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Optimal strictly positive real controllers using direct optimization

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Abstract

In this paper, we consider the $\mathscr{H}_2$-optimal control problem subject to the constraint that the resulting controller be strictly positive real. A direct numerical optimization approach is adopted in conjunction with a controller parametrization that is linear in the unknown parameters. The SPR constraint is easily expressed at each frequency in the form of a linear inequality. The method is applied to a numerical example from the literature and good results are achieved. In particular, the proposed method is particularly adept at determining low order controllers.

1. Introduction

Many physical systems enjoy the input–output property known as passivity which is a mathematical statement that such systems do not produce energy. A stronger property than passivity is strict passivity which characterizes systems that dissipate energy. The passivity theorem is a general result which states that the negative feedback interconnection of a passive system and a strictly passive system is always input–output stable [1]. This provides a strong basis for robust control since the passivity property of a system to be controlled can often be guaranteed in the face of parameter uncertainty. An important class of passive systems are flexible mechanical systems with collocated force (torque) actuators and rate (angular rate) sensors [2].

Linear time-invariant passive systems are characterized by positive real transfer functions. In the case of mechanical systems with collocation this property is independent

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of the natural frequencies, damping ratios, and the number of modelled modes. An important class of systems are characterized by strictly positive real (SPR) transfer functions [3]. It is known that an SPR system always stabilizes a passive system. Hence robust stability is automatically achieved by limiting controller design to the SPR case.

Given this important stability result, several authors have looked at systematic ways of designing SPR transfer functions. In [2], the Kalman–Yakubovich lemma in conjunction with free weighing matrices was used to design an SPR controller. It was not an observer-based compensator. McLaren and Slater [4] modified its structure so that it did have this property. Lozano-Leal and Joshi [5] examined LQG weight selection so that the solution of the resulting optimal control problem possessed the SPR property. Their work was extended in [6] so that the weight selection also guaranteed that an $\mathscr{H}_\infty$-norm bound was also achieved. None of these approaches determined optimal controllers subject to the SPR constraint for general classes of problems (i.e., problems for which the weight selection was not a priori constrained).

The determination of the best SPR controller given an $\mathscr{H}_2$-norm performance index was treated in [7,8]. Both papers used linear matrix inequalities to formulate the SPR constraint and forced the controller to have the same order as the plant. The controller was also forced to have the structure of an observer-based compensator and the observer gain was chosen to be the same as the $\mathscr{H}_2$-optimal (LQG) controller. The solutions presented in [8] showed improvements over those in [7] for a common numerical example (a simply supported beam) which was introduced in [6]. It should be noted that it is an open question as to whether the optimal SPR controller for the $\mathscr{H}_2$ problem has an observer-based structure. Furthermore, its order may not necessarily be the same as that of the plant. All of the methods noted above assume the SPR compensator to have the same order as the plant.

An alternative approach was taken in [9] where it was suggested that a (stable) optimal controller be approximated by the closest SPR transfer function which possessed the same poles. Closeness was measured using the $\mathscr{H}_2$-norm and an argument was made that if the controllers were close so were the corresponding closed-loop systems.

In this paper, we take a numerical optimization approach to finding optimal SPR controllers. The controller is parametrized using a Ritz-type expansion which is linear in the unknown parameters. This may be interpreted as a polynomial in the z-plane with analyticity in the unit disk after a suitable mapping from the s-plane to the z-plane. A major advantage of this approach is the linear nature of the inequality constraints produced by enforcing the SPR constraint at each frequency. The cost function to be minimized is taken to be the closed-loop $\mathscr{H}_2$-norm of a suitable transfer function but more general problems are easily treated using the proposed method. The nonlinear optimization problem with linear inequality constraints is tackled using the well known code NPSOL [10], which was developed at Stanford University.

2. Controller design problem

Following [7] and [8], we consider the generalized plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2 u, \tag{1}$$

$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}u, \tag{2}$$

$$y = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}, \tag{3}$$

where $\mathbf{x}(t) \in \Re^n$ is the state vector, $u(t) \in \Re$ is the control input, $y(t) \in \Re$ is the measurement output, $\mathbf{w} \in \Re^p$ is the disturbance input, and $\mathbf{z} \in \Re^m$ is the regulated output. It is assumed that:

(1) $(\mathbf{A}, \mathbf{B}_1)$ is controllable and $(\mathbf{C}_1, \mathbf{A})$ is observable;
(2) $(\mathbf{A}, \mathbf{B}_2)$ is controllable and $(\mathbf{C}_2, \mathbf{A})$ is observable;
(3) $\mathbf{D}_{12}^{\mathrm{T}}\mathbf{C}_1 = \mathbf{0}$ and $\mathbf{D}_{12}^{\mathrm{T}}\mathbf{D}_{12} > 0$;
(4) $\mathbf{D}_{21}\mathbf{B}_1^{\mathrm{T}} = \mathbf{0}$ and $\mathbf{D}_{21}\mathbf{D}_{21}^{\mathrm{T}} > 0$.

The controller to be determined is given by $C(s) = \mathbf{C}_c(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{B}_c$ where $u(s) = -C(s)y(s)$. Hence, the closed-loop transfer matrix is $\mathbf{T}_{zw}(s) = \mathbf{C}_z(s\mathbf{1} - \mathbf{A}_z)^{-1}\mathbf{B}_z$, where

$$\mathbf{A}_z = \begin{bmatrix} \mathbf{A} & -\mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix}, \quad \mathbf{B}_z = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_c\mathbf{D}_{12} \end{bmatrix}, \quad \mathbf{C}_z = [\mathbf{C}_1 \quad -\mathbf{D}_{12}\mathbf{C}_c]. \tag{4}$$

To fix ideas, the quantity to be minimized is the $\mathscr{H}_2$-norm of $\mathbf{T}_{zw}(s)$ which can be calculated using

$$\|\mathbf{T}_{zw}\|_2^2 = \operatorname{tr} \mathbf{B}_z^{\mathrm{T}}\mathbf{P}\mathbf{B}_z, \quad \mathbf{P}\mathbf{A}_z + \mathbf{A}_z^{\mathrm{T}}\mathbf{P} = -\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z. \tag{5}$$

In the absence of any constraints on $C(s)$, the minimizing controller is the well known linear quadratic gaussian (LQG) solution.

We are interested in strictly proper controllers which are also SPR. The strictly proper constraint ensures that the norm in Eq. (5) is finite. Strictly proper SPR transfer functions $C(s)$ are characterized by the following properties [11]:

(i) $C(s)$ is real for real s and $C(s)$ is analytic for $Re\{s\} \geqslant 0$;
(ii) $Re\{C(\mathrm{j}\omega)\} > 0, \ -\infty < \omega < \infty$;
(iii) $\lim_{\omega\to\infty} \omega^2 Re\{C(\mathrm{j}\omega)\} > 0$.

Transfer functions which satisfy (i) and (ii) but not necessarily (iii) are said to weak SPR. It has been noted [12] that a weak SPR system stabilizes a positive real one and more recently it has been proven that a weak SPR system stabilizes a general, possibly nonlinear, passive system [13]. For these reasons, we will confine our search to weak SPR systems.

The problem that we wish to solve is find the weak SPR controller which minimizes $\|\mathbf{T}_{zw}\|_2$ amongst this class. To date, an exact solution to this problem has not been found. However, in [5] it was noted that the optimal solution is the LQG solution if the parameters $\mathbf{D}_{21}$, $\mathbf{D}_{12}$, $\mathbf{C}_1$, and $\mathbf{B}_1$ are suitably chosen. More general solutions were obtained in [7,8] by assuming an observer-based compensator with the same order as the plant and the same observer gain $\mathbf{B}_c$ as the corresponding LQG solution. The solution presented in [8] outperformed that in [7] for the numerical example introduced in [6].

3. Numerical optimization problem

Since SPR controllers that render the closed-loop H_2 norm finite are strictly proper stable transfer functions, we look for an appropriate parametrization. This should be such that the strictly positive real constraint is easily applied. Let $\mathscr{D}$ denote the open unit disk in the complex plane. The space of bounded analytic functions in $\mathscr{D}$ is denoted by $\mathscr{H}_\infty(\mathscr{D})$. It is well known [14] that such functions can be represented by the expansion

$$g(z) = \sum_{n=1}^{\infty} h_n z^{n-1}. \tag{6}$$

Functions $g(z)$ that are analytic in $\mathscr{D}$ and $\sup_{r<1} \int_0^{2\pi} |g(re^{j\theta})|^2 \, d\theta < \infty$ comprise the Hardy space $\mathscr{H}_2(\mathscr{D})$. They can be represented by Eq. (6) subject to the constraint $\sum_{n=1}^{\infty} h_n^2 < \infty$.

The mapping $z = (s-1)/(s+1)$ maps the open right-half plane onto the open unit disk. It also maps the imaginary axis onto the unit circle; the point $s = j\omega$ is mapped onto $e^{j\theta}$ with

$$\omega = a \frac{\sin\theta}{1 - \cos\theta}. \tag{7}$$

It is also known that $g[(s-1)/(s+1)] \in \mathscr{H}_\infty$ when $g(z) \in \mathscr{H}_\infty(\mathscr{D})$ and this generates all of $\mathscr{H}_\infty$. Also if $g(z) \in \mathscr{H}_2(\mathscr{D})$ then $g[(s-1)/(s+1)]/(s+1) \in \mathscr{H}_2$ and this can be used to generate all of $\mathscr{H}_2$. Since, we seek to parametrize $\mathscr{H}_2$ rational functions, the following expansion is proposed:

$$C(s) = \sum_{n=1}^{N} h_n \frac{(s-1)^{n-1}}{(s+1)^n}. \tag{8}$$

We note that parametrizations such as this have been adopted for numerical controller optimization in [15].

Note that $C(s)$ satisfies property (i) for SPR functions by construction. Property (ii) can be expressed in terms of θ:

$$\begin{aligned} Re\{C(j\omega)\} &= Re\{g(e^{j\theta})/(j\omega+1)\} \\ &= \sum_{n=1}^{N} h_n[\omega\cos(n-1)\theta + \sin(n-1)\theta]/(1+\omega^2) > 0, \end{aligned} \tag{9}$$

where ω is given in Eq. (7). We propose to enforce this constraint at M equally spaced values from $\theta = 0$ to $\theta = \pi$, i.e., at $\theta_k = k\pi/M$, $k = 1, \ldots, M$. Defining $\mathbf{h} = [h_1 \ \cdots \ h_n]^{\mathrm{T}}$, each constraint in Eq. (9) is of the form

$$\mathbf{a}_k^{\mathrm{T}}\mathbf{h} > 0, \quad \mathbf{a}_k = \mathrm{col}_n\{[\omega_k \cos(n-1)\theta_k + \sin(n-1)\theta_k]/(1+\omega_k^2)\}, \quad k = 1, \ldots, M. \tag{10}$$

Given $\mathbf{h}$, the controller parameters $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c)$ are readily formed from Eq. (8) so that the quantity to be minimized is

$$J(\mathbf{h}) = \|\mathbf{T}_{zw}\|_2, \tag{11}$$

where $\|\mathbf{T}_{zw}\|_2$ is available from Eqs. (4) and (5). We propose to use a numerical optimization approach to minimize $J(\mathbf{h})$ subject to the linear constraints in Eq. (10). The optimization program NPSOL will be employed which is based on the sequential quadratic programming algorithm.

It is also known that the expansion

$$C(s) = \sum_{n=1}^{N} h_n \frac{(s-a)^{n-1}}{(s+a)^n} \tag{12}$$

with $a>0$ is dense in $\mathscr{H}_2$ as $N \to \infty$. In our numerical examples, we use $a = 1$ but the formulation is easily modified to use different values of a.

4. Numerical examples

We adopt the numerical example originally proposed in [6] and also used in [7,8]. It consists of an Euler–Bernoulli beam of length $\ell = \pi$ that is simply supported at each end. The bending deflection of the beam at position p and time t is denoted by $d(p,t)$ which can be expressed using the modal expansion

$$d(p,t) = \sum_{\alpha=1}^{\infty} \sin(\alpha p) q_\alpha(t). \tag{13}$$

Assuming there is a point force actuator applying force $u(t)$ at $p = p_a = 0.55\ell$, the modal equations are given by

$$\ddot{q}_\alpha + 2\zeta\omega_\alpha \dot{q}_\alpha + \omega_\alpha^2 q_\alpha = \sin(\alpha p_a) u(t), \quad \alpha = 1, 2, 3, \ldots,$$

where $\omega_\alpha = \alpha^2$ and the damping ratio is given by $\zeta = 0.01$. It is assumed that there is a velocity sensor collocated with the force actuator which measures $\dot{d}(p_a, t)$. If only five modes are retained in the expansion, then the state vector can be taken as $\mathbf{x} = [q_1 \ \dot{q}_1 \ \ \cdots \ \ q_5 \ \dot{q}_5]^{\mathrm{T}}$. The matrices describing the plant model are then given by

$$\mathbf{A} = \mathrm{diag}\left\{ \begin{bmatrix} 0 & 1 \\ -\omega_\alpha^2 & -2\zeta_\alpha\omega_\alpha \end{bmatrix} \right\}, \quad \alpha = 1, \ldots, 5,$$

$$\mathbf{C}_2 = \mathbf{B}_2^{\mathrm{T}} = \mathrm{row}\{[0 \ \ b_\alpha]\}, \quad \alpha = 1, \ldots, 5, \quad b_\alpha = \sin(\alpha p_a),$$

Table 1
Optimal cost vs. approximation order

N	Example 1	Example 2
	$\Vert\mathbf{T}_{zw}\Vert_2$	$\Vert\mathbf{T}_{zw}\Vert_2$
1	2.2205	1.3870
2	2.0248	1.2433
3	2.0194	1.2427
4	2.0012	1.2397
5	1.9973	1.2390
6	1.9921	1.2390
7	1.9920	1.2388
8	1.9920	1.2388
9	1.9918	1.2388
10	1.9917	1.2388
11	1.9917	1.2388
12	1.9916	1.2387
LQG	1.9843	1.2376
$C(s) = 0$	4.1711	4.000
Ref. [7]	2.2162	1.3856
Ref. [8]	1.9844	N/A

$$\mathbf{B}_1 = [\mathbf{B}_2 \ \ \mathbf{0}], \quad \mathbf{D}_{21} = [0 \ \ 1.9].$$

We shall present two examples. In Example 1, the regulated output is defined in terms of the velocity at $p_e = 0.7\ell$:

$$\mathbf{C}_1 = \begin{bmatrix} 0 & c_1 & \cdots & 0 & c_5 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad c_\alpha = \sin(\alpha p_e),$$

$$\mathbf{D}_{12} = [0 \ \ 1.9]^{\mathrm{T}}.$$

In Example 2, the regulated output is defined in terms of the position at $p_e = 0.7\ell$:

$$\mathbf{C}_2 = \begin{bmatrix} c_1 & 0 & \cdots & c_5 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_{12} = [0 \ \ 0.5]^{\mathrm{T}}.$$

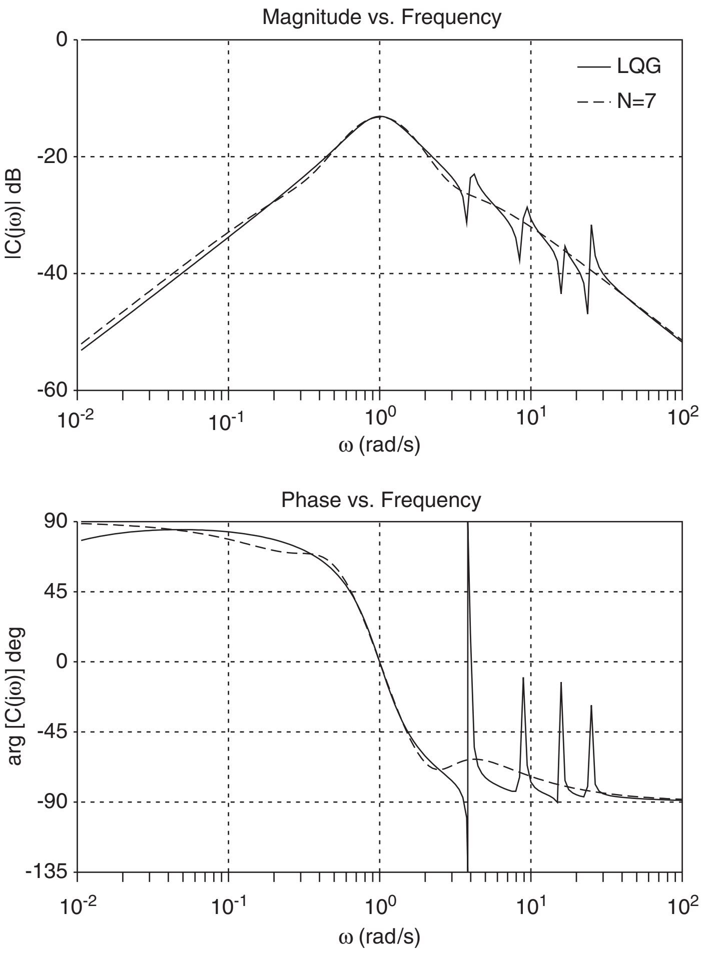

Fig. 1. Example 1, Bode plots.

For each example, we solve a series of problems with progressively increasing order of the approximation. The optimal solution for order N is used as the initial guess for optimizing the problem of order $N+1$. For N = 1, we take the initial guess to be $h_1 = 1$ and for $N > 1$, we take $h_N = 0$. The SPR constraint is applied at $M = 200$ values on the unit circle.

The values of the optimal cost achieved as function of approximation order are given in Table 1 for both examples. The LQG cost is also given as are the values achieved by the methods presented in [7,8]. Our solutions show monotonically decreasing cost as a function of the approximation order which is to be expected. The converged cost is very close to the lower bound established by the LQG case. It is also apparent that relatively low order controllers, say $N = 5$, come close to achieving this lower bound. Indeed, one of the main advantages of the proposed method is in determining low order controllers and establishing the sensitivity of the cost with respect to order. Our optimal solutions outperform the results of [7] for $N > 1$. For Example 1, the result of [8] is better than our converged result and there is no result available for comparison for Example 2.

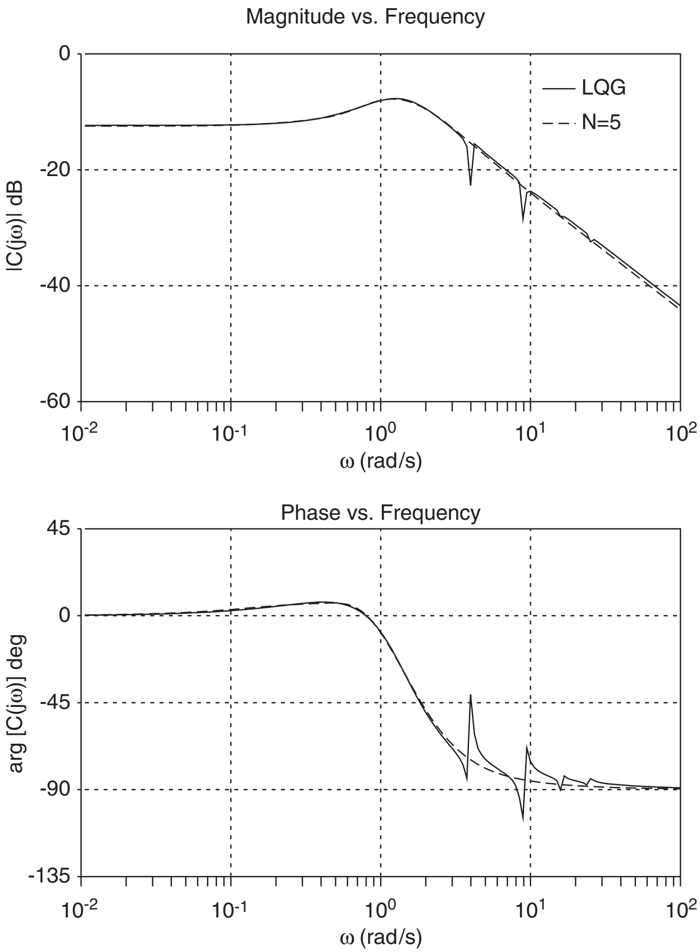

Fig. 2. Example 2, Bode plots.

Bode plots of the optimal controller are shown in Fig. 1 for Example 1 with $N = 7$ and in Fig. 2 for Example 2 with $N = 5$. Satisfaction of the SPR constraint is clearly exhibited in both cases. Also shown is the LQG case which is not SPR in both examples. Our optimal solutions appear to yield SPR approximations to the LQG controller in both cases.

5. Concluding remarks

A method of determining optimal strictly positive real controllers using direct numerical optimization has been presented. A controller parametrization was adopted that was linear in the unknown parameters. This enabled the SPR constraint at each frequency to be expressed as a linear inequality. The numerical optimization was performed using the general purpose computer program NPSOL. Optimal solutions were obtained for various orders of approximation and they exhibited convergence with respect to the optimal cost. A major benefit of the proposed technique is the ability to obtain solutions for low order controllers relative to the plant order. Although we have used the $\mathscr{H}_2$-norm as a performance measure, the technique is readily applied to other problems for which the cost function can be calculated numerically given the controller parameters.

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