that many authors choose to state the definition of SPR using frequency domain conditions. This is a very tempting option. However, care must be exercised in this case since it is very easy to overlook some of the conditions that must be satisfied by an SPR function.

The objective of this paper is 2-fold. In the first place, we state the definition of an SPR transfer function, and clearly state necessary and sufficient conditions for SPR in the frequency domain. Our intention here is to clarify some obscure points in these concepts which seem to have been distorted and erroneously used over the years. Once the definition has been clarified, we study the following problem: given a polynomial a of degree n, and with all of its roots in the left half of the complex plane, find the set of all possible polynomials b that make the rational function b/a SPR. Moreover, we show that if a and b have the same degree, then the set of all possible b that make b/a SPR can be parametrized in terms of $n+1$ real numbers which satisfy a simple constraint. In the case where b has order $n+1$ (i.e. b/a is improper), $n+2$ real numbers are necessary. This is an important problem with applications in several areas, and constitutes an extension of a previous article by the authors (**4**). The same problem was considered there with restriction to the strictly proper case (i.e. b had degree $n-1$). The present extensions are nontrivial, particularly in the improper case.

II. SPR Transfer Functions

In this section we state the definition of SPR and formulate equivalent conditions in the frequency domain. In the sequel, $\mathscr{P}^n$ denotes the set of polynomials of nth degree in the indetermined variable s, with coefficients in the field $\Re$ of real numbers. $\Re^n$ represents the linear space of n-tuples in $\Re$.

Definition 1: A rational function $H(s) = b(s)/a(s)$, where $b(s) \in \mathscr{P}^m$ and $a(s) \in \mathscr{P}^n$ is said to be SPR if for some $\varepsilon > 0$, $Re[H(s-\varepsilon)] \geqslant 0$ for all $Re[s] > 0$. It is straightforward to show that this implies that $|m-n| \leqslant 1$.

Remarks: This definition of SPR was introduced by J. H. Taylor in Ref. (**5**) and shown to be equivalent to the so-called Kalman–Yakubovich Lemma.† The original motivation comes from network theory: an SPR function corresponds to the driving point impedance of a *dissipative* network, i.e. a network composed of resistors, lossy inductors and lossy capacitors.

We now define three different classes of functions, with increasing order of complexity.

Definition 2: Consider a rational function $H(s) = b(s)/a(s)$, where $b(s) \in \mathscr{P}^m$ and $a(s) \in \mathscr{P}^n$. Then, $H(s)$ is said to be in the class $\mathscr{Q}$ if and only if

(i) a is a Hurwitz polynomial (i.e. all of its roots lie in the open left half of the complex plane)
(ii) $Re[H(j\omega)] > 0$, $\forall\omega[0, \infty)$.

† Only strictly proper transfer functions were discussed in Ref. (**5**).

Definition 3: $H(s)$ is said to be *weak* SPR if it is in the class $\mathcal{Q}$ and

(i) the degrees of the numerator and denominator polynomials differ by, at most, one
(ii) if $\partial(b) > \partial(a)$, then

$$\lim_{\omega \to \infty} [H(j\omega)]/j\omega > 0 \tag{1}$$

where $\partial(b)$ denotes the degree of the polynomial b, and similarly for a.

Definition 4: $H(s)$ is said to be SPR if it is weak SPR and in addition, if $\partial(b) \neq \partial(a)$, then one of the following conditions must be satisfied

(i) if $\partial(b) < \partial(a)$, then

$$\lim_{\omega \to \infty} \omega^2 Re[H(j\omega)] > 0 \tag{2}$$

(ii) if $\partial(b) > \partial(a)$, then

$$\lim_{\omega \to \infty} Re[H(j\omega)] > 0. \tag{3}$$

Remarks: The equivalence between Definitions 1 and 4 was established by Iannou and Tao in Ref. **(6)**. Whenever confusion seems likely, SPR transfer functions will be referred to as *strong* SPR. Notice that according to Definitions 3 and 4, if $\partial(a) = \partial(b)$ there is no distinction between weak and strong SPR transfer functions. Moreover, if $\partial(b) = \partial(a)$, then $H(s)$ is SPR if and only if it is in the class $\mathcal{Q}$.

Before we discuss some of the differences among these definitions, we prove Lemma 1 below. This lemma is often loosely quoted as being satisfied by SPR transfer functions, even when SPR functions are defined to be the functions in the class $\mathcal{Q}$. Our proof and examples will clearly show that this is however not the case.

Lemma 1: Consider a rational function $H(s) = b(s)/a(s)$, where $b(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0$, $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$. We have

(i) $H(s)$ is weak SPR if and only if $H^{-1}(s)$ is weak SPR
(ii) $H(s)$ is SPR if and only if $H^{-1}(s)$ is SPR.

Proof: Assume $H(s)$ is weak SPR. It is clear that if the numerator and denominator of $H(s)$ differ by, at most, one, then the same is true of $H^{-1}(s)$. Assume now that $H(j\omega) = A + jB$. Then, $H^{-1}(j\omega) = (A - jB)/(A^2 + B^2)$. Therefore, the real parts of $H(j\omega)$ and $H^{-1}(j\omega)$ have the same sign and then, property (ii) and Definition 2 is satisfied by $H(j\omega)$ if and only if it is satisfied by $H^{-1}(j\omega)$.

We now show that $b(s)$ must be Hurwitz. By the principle of the argument we know that if C_1 is the image under $H(s)$ of the standard D contour, defined as the closed path which is the boundary of the right half of the disk of radius $R \to \infty$ with center zero, then C_1 satisfies the encirclement condition

$$N = Z - P \tag{4}$$

where N is the number of encirclements of C_1 about the origin of the complex plane in the counterclockwise direction when s moves around D in the same direction. Z and $P = 0$ are the number of zeros and poles of $H(s)$ enclosed by D. If $H(s)$ is weak SPR, then $Re[H(j\omega)] > 0$, $\forall\, \omega$, so as s moves along the imaginary axis from $-\infty$ towards ∞ following the D contour, $Re[H(j\omega)]$ remains positive. Let γ denote the semicircle $Re^{j\theta}$, $-\pi/2 \leqslant \theta \leqslant \pi/2$ of the D contour. Taking R sufficiently large, and recalling that the degrees of a and b differ by at most one, $H(s)$ approximates one of the following functions: (i) $H(s) = 0$ (if $\partial(b) < \partial(a)$); $H(s) = b_n/a_n$ (if $\partial(b) = \partial(a)$); and $H(s) = b_m s$ (if $\partial(b) > \partial(a)$).

It is clear that in the first case $Re[H(s)] = 0$, $\forall\, s \in \gamma$. Similarly in case (ii)

$$\lim_{\omega \to \infty} H(j\omega) = b_n/a_n = \lim_{R \to \infty} H(Re^{j\theta}), \quad \theta \in [-\pi/2, \pi/2]$$

which cannot be less than zero because $H(s)$ is weak SPR. Therefore, since $a_n = 1 > 0$ and $b_n \neq 0$ by assumption, we must have $b_n/a_n = b_n > 0$. Finally in case (iii) we have $\lim_{\omega \to \infty}[H(j\omega)]/j\omega = b_m$. Thus $b_m > 0$ by Eq. (1) in Definition 3. It follows that $Re[H(s)] > 0$, $\forall\, s \in \gamma$. It is then clear that in all cases we have $Re[H(s)] \geqslant 0$, $\forall\, s \in \gamma$, and then the number of encirclements of the origin by C_1 is $N = 0$. Also, since $P = 0$, Eq. (4) implies that $Z = 0$ and $b(s)$ is Hurwitz. To complete the proof we must show that when $\partial(a) > \partial(b)$, $H^{-1}(j\omega)$ satisfies Eq. (1). To this end we notice that since $a(s)$ and $b(s)$ are Hurwitz, their coefficients a_i and b_j have all the same sign for all $i = 1, \ldots, n$, $j = 1, \ldots, n-1$. Thus $a_0 > 0$, since $a_n = 1$ by assumption. Also $H(0) = a(0)/b(0) = a_0/b_0 > 0$, since $H(s)$ is weak SPR. Thus, $b_0 > 0$, and consequently $b_j > 0$ for all $j = 1, \ldots, n-1$. It follows that $\lim_{\omega \to \infty} H(j\omega)/j\omega = b_m^{-1} > 0$. This completes the proof of part (a).

(b) We must consider separately three cases, namely: (i) $\partial(b) = \partial(a)$; (ii) $\partial(b) < \partial(a)$; and (iii) $\partial(b) > \partial(a)$. Case (i) is straightforward since in this case $H(s)$ is SPR if and only if it is weak SPR (see the remarks after Definition 5). Consider now case (ii) and suppose without loss of generality that n is even. We have

$$H(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

and

$$\lim_{\omega \to \infty} \omega^2 Re[H(j\omega)] = a_{n-1}b_{n-1} - b_{n-2}. \tag{5}$$

Similarly

$$H^{-1}(s) = \frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_{n-1}s^{n-1} + \cdots + b_0}$$

and

$$\lim_{\omega \to \infty} Re[H^{-1}(j\omega)] = \frac{a_{n-1}b_{n-1} - b_{n-2}}{b_n^2}. \tag{6}$$

Therefore comparing Eqns (5) and (6) we conclude that $H(s)$ satisfies Eq. (2) if and only if $H^{-1}(s)$ satisfies Eq. (3), and the result follows. Case (iii) is entirely similar.

Remarks: Many authors define SPR functions to be what we have called *functions in the class* $\mathcal{Q}$. The problem with this definition is that without the stronger assumptions made in Definition 3, the behaviour of $H(s)$ for values of s in the right half of the complex plane is not captured by $H(j\omega)$. In particular, condition (i) in Definition 3 ensures that $H(s)$ is positive real. Consider the following examples.

Example 1. Let $H(s) = H_1(s)+H_2(s)$. Thus

$$Re[H(j\omega)] = Re[H_1(j\omega)]+Re[H_2(j\omega)].$$

Now assume that $H_1(s) = 1/(s+1)$ and $H_2(s) = s^3$. We have, $H_2(j\omega) = -j\omega^3$, and so $Re[H_2(j\omega)] = 0$. It follows that

$$Re[H(j\omega)] = Re[H_1(j\omega)] > 0.$$

Thus, $H(s) \in \mathcal{Q}$. However, $H(s) = (s^4+s^3+1)/(s+1)$ which is nonminimum phase (i.e. its numerator has roots in the right half of the complex plane), and improper with relative degree 3. It follows that $H(s)$ does not represent the driving point impedance of a dissipative network and it should not be labeled SPR.

Example 1 may seem artificial because the function $H(s)$ is improper with relative degree 3. This was done to show the need for assumption (i) in Definition 3.

Example 2. Let $H(s) = H_1(s)+H_3(s)$, and assume that $H_1(s)$ is as in example 1, and $H_3(s) = -s$. Thus, $H_3(j\omega) = -j\omega$ and again $Re[H_3(j\omega)] = 0$. We have $H(s) \in \mathcal{Q}$, $H(s) = (-s^2-s+1)/(s+1)$ which is also nonminimum phase. However, the relative degree is in this case 1. Even though $H(s) \in \mathcal{Q}$ [or $H(s)$ is SPR, according to some authors], it is clear that no passive network can be constructed with a driving point impedance $H(s)$. Note that condition (ii) in Definition 3 is not satisfied.

It is clear from the previous examples that the inverse of a function in the class $\mathcal{Q}$ is not necessarily in the same class. The necessity of assumptions (i) and (ii) in the frequency domain conditions for SPR was pointed out by Ioannou and Tao in Ref. (**6**). See also (**5**) for some historical remarks regarding this condition and its relation with the Kalman–Yakubovich lemma, and (**7**) for further discussion of the importance of this condition in the stabilization of a class of nonlinear systems containing feedback.

III. Case I: $\partial(b) = \partial(a)$

In this section we solve the following problem: given a Hurwitz polynomial $a \in \mathcal{P}^n$, find the set of all possible polynomials $b \in \mathcal{P}^n$ that make b/a SPR. Moreover, we show that this set can be parametrized using $n+1$ real numbers that satisfy a simple constraint.

We first define the notation used throughout this section. Given a Hurwitz polynomial $a \in \mathcal{P}^n$, we define the following sets

$$\mathcal{B} = \{b \in \mathcal{P}^m : H(s) = b(s)/a(s) \in \mathcal{Q}\}.$$

$$\mathcal{S}_p = \{b \in \mathcal{B} : n = m\}.$$

Given Hurwitz polynomials a, $\mathcal{B}$ is the set of all polynomials $b \in \mathcal{P}^m$, for arbitrary m, that make the rational function b/a belong to the class $\mathcal{Q}$, and $\mathcal{S}_p$ is the subset of $\mathcal{B}$

consisting of all polynomials $b \in \mathscr{B}$ that satisfy $\partial(b) = \partial(a)$. It is then immediate from Definitions 2–4 that $\mathscr{S}_p$ consists of the set of all polynomials $b \in \mathscr{B}$ that make b/a SPR and proper.

Part (i) of the following theorem is taken from Ref. (**4**). We will outline the proof for the sake of completeness, and because it is fundamental in the rest of the paper.

Theorem I

Let $a \in \mathscr{P}^n$ be Hurwitz. We have

(i) $b(s) \in \mathscr{P}^m$ belongs to the set $\mathscr{B}$ if and only if there exist functions $u(s)$, $v(s)$, $r(s)$, and $k(s)$, such that

$$b(s) = a(s)r(s) + [u(s) - v(s)]k(s) \tag{7}$$

where $u(s)$ and $v(s)$ are, respectively, even and odd polynomials which satisfy the Bezout identity

$$a_e(s)u(s) + a_o(s)v(s) = 1. \tag{8}$$

Here a_e and a_o denote the even and odd parts of $a(s)$. The function $r(s)$ is an arbitrary odd polynomial, and $k(s)$ is an even polynomial that satisfies the inequality $k(j\omega) > 0, \forall\, \omega \in \mathfrak{R}$.

(ii) For given $b(s)$, the even polynomial $k(s)$ is uniquely determined.

(iii) For each $b(s)$, the polynomial $r(s)$ is uniquely determined by $k(s)$.

Proof: (i) Partitioning a and b into its even and odd parts we have:

$$H(s) = \frac{b(s)}{a(s)} = \frac{b(s)a(-s)}{a(s)a(-s)} = \frac{[b_e a_e - b_o a_o] + [a_e b_o - b_e a_o]}{a_e^2 - a_o^2}. \tag{9}$$

Here $[b_e a_e - b_o a^o]$ and $[a_e b_o - b_e a_o]$ are the even and odd parts of the numerator of the right hand side of Eq. (9), and since the denominator $[a_e^2 - a_o^2]$ is always real and positive, $H(s) \in \mathscr{Q}$ if and only if

$$b_e(s)a_e(s) - b_o(s)a_o(s) = k(s) \tag{10}$$

for an even polynomials $k(s)$ which satisfies $k(j\omega) > 0, \forall\, \omega \in \mathfrak{R}$.

Given $k(s)$, all possible solutions of Eq. (9) can be obtained by adding the homogeneous solution and a particular solution. It is immediate that $b_{eh} = a_o r$, and $b_{oh} = a_e r$ satisfy the homogeneous equation $b_e(s)a_e(s) - b_o(s)a_o(s) = 0$ for any polynomial $r(s)$. To find a particular solution we use the fact that the even and odd parts of $a(s)$ are coprime [see Lemma 1 in (**4**)], and thus there exist an even function u and an odd function v that satisfy Eq. (8). Multiplying Eq. (8) by $k(s)$ we obtain $a_e(s)u(s)k(s) + a_o(s)v(s)k(s) = k(s)$, which, when compared to Eq. (10), yields

$$b_{ep}(s) = u(s)k(s), \quad b_{op}(s) = -v(s)k(s). \tag{11}$$

It follows that

$$b_e(s) = b_{eh}(s) + b_{ep}(s) = a_o(s)r(s) + u(s)k(s)$$

$$b_o(s) = b_{oh}(s) + b_{op}(s) = a_e(s)r(s) - v(s)k(s)$$

and the result follows by forming $b = b_e - b_o$. See Ref. (**4**) for further details.

(ii) This is immediate from Eq. (10). Suppose that there exists two polynomials $k_1(s)$ and $k_2(s)$ that satisfy this equation. Then, we have $k_1 - k_2 = (b_e a_e - b_o a_o) - b_e a_e - (b_o a_o) = 0$. Thus, $k_1 = k_2$.

(iii) Given $a(s) \neq 0$, suppose that there exist two polynomials $r_1(s)$ and $r_2(s)$ which satisfy Eq. (7) for the same $k(s)$. We have

$$b(s) = a(s)r_1(s) + [u(s) - v(s)]k(s) = a(s)r_2(s) + [u(s) - v(s)]k(s).$$

Thus, $a(s)[r_1(s) - r_2(s)] = 0$, and using the fact that the set of real polynomials is an integral domain (i.e. it has no zero divisors), we have $[r_1 - r_2] = 0$, and so $r_1 = r_2$.

Definition 6: Let $h(s) \in \mathscr{P}^n$ be defined by

$$h(s) = h_n s^n + h_{n-1} s^{n-1} + \cdots + h_o.$$

With this polynomial we will associate the vector of coefficients $\hat{\mathbf{h}} \in \Re^{n+1}$, given by

$$\hat{\boldsymbol{h}} = [h_n \quad h_{n-1} \quad \cdots \quad h_o]^T. \qquad \blacksquare$$

Theorem II

Given a Hurwitz polynomial a, of degree $n \geqslant 2$, the set $\mathscr{S}_p \subset \mathscr{B}$ can be parametrized using $n+1$ real numbers $k_1, k_2, \ldots, k_{n+1}$, which are chosen such that

$$k(s) = \sum_{i=1}^{n+1} (-1)^{n-1} s^{2(i-1)} k_i > 0, \tag{12}$$

$$k(\jmath\omega) > 0, \quad \forall \omega \geqslant 0 \tag{13}$$

but are otherwise arbitrary. Moreover, denoting $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$, and $b(s) = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0$, then any member of $\mathscr{S}_p$ can be determined by

$$\hat{\boldsymbol{b}} = X^{-1}\hat{\boldsymbol{k}} \tag{14}$$

where $\hat{\mathbf{b}}$ and $\hat{\mathbf{k}}$ are the associated vectors of coefficients of the polynomials $b(s)$ and $k(s)$, and the matrix X has the form:

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ -a_{n-2} & a_{n-1} & -1 & 0 & \ldots & 0 \\ -a_{n-4} & a_{n-3} & -a_{n-2} & a_{n-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_0 & -a_1 & a_2 & -a_3 & \ldots & \vdots \\ 0 & 0 & -a_0 & a_1 & \ldots & -a_2 \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{bmatrix}. \tag{15}$$

$\blacksquare$

Proof: Given $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$, any $b \in \mathscr{S}_p$ has the form $b(s) = n_n s^n +$

$b_{n-1}s^{n-1}+\cdots+b_0$. We can separate a and b into their even and odd parts. Substituting for a_e, a_o, b_e and b_o in Eq. (10), and matching powers of s leads to the identity

$$\hat{k} = X\hat{b}$$

where $\hat{\mathbf{b}}$ and $\hat{\mathbf{k}}$ are the associated vectors of coefficients of the polynomials $b(s)$ and $k(s)$, and X is the square matrix defined by Eq. (15). By using similarity transformations, the matrix X can be converted into the Sylvester matrix originated by the polynomials a_e and a_o, (**8**). It then follows from the fact that a_e and a_o are coprime for any Hurwitz polynomial, that X is nonsingular, and so invertible. Thus, it is clear from Eq. (10) that if $k(\cdot)$ satisfies the constraint $k(j\omega) > 0$, $\forall\, \omega \in \Re$, then the polynomial $b(s)$ originated by Eq. (14) belongs to $\mathscr{S}_p$.

To complete the proof we must show that this equation generates the entire set $\mathscr{S}_p$. To see this notice that according to Theorem 1, every polynomials $b \in \mathscr{S}_p \subset \mathscr{B}$ can be written in the form given by Eq. (7) with $u(\cdot)$, $v(\cdot)$, $r(\cdot)$, and $k(\cdot)$ defined as in Theorem 1. Moreover, given $k(\cdot)$, each polynomial $b \in \mathscr{S}_p$ is uniquely determined by $r(s)$. Assume now without loss of generality that n is even. Since $k(s)$ satisfies Eq. (10), we have $\partial(k) = 2n$. Also, $\partial(u) = n-2$, and $\partial(v) = n-1$, as can be easily seen by application of the Euclidean algorithm (**9**). Thus, we have $\partial[(u-v)k] = 3n-1$. It follows from Eq. (7) that $\partial(b) = n$ if and only if $\partial(r) = 2n-1$, and $r(s)$ is chosen so that the coefficients of the first $2n-1$ powers contained in $a(s)r(s)$ are cancelled with the corresponding coefficients of $(u-v)k$. That such a polynomial $r(s)$ exists can be shown using the constructive procedure used in Ref. (**4**), or by noticing that, by part (ii) of theorem 1, any $b(s) \in \mathscr{S}_p$ uniquely determines $k(s)$, given by Eq. (10). Thus, since $\mathscr{S}_p \subset \mathscr{B}$, part (i) of theorem 1 implies the existence of an odd polynomial $r(s)$ that produces the desired $b(s)$.

IV. Case II: $\partial(b) > \partial(a)$

In this section we show that, given a Hurwitz polynomial $a \in \mathscr{P}^n$, the set of all possible $b \in \mathscr{P}^{n+1}$ that make b/a SPR can be parametrized using $n+2$ real numbers with simple constraints. We also show how to separate the set of *weak* and *strong* SPR, More explicitly, we will obtain weak SPR functions as a special case of the set strong SPR, by forcing one of the $n+2$ parameters to be identically zero.

Given a Hurwitz polynomial $a \in \mathscr{P}^n$, we define the following two sets

$$\mathscr{W}\mathscr{S}_I = \{b \in \mathscr{P}^{n+1} \cap \mathscr{B}\colon \lim_{\omega \to \infty} [H(j\omega)]/j\omega = \xi > 0\}$$

$$\mathscr{S}_I = \{b \in \mathscr{W}\mathscr{S}_I\colon \lim_{\omega \to \infty} Re[H(j\omega)] > 0\}.$$

In words: $\mathscr{W}\mathscr{S}_I$ is the set of all polynomials $b \in \mathscr{P}^n$ that satisfy: (i) b/a belongs to the class $\mathscr{Q}$; (ii) $\partial(b) = \partial(a)+1$; and (iii) satisfy the condition $\lim_{\omega\to\infty}[H(j\omega)]/j\omega = \xi > 0$. Equivalently, $\mathscr{W}\mathscr{S}_I$ is the set of all polynomials b that make b/a improper and weak SPR. Similarly, $\mathscr{S}_I$ is the subset of $\mathscr{W}\mathscr{S}_I$ that makes b/a (strong) SPR.

Before we present the main result of this section, we notice that the approach taken in Section III is not feasible here. To see this, notice that if $\partial(b) = n+1$, while $\partial(a) = n$ we have, $a(s) = s^n + a_{n-1}s^{n-1}+\cdots+a_0$, and $b(s) = b_{n+1}s^{n+1}+b_ns^n+\cdots+b_0$. Suppose now that n is even. In this case $\partial(a_e) = n$, $\partial(a_o) = n-1$, $\partial(b_e) = n$, and $\partial(b_o) = n+1$. Thus, from Eq. (10), $\partial(k) = \partial(b_e)+\partial(a_e) = \partial(b_o)+\partial(a_o) = 2n$. Similarly, if n is odd, then $\partial(a_e) = n-1$, $\partial(a_o) = n$, $\partial(b_e) = n+1$, and $\partial(b_o) = n$, and so, once again Eq. (10) shows that $\partial(k) = 2n$. Expanding the product $b_e a_e - b_o a_o = k$ and matching coefficients we obtain:

$$\begin{bmatrix} k_{n+1} \\ k_n \\ \vdots \\ k_1 \end{bmatrix} = Y \begin{bmatrix} b_{n+1} \\ b_n \\ \vdots \\ b_0 \end{bmatrix}, \quad Y = [W \quad X]. \tag{16}$$

Here X is given by Eq. (15) and $W \in \Re^{n+1}$ is the column matrix of the form

$$W = [-a_{n-1} \quad a_{n-3} \quad \cdots \quad a_1 \quad 0 \cdots 0]^T \tag{17}$$

if n is even, and

$$W = [-a_{n-1} \quad a_{n-3} \quad \cdots \quad a_0 \quad 0 \cdots 0]^T \tag{18}$$

if n is odd. Therefore, in this case the matrix Y is rectangular and consequently not invertible. It follows that the b_is are not uniquely determined by the k_is.

The following theorem shows that a parametrization of all improper SPR transfer functions with a common denominator can still be found.

Theorem III

Given a Hurwitz polynomial $a \in \mathscr{P}^n$, $n \geqslant 2$, the set $\mathscr{S}_I$ can be parametrized using $n+2$ real numbers $k_1, k_2, \ldots, k_{n-1}, \phi$ chosen such that

$$k(s) = \sum_{i=1}^{n+1} (-1)^{n-1} s^{2(i-1)} k_i > 0, \tag{19}$$

$$k(j\omega) > 0, \quad \forall\, \omega \geqslant 0 \tag{20}$$

$$\phi > 0 \tag{21}$$

but are otherwise arbitrary. Moreover, denoting $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a^0$, and $b(s) = b_{n+1}s^{n+1} + b_n s^n + \cdots + b_0$, then any member of $\mathscr{S}_I$ can be written as follows

$$\hat{\boldsymbol{b}} = \phi[1 \quad d_n \quad d_{n-1} \quad \ldots \quad d_0]^T \tag{22}$$

where the d_is, $i = 0, \ldots, n$ are the coefficients of the vector associated with the polynomial $d(s) = d_n s^n + d_{n-1}s^{n-1} + \cdots + d_0$, defined by

$$\hat{\boldsymbol{d}} = X^{-1}(\phi^{-1}\hat{\boldsymbol{k}} - W). \tag{23}$$

Proof: Given $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$, if $b \in \mathscr{S}_I$ then we can write

$$H(s) = \frac{b_{n+1}s^{n+1} + b_n s^n + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

where $b_{n+1} = \lim_{\omega\to\infty}[H(j\omega)]/j\omega > 0$ by (i) in Definition 4. Thus, we can write

$$H(s) = b_{n+1}\frac{s^{n+1}+d_n s^n + d_{n-1}s^{n-1}+\cdots+d_0}{s^n + a_{n-1}s^{n-1}+\cdots+a_0}. \tag{24}$$

$$= b_{n+1}\tilde{H}(s) \tag{25}$$

It is straightforward to show that $H(s)$ is SPR if and only if $\tilde{H}(s)$ is SPR. We now define $\phi = b_{n+1}$, an arbitrary positive real number, and proceed as in Theorem 2 with $k(s) = \phi[\tilde{h}_e(s)a_e(s) - \tilde{h}_o(s)a_o(s)]$, where $\tilde{h}_e$ and $\tilde{h}_o$ are the even and odd part of the numerator polynomial of $\tilde{H}$. Expanding the product and matching coefficients we obtain

$$\hat{\boldsymbol{k}} = \phi(X\hat{\boldsymbol{d}} + W)$$

Thus, $\hat{\mathbf{d}} = X^{-1}(\phi^{-1}\hat{\mathbf{k}} - W)$ and then $\hat{\mathbf{b}}$ is given by Eq. (22). The result follows by the same arguments of Theorem 2.

Remarks: The preceeding theorem is not thorough enough to distinguish between SPR and the less stringent notion of weak SPR. This distinction is of great theoretical importance. Notice that an SPR function (and not a weak SPR one) represents the driving point impedance of a *realizable* passive network, i.e. one formed by combining resistances, lossy inductors, and lossy capacitors. Our next result will clearly separate these two cases.

Theorem IV

Given a Hurwitz polynomial a of degree $n \geqslant 2$, the polynomial $b \in \mathscr{P}^{n+1}$ belongs to the set $\mathscr{S}_1 \subset \mathscr{W}\mathscr{S}_1$ if and only if $k_{n+1} \neq 0$ in the parametrization of Theorem 3.

Proof: We have

$$Re[\tilde{H}(j\omega)] = \frac{\tilde{h}_e(j\omega)a_e(j\omega) - \tilde{h}_o(j\omega)a_o(j\omega)}{a_e^2(j\omega) - a_o^2(j\omega)}$$

and since $\partial(k(s)) = \partial([a_e^2(s) - a_o^2(s)]) = 2n$, we have

$$\lim_{\omega\to\infty} Re[H(j\omega)] = \phi \lim_{\omega\to\infty} Re[\tilde{H}(j\omega)] = k_{n+1}$$

and the result follows.

V. Examples

5.1 *Example 1*

Consider the polynomial $a = s^2 + a_1 s + a_0$. We are interested in the set of all polynomials $b(s)$ of degree $\partial(b) = 2$ that make the rational function b/a SPR. Define now $k(s) = k_3 s^4 - k_2 s^2 + k_1$. By the Sturm's Theorem (**8**) $k(s)$ satisfies the positivity condition $k(j\omega) > 0$, $\forall\, \omega \geqslant 0$ if and only if $k_3 > 0$, $k_1 > 0$, and $k_2 > -2\sqrt{k_1 k_3}$. We have

$$\begin{bmatrix} k_3 \\ k_2 \\ k_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_0 & a_1 & -1 \\ 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = X\hat{\boldsymbol{b}}$$

therefore

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = X^{-1}\hat{k} = (a_1 a_0)^{-1} \begin{bmatrix} a_1 a_0 & 0 & 0 \\ a_0^2 & a_0 & 1 \\ 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} k_3 \\ k_2 \\ k_1 \end{bmatrix}$$

and

$$H(s) = \frac{k_3 s^2 + (a_1 a_0)^{-1}(a_0^2 k_3 + a_0 k_2 + k_1)s + k_1/a_0}{s^2 + a_1 s + a_0}.$$

5.2 *Example 2*

Given the same polynomial $a(s)$ used in Example 1, we look for the set of all polynomials $b(s)$ of degree $\partial(b) = 3$ that make b/a SPR. In this case, Eq. (23) implies that

$$\hat{d} = X^{-1}(\phi^{-1}\hat{k} - W)$$

$$\Rightarrow \begin{bmatrix} d_2 \\ d_1 \\ d_0 \end{bmatrix} = \frac{1}{\phi a_1 a_0} \begin{bmatrix} a_1 a_0 & 0 & 0 \\ a_0^2 & a_0 & 1 \\ 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} k_3 + \phi a_1 \\ k_2 \\ k_1 \end{bmatrix}.$$

Thus,

$$H(s) = \frac{\phi s^3 + (k_3 + \phi a_1)s^2 + (a_1 a_0)^{-1}[a_0^2(k_3 + \phi a_1) + a_0 k_2 + k_1]s + k_1/a_0}{s^2 + a_1 s + a_0}.$$

Where we have used Eq. (22) to obtain $\hat{b}$. This expression contains all SPR functions with denominator $a(s)$. The set of weak SPR functions with the same denominator is obtained by letting $k_3 = 0$.

VI. Conclusions

A detailed study of the properties of SPR transfer functions was carried out. Several inconsistencies in the use of popular frequency domain conditions for SPR were pointed out, and supported by simple examples. A parametrization of all possible polynomials b that make the ratio b/a weak or strong SPR for a given Hurwitz polynomial a was obtained. Moreover, the important distinction between weak and strong SPR was clarified and given a simple solution, for the first time, for improper rational functions.

References

(1) E. A. Guillemin, "Synthesis of Passive Networks", Wiley, NY, 1957.
(2) M. E. Van Valkenburg, "Introduction to Modern Network Synthesis", Wiley, NY, 1960.
(3) K. Narendra and J. H. Taylor, "Frequency Domain Criteria for Absolute Stability", Academic Press, NY, 1973.
(4) H. J. Marquez and C. J. Damaren, "On the design of strictly positive real transfer functions", *IEEE Trans. Circ. Syst. Part I: Fundament. Theory Applic.* Vol. 42, No. 4, pp. 214–218, 1995.

(5) J. H. Taylor, "Strictly positive-real functions and the Lefshetz–Kalman–Yakubovich (LKY) Lemma", *IEEE Trans. Circ. Syst.*, pp. 310–311, 1974.
(6) P. Ioannou and G. Tao, "Frequency domain conditions for strictly positive real functions", *IEEE Trans. Automat. Contr.* Vol. AC-32, pp. 53–54, 1987.
(7) H. J. Marquez and C. J. Damaren, "Comments on strictly positive real transfer functions revisited", *IEEE Trans. Automat. Contr.* Vol. AC-40, No. 3, pp. 478–479, 1995.
(8) E. I. Jury, "Inners and Stability of Dynamic Systems", Wiley, NY, 1974.
(9) J. Gilbert and L. Gilbert, "*Elements of Modern Algebra*", PWS-KENT, 1992.