Design of Gain-Scheduled Strictly Positive Real Controllers Using Numerical Optimization for Flexible Robotic Systems

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The design of gain-scheduled strictly positive real (SPR) controllers using numerical optimization is considered. Our motivation is robust, yet accurate motion control of flexible robotic systems via the passivity theorem. It is proven that a family of very strictly passive compensators scheduled via time- or state-dependent scheduling signals is also very strictly passive. Two optimization problems are posed; we first present a simple method to optimize the linear SPR controllers, which compose the gain-scheduled controller. Second, we formulate the optimization problem associated with the gain-scheduled controller itself. Restricting our investigation to time-dependent scheduling signals, the signals are parameterized, and the optimization objective function seeks to find the form of the scheduling signals, which minimizes a combination of the manipulator tip tracking error and the control effort. A numerical example employing a two-link flexible manipulator is used to demonstrate the effectiveness of the optimal gain-scheduling algorithm. The closed-loop system performance is improved, and it is shown that the optimal scheduling signals are not necessarily linear. [DOI: 10.1115/1.4001335]

1 Introduction

In the past 2 decades, the stability of nonlinear controllers has been investigated in many contexts. The stability of gain-scheduled controllers or linear parameter-varying (LPV) systems employed as controllers was investigated in, for example, Refs. [1,2]. There are generally restrictions on LPV controllers, for example, the scheduling signals should vary slowly, capture the plant's nonlinearities, and any reference trajectory that the plant must follow must not excite unmodeled dynamics. The stability of systems controlled via LPV controllers generally rely on the small gain theorem. Other authors, such as in Ref. [3], have investigated the stability of gain-scheduled $\mathcal{H}_\infty$ controllers, again employing the small gain theorem, as well as linear matrix inequality (LMI) techniques.

Although the small gain theorem is a very powerful result in the input-output stability theory, the passivity theorem is an equivalently important result. The passivity theorem states that a passive system connected in negative feedback with a very strictly passive system (also referred to as an input strictly passive system with finite gain) is L_2-stable [4]. The passive system to be controlled may be time-varying or nonlinear, as may the very strictly passive system representing the compensator. Unlike the small gain theorem, which inherently restricts the gain of the controller, the passivity theorem permits the controller to have very large gain, resulting in better closed-loop performance.

Nonlinear flexible systems, such as flexible robotic manipulators, are an interesting class of passive systems. The mapping between joint torques and joint rates is known to be passive due to the collocation of the input and output. Passivity of the manipulator is independent of the mass and stiffness characteristics, as well as the mode shapes and vibration frequencies. Therefore, spillover instabilities are avoided when the system is controlled via a very strictly passive compensator. Usually, the control objective of flexible manipulators is end-effector velocity tracking (as well as position tracking), for which a passive input-output mapping is possible by defining a modified input-output map, as presented in Ref. [5].

The purpose of this paper is to employ numerical optimization techniques to optimally design a gain-scheduled controller composed of a family of linear SPR controllers. We will present methods to independently optimize the linear SPR controllers, and optimize the scheduling signals of the overall gain-scheduled controller. We will also show that a gain-scheduled controller composed of linear SPR controllers scheduled with finite energy, bounded scheduling signals is very strictly passive, hence, stability of the closed-loop is guaranteed by the passivity theorem. Our main objective is to decrease the end-effector tracking error while executing a complicated spatial maneuver. Simulation results will be presented.

2 Input-Output Properties and Dynamics of Flexible Robotic Systems

For simplicity, we will restrict ourselves to the control of a planar, two-link robotic manipulator possessing flexible links, as shown in Fig. 1. In general, the algorithms to be presented in future sections may be applied to a flexible manipulator with more than two links, however, a two-link system is sufficient to demonstrate the effectiveness of the algorithm developed.

The nonlinear dynamic equations of motion for a two-link flexible robotic manipulator (and in general any flexible robotic manipulator with more than two links) can be written as

$$\mathbf{M}(\boldsymbol{\theta},\mathbf{q}_e)\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \hat{\mathbf{B}}(\boldsymbol{\tau} + \mathbf{w}_1) + \mathbf{f}_{\text{non}}(\boldsymbol{\theta},\mathbf{q}_e,\dot{\boldsymbol{\theta}},\dot{\mathbf{q}}_e)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, $\hat{\mathbf{B}}=[\mathbf{1}\ \ \mathbf{0}]^T$, and $\mathbf{f}_{\text{non}}=[\mathbf{f}_{\text{non},\theta}^T\ \ \mathbf{f}_{\text{non},e}^T]^T$ capture the nonlinear inertia forces stemming from centrifugal and Coriolis accelerations. The generalized coordinates, joint torques, and disturbance torques are $\mathbf{q}=[\boldsymbol{\theta}^T\ \ \mathbf{q}_e^T]^T$, $\boldsymbol{\tau}=[\tau_1\ \ \tau_2]^T$, and $\mathbf{w}_1$ for the two-link model being considered, where $\boldsymbol{\theta}=[\theta_1\ \ \theta_2]^T$ are the columnized joint angles, and $\mathbf{q}_e$ are the columnized elastic coordinates associated with discretization of flexible links [6].

We will be concerned with the control of the robot tip velocity $\dot{\boldsymbol{\rho}}_{\mu=1}=[v_x\ \ v_y]^T$, where

$$\dot{\boldsymbol{\rho}}_\mu = \mathbf{J}_\theta(\boldsymbol{\theta},\mathbf{q}_e)\dot{\boldsymbol{\theta}} + \mu\mathbf{J}_e(\boldsymbol{\theta},\mathbf{q}_e)\dot{\mathbf{q}}_e$$

Here, $\mathbf{J}_\theta$ is the rigid Jacobian mapping joint rates to spatial velocities, as if the manipulator were rigid, and $\mathbf{J}_e$ is the elastic Jacobian, which maps the elastic rates to the tip rate, as if the joints were locked. The mapping between the joint torques and the actual tip rate is not passive; in Ref. [5], it was shown that for manipulators carrying large payloads, the modified input-output mapping between $\mathbf{u}=\mathbf{J}_\theta^{-T}\boldsymbol{\tau}$ and the μ-tip rate

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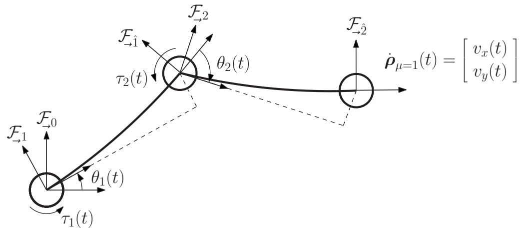

Fig. 1 Planer two-link flexible manipulator

$$\dot{\boldsymbol{\rho}}_{\mu} = \mathbf{J}_{\theta}(\boldsymbol{\theta},\mathbf{q}_e)\dot{\boldsymbol{\theta}} + \mu\mathbf{J}_e(\boldsymbol{\theta},\mathbf{q}_e)\dot{\mathbf{q}}_e = \mu\dot{\boldsymbol{\rho}}_{\mu=1} + (1-\mu)\mathbf{J}_{\theta}(\boldsymbol{\theta},\mathbf{q}_e)\dot{\boldsymbol{\theta}} \tag{1}$$

is passive when $0 \le \mu < 1$. This modified input-output mapping is necessary in order to implement the passivity theorem. The true-tip position is captured by $\mu=1$, which, as previously mentioned, when combined with $\mathbf{u}$, does not represent a passive map. In general, μ should be picked to be as close to 1 as possible, thus having $\dot{\boldsymbol{\rho}}_{\mu}$ closely resemble the true-tip velocity. The numerical simulation to be presented in Sec. 7 will use $\mu=0.8$.

In the future, we not only implement rate control, but also position control in the form of proportional control. From the definition of $\dot{\boldsymbol{\rho}}_{\mu}$ in Eq. (1), by assuming $\mathbf{J}_{\theta}(\boldsymbol{\theta},\mathbf{q}_e)=\mathbf{J}_{\theta}(\boldsymbol{\theta},\mathbf{0})$, we can approximate $\boldsymbol{\rho}_{\mu}$ as $\boldsymbol{\rho}_{\mu} \doteq \mu\boldsymbol{\rho}_{\mu=1}+(1-\mu)\mathcal{F}_r(\boldsymbol{\theta})$. $\mathcal{F}_r(\boldsymbol{\theta})$ is the rigid forward kinematics map, as defined in Ref. [6].

3 Desired Trajectory and General Control Form

Our control objective is to have the two-link manipulator previously discussed follow a prespecified tip trajectory $\boldsymbol{\rho}_d$. We can define $\boldsymbol{\rho}_d$ and $\dot{\boldsymbol{\rho}}_d$ based on an equivalent rigid robot joint trajectory, mapped through the forward kinematics $\boldsymbol{\rho}_d=\mathcal{F}_r(\boldsymbol{\theta}_d)$. In future sections, we will employ the following desired joint trajectory in order to calculate $\boldsymbol{\rho}_d$ and $\dot{\boldsymbol{\rho}}_d$:

$$\boldsymbol{\theta}_d = \begin{cases} \boldsymbol{\theta}_{d,12}, & t_1 \le t < t_2 \\ \boldsymbol{\theta}_{d,23}, & t_2 \le t < t_3 \\ \vdots & \vdots \\ \boldsymbol{\theta}_{d,N-1N}, & t_{N-1} \le t \le t_N \end{cases}$$

where

$$\boldsymbol{\theta}_{d,i\ i+1} = \left[10\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^3 - 15\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^4 + 6\left(\frac{t-t_i}{t_{i+1}-t_i}\right)^5\right](\boldsymbol{\theta}_{i+1} - \boldsymbol{\theta}_i) + \boldsymbol{\theta}_i \tag{2}$$

for $i=1,2\ldots N-1$, $\boldsymbol{\theta}_i$ are various set point positions, t_i are the times at which the manipulator is to pass through the set points, and N are the number of set point. The manipulator comes to a complete stop at each set point before continuing onto the next set point. It is important to realize that Eq. (2) is only used to provide a smooth desired tip trajectory $\boldsymbol{\rho}_d$ [7]. It is not expected that the joint angles will follow the desired joint trajectory.

In most practical applications, control would be a combination of feedforward and feedback control, $\boldsymbol{\tau}=\boldsymbol{\tau}_{\text{ff}}+\boldsymbol{\tau}_{\text{fb}}$. Feedforward control is essentially a method by which a portion of the nonlinear dynamics present in a system are canceled out. In the case of a two-link flexible manipulator carrying a massive payload, an effective feedforward is simply the rigid inverse dynamics of the system

$$\boldsymbol{\tau}_{\text{ff}} = \mathbf{M}_{\theta\theta}(\boldsymbol{\theta}_d,\mathbf{0})\ddot{\boldsymbol{\theta}}_d - \mathbf{f}_{\text{non},\theta_d}$$

where $\mathbf{f}_{\text{non},\theta_d}$ is the joint angle partition of $\mathbf{f}_{\text{non}}$ evaluated at the desired trajectory. Our feedback control will be a combination of position and rate control

$$\boldsymbol{\tau}_{\text{fb}} = -\mathbf{J}_{\theta}^T[\mathbf{K}_p(\boldsymbol{\rho}_{\mu}-\boldsymbol{\rho}_d) + \boldsymbol{G}(\dot{\boldsymbol{\rho}}_{\mu}-\dot{\boldsymbol{\rho}}_d)]$$

where $\mathbf{K}_p$ represents proportional control (which will not be included in the gain-scheduling algorithm to be presented, but remain constant) and $\boldsymbol{G}$ is a system operator representing a linear or nonlinear rate controller. The focus of this paper is designing $\boldsymbol{G}$ to be an optimal gain-scheduled very strictly passive controller.

4 Local Optimization—The Optimal Design of SPR Controllers

Our objective is to optimally design a very strictly passive gain-scheduled controller for a flexible robotic system, guaranteeing stability via the passivity theorem. Before we can fully realize this objective, we must investigate a similar preliminary objective, that being the local design of the N linear SPR controllers (i.e., the family members, or the basis), which compose the gain-scheduled controller [8–12]. As discussed in Ref. [13], $\mathbf{G}(s)+\delta\mathbf{1}$ is very strictly passive where the transfer matrix $\mathbf{G}(s)$ is SPR.

4.1 Local Optimization Problem Statement. The two-link manipulator we wish to control is a nonlinear system. Our preliminary or local optimization objective is to optimally design various linear SPR controllers, which are optimal in the vicinity of a specific joint configuration. Therefore, we will linearize the two-link manipulator about a particular joint configuration, $\mathbf{q}_d = [\boldsymbol{\theta}_d^T\ \ \mathbf{0}]^T$, which can be described by a state-space model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}$$

$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}$$

where $\mathbf{x} \in \mathbb{R}^{12}$ are the system states (composed of the number of joint angles, the elastic coordinates, and both their rates), $\mathbf{u} \in \mathbb{R}^2$ is the control input, $\mathbf{y} \in \mathbb{R}^2$ are the noisy system measurements (that being $\dot{\boldsymbol{\rho}}_{\mu}+\mathbf{w}_2$), $\mathbf{z} \in \mathbb{R}^4$ is the regulated output (that being the actual tip position and rate, $\boldsymbol{\rho}_{\mu=1}$ and $\dot{\boldsymbol{\rho}}_{\mu=1}$), and $\mathbf{w}^T=[\mathbf{w}_1^T\ \ \mathbf{w}_2^T]$, $\mathbf{w} \in \mathbb{R}^4$ represents system disturbances/noise. We will assume that:

1. $(\mathbf{A},\mathbf{B}_1)$ is controllable and $(\mathbf{C}_1,\mathbf{A})$ is observable,
2. $(\mathbf{A},\mathbf{B}_2)$ is controllable and $(\mathbf{C}_2,\mathbf{A})$ is observable,
3. $\mathbf{D}_{12}^T\mathbf{C}_1=\mathbf{0}$ and $\mathbf{D}_{12}^T\mathbf{D}_{12}>\mathbf{0}$, and
4. $\mathbf{D}_{21}\mathbf{B}_1^T=\mathbf{0}$ and $\mathbf{D}_{21}\mathbf{D}_{21}^T>\mathbf{0}$.

Consider a general SPR controller $\mathbf{u}(t)=-\boldsymbol{G}\mathbf{y}(t)$ in state-space form

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{y}$$

$$\mathbf{u} = -\mathbf{C}_c\mathbf{x}_c$$

Combining the linearized plant and controller yields the closed-loop system dynamics

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{x}}_c\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A} & -\mathbf{B}_2\mathbf{C}_c\\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c\end{bmatrix}}_{\mathbf{A}_{zw}}\underbrace{\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_c\end{bmatrix}}_{\mathbf{x}_{zw}} + \underbrace{\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_c\mathbf{D}_{21}\end{bmatrix}}_{\mathbf{B}_{zw}}\mathbf{w}$$

$$\mathbf{z} = \underbrace{[\mathbf{C}_1\ \ -\mathbf{D}_{12}\mathbf{C}_c]}_{\mathbf{C}_{zw}}\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_c\end{bmatrix}$$

Our preliminary or local optimization objective function will be *minimization of the closed-loop* $\mathcal{H}_2$-norm of the linearized system while varying design variables associated with the parameterization of a SPR controller. The closed-loop $\mathcal{H}_2$-norm can be calculated via

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$$\mathcal{J}_2 = \sqrt{\text{tr } \mathbf{B}_{zw}^T \mathbf{P} \mathbf{B}_{zw}}$$

The matrix $\mathbf{P}=\mathbf{P}^T>\mathbf{0}$ is found by solving the Lyapunov equation

$$\mathbf{P}\mathbf{A}_{zw} + \mathbf{A}_{zw}^T\mathbf{P} = -\mathbf{C}_{zw}^T\mathbf{C}_{zw}$$

A family of locally optimized SPR controllers will make up the optimal gain-scheduled controller basis.

4.2 Parameterization of the SPR Controllers. To ensure SPRness of the basis controllers, we will employ the parameterization proposed by Marquez and Damaren [14]. Consider the following diagonal controller:

$$\mathbf{G}(s) = \mathbf{C}_c(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{B}_c = \frac{1}{s^3 + as^2 + bs + c} \times \begin{bmatrix} n_{2,11}s^2 + n_{1,11}s + n_{0,11} & 0 \\ 0 & n_{2,22}s^2 + n_{1,22}s + n_{0,22} \end{bmatrix} \quad (3)$$

The above transfer matrix will be SPR if the two transfer functions within $\mathbf{G}(s)$ are SPR independently. The SPRness of each is guaranteed if the denominator polynomial is Hurwitz and the numerator polynomial coefficients satisfy

$$n_{2,ii} = \frac{bck_{1,ii} + ck_{2,ii} + ak_{3,ii}}{abc - c^2},$$

$$n_{1,ii} = \frac{c^2k_{1,ii} + ack_{2,ii} + a^2k_{3,ii}}{abc - c^2}, \quad n_{0,ii} = \frac{k_{3,ii}}{c}$$

where $i=1,2$ and the parameters $k_{1,ii}$, $k_{2,ii}$, and $k_{3,ii}$ satisfy

$$k_{1,ii} \geq 0, \quad k_{2,ii} > -2\sqrt{k_{1,ii}k_{3,ii}}, \quad k_{3,ii} > 0$$

Thus, the local optimization design variables are the denominator polynomial coefficients, a, b and c, as well as the parameters $k_{1,11}$, $k_{2,11}$, $k_{3,11}$, $k_{1,22}$, $k_{2,22}$, and $k_{3,22}$.

To summarize, given the controller parameterization above, the local optimization problem is posed as follows.

Design variables:

$$\mathbf{x} = \begin{bmatrix} a & b & c & k_{1,11} & k_{2,11} & k_{3,11} & k_{1,22} & k_{2,22} & k_{3,22} \end{bmatrix}$$

Constraints: $\text{Re}\{\lambda_j(\mathbf{A}_c)\}<0$ for $j=1,\ldots,3$, $k_{1,ii}\geq 0$, $k_{2,ii}> -2\sqrt{k_{1,ii}k_{3,ii}}$, $k_{3,ii}>0$ for $i=1,2$.

Objective function: Closed-loop $\mathcal{H}_2$-norm, $\mathcal{J}_2=\sqrt{\text{tr } \mathbf{B}_{zw}^T\mathbf{P}\mathbf{B}_{zw}}$.

The numerical optimization algorithm employed to solve the preliminary optimization problem stated above is a sequential quadratic programming (SQP) algorithm. Constraints are enforced via Lagrange multipliers, while gradient information is approximated using a finite difference scheme [15].

5 Passivity Properties of a Gain-Scheduled Controller Composed of SPR Controllers

Before discussing how we will optimize the scheduling signals, s_i, of a gain-scheduled SPR controller, we will investigate the passivity properties of such a controller. Consider the gain-scheduled feedback control system in Fig. 2. The plant $\boldsymbol{G}_0$ is the two-link manipulator we wish to control, and it is prewrapped with proportional control (which remains constant at all times). There are N linear SPR controllers $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots, \boldsymbol{G}_N$ of the form $\mathbf{y}_i(t)=\boldsymbol{G}_i\mathbf{u}_i(t)$, $i=1,2,\ldots,N$. Each controller is a rate controller and satisfies the definition of a very strictly passive system [4]:

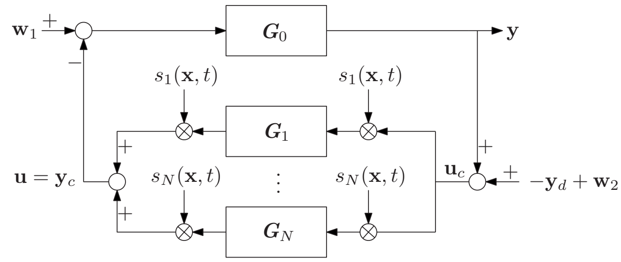

Fig. 2 Gain-scheduled feedback control system

$$\int_0^T \mathbf{u}_i^T(t)\boldsymbol{G}_i\mathbf{u}_i(t)dt = \int_0^T \mathbf{u}_i^T(t)\mathbf{y}_i(t)dt \geq \delta_i\int_0^T \mathbf{u}_i^T(t)\mathbf{u}_i(t)dt + \epsilon_i\int_0^T \mathbf{y}_i^T(t)\mathbf{y}_i(t)dt = \delta_i\|\mathbf{u}_i\|_{2T}^2 + \epsilon_i\|\mathbf{y}_i\|_{2T}^2, \ \forall\, \mathbf{u}_i \in L_{2e}, \ \forall\, T\geq 0, \ \delta_i>0, \ \epsilon_i>0 \quad (4)$$

Being very strictly passive, the basis controllers possess finite gain, that is the $\mathcal{H}_\infty$-norm of the transfer matrix is finite, $\gamma_i = \|\boldsymbol{G}_i\|_\infty < \infty$. The controller input-output map can be written as

$$\mathbf{u}(t) = \boldsymbol{G}(\dot{\boldsymbol{\rho}}_\mu(t) - \dot{\boldsymbol{\rho}}_d(t)) \equiv \mathbf{y}_c(t) = \boldsymbol{G}\mathbf{u}_c(t)$$

but can also be written in terms of the individual controller outputs and scheduling signals

$$\mathbf{y}_c(t) = \sum_{i=1}^N s_i(\mathbf{x}(t),t)\mathbf{y}_i(t), \quad \mathbf{y}_i(t) = \boldsymbol{G}_i\mathbf{u}_i(t), \quad \mathbf{u}_i(t) = s_i(\mathbf{x}(t),t)\mathbf{u}_c(t)$$

Here we have stated that the scheduling signals may be a function of the states of the plant $\mathbf{x}$ and time t. The scheduling signals may also be a function of just the states $s_i(\mathbf{x}(t))$, or just time $s_i(t)$. In general, we will assume the signals satisfy

$$\sum_{i=1}^N s_i^2(\mathbf{x}(t),t) \geq \alpha > 0, \quad s_i \in L_{2e} \cap L_\infty$$

which ensures that at least one scheduling signal is active at all times, each signal is square-integrable on a finite time interval, and each signal is bounded.

In order to guarantee stability via the passivity theorem, we must prove that the proposed gain-scheduled controller is very strictly passive. We will start by showing that the gain-scheduled controller is input strictly passive, as presented in Ref. [16], then show that the gain-scheduled controller possesses finite gain. In the following proofs, we will neglect writing function arguments to be concise.

THEOREM 5.1. *A gain-scheduled controller $\boldsymbol{G}$ composed of a family of very strictly passive (VSP) controllers $\boldsymbol{G}_i$ is input strictly passive (ISP).*

Proof.

$$\int_0^T \mathbf{u}_c^T\boldsymbol{G}\mathbf{u}_c dt = \int_0^T \mathbf{u}_c^T\mathbf{y}_c dt = \int_0^T \mathbf{u}_c^T\left(\sum_{i=1}^N s_i\mathbf{y}_i\right)dt = \sum_{i=1}^N\int_0^T s_i\mathbf{u}_c^T\mathbf{y}_i dt = \sum_{i=1}^N\int_0^T \mathbf{u}_i^T\mathbf{y}_i dt \geq \sum_{i=1}^N \delta_i\int_0^T \mathbf{u}_i^T\mathbf{u}_i dt = \sum_{i=1}^N \delta_i\int_0^T s_i^2\mathbf{u}_c^T\mathbf{u}_c dt \geq \delta\int_0^T \mathbf{u}_c^T\mathbf{u}_c dt, \quad \delta = \alpha \min \delta_i$$

Thus

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$$\int_0^T \mathbf{u}_c^T \boldsymbol{G} \mathbf{u}_c dt \geq \delta \int_0^T \mathbf{u}_c^T \mathbf{u}_c dt$$

We would now like to extend the above result and show that $\boldsymbol{G}$ is very strictly passive by showing that $\boldsymbol{G}$ has finite gain.

THEOREM 5.2. *A gain-scheduled controller $\boldsymbol{G}$ composed of a family of very strictly passive controllers possesses finite gain.*

Proof. Consider an arbitrary time-dependent signal $\mathbf{b} \in L_{2e}$, $\forall t \geq 0$ similar to the output of any one SPR controller $\boldsymbol{G}_i$

$$\int_0^T s_i^2 \mathbf{b}^T \mathbf{b} dt \leq \sup_t |s_i|^2 \int_0^T \mathbf{b}^T \mathbf{b} dt$$

Taking the square-root of both sides, it follows that

$$\|s_i \mathbf{b}\|_{2T} \leq \|s_i\|_\infty \|\mathbf{b}\|_{2T}$$

Consider now the output of the gain-scheduled controller

$$\mathbf{y}_c = \sum_{i=1}^N s_i \mathbf{y}_i = \sum_{i=1}^N s_i \boldsymbol{G}_i \mathbf{u}_i = \sum_{i=1}^N s_i \boldsymbol{G}_i (s_i \mathbf{u}_c)$$

The L_{2T}-norm of $\mathbf{y}_c$ is

$$\begin{aligned}
\|\mathbf{y}_c\|_{2T} &= \left\| \sum_{i=1}^N s_i \boldsymbol{G}_i (s_i \mathbf{u}_c) \right\|_{2T} \\
&\leq \sum_{i=1}^N \|s_i \boldsymbol{G}_i (s_i \mathbf{u}_c)\|_{2T} \text{ (via the triangle inequality)} \\
&\leq \sum_{i=1}^N \|s_i\|_\infty \|\boldsymbol{G}_i (s_i \mathbf{u}_c)\|_{2T} \leq \sum_{i=1}^N \|s_i\|_\infty \gamma_i \|s_i \mathbf{u}_c\|_{2T} \\
&\leq \sum_{i=1}^N \|s_i\|_\infty^2 \gamma_i \|\mathbf{u}_c\|_{2T}
\end{aligned}$$

Assuming $\mathbf{u}_c \in L_2$, $\mathbf{u}_c \neq \mathbf{0}$, and letting $T \to \infty$ gives

$$\frac{\|\mathbf{y}_c\|_2}{\|\mathbf{u}_c\|_2} \leq \sum_{i=1}^N \|s_i\|_\infty^2 \gamma_i < \infty$$

that is to say, the gain-scheduled controller has finite gain.

By combining the input strictly passive and finite gain characteristics of Theorems 5.1 and 5.2, the gain-scheduled controller is very strictly passive. Therefore, we may utilize this controller in a negative feedback interconnection with a passive system, and are guaranteed stability via the passivity theorem.

6 The Gain-Scheduling Optimization Problem

Our objective is to design some sort of optimal SPR gain-scheduled controller to control a two-link flexible manipulator as it moves between various set points in time T. The manipulator tip is to follow a desired trajectory $\boldsymbol{\rho}_d$, based on desired joint angles $\boldsymbol{\theta}_d$, mapped through the manipulator forward kinematics. Previously in Sec. 4, a method for designing optimal SPR controllers to be used as the basis for the gain-scheduled controller was presented. The basis controllers are coupled to linearization of the plant about a set point. We now seek to utilize a family of these controllers within a gain-scheduling algorithm, each separately optimized about different operating points. Our main optimization objective is to attain better end-effector tracking given a desired tip position and tip velocity, $\boldsymbol{\rho}_d$ and $\dot{\boldsymbol{\rho}}_d$.

In particular, we will choose three linearization points to design three SPR controllers, $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_3$. Thus, we require three scheduling signals, s_1, s_2, and s_3. We will specify that the scheduling signals will only be a function of time: $\mathbf{y}_c(t) = \Sigma_{i=1}^3 s_i(t) \mathbf{y}_i(t)$, $s_i \in L_{2e} \cap L_\infty$. The three linearization points are coupled to three joint angles, $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\boldsymbol{\theta}_3$, which the manipulator is to pass through at times t_1, t_2, and t_3.

6.1 Scheduling Signal Parameterization and the Optimization Design Variables, Constraints, and Objective Function.

Historically, gain-scheduling algorithms employ linear scheduling signals. Rather than having linear scheduling signals interpolate between the basis controllers (i.e., the family of optimal SPR controllers), we will have a numerical optimizer determine optimal time-dependent scheduling signals, which may or may not be linear. We will elect to use the following polynomial as a candidate for optimization and gain scheduling:

$$s_i(t) = a_{i0} + a_{i1} t + a_{i2} t^2 + a_{i3} t^3 + a_{i4} t^4, \quad i = 1,2,3 \tag{5}$$

An explicit fourth-degree time-dependent polynomial has been chosen because it has sufficient richness to create nonlinear scheduling signals. Each scheduling signal will be "on" or "off" at specific times during the robot trajectory

$$s_1(t) = \begin{cases} 1 & t = t_1 \\ a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3 + a_{14} t^4 & t_1 < t < t_2 \\ 0 & t \geq t_2 \end{cases} \tag{6a}$$

$$s_2(t) = \begin{cases} 0 & t = t_1 \\ a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3 + a_{24} t^4 & t_1 < t < t_3 \\ 1 & t = t_2 \\ 0 & t \geq t_3 \end{cases} \tag{6b}$$

$$s_3(t) = \begin{cases} 0 & t \leq t_2 \\ a_{30} + a_{31} t + a_{32} t^2 + a_{33} t^3 + a_{34} t^4 & t_2 < t < t_3 \\ 1 & t \geq t_3 \end{cases} \tag{6c}$$

We have constrained all of the scheduling signals to be 1 when the robot is exactly at the corresponding set point, and zero when time has passed the neighboring set points set time t_i. This will ensure that when the robot is between two set points, only the two controllers designed between the current positions are controlling the motion, rather than all the controllers in the family. For example, when $t_2 < t < t_3$, it makes much more sense for just controllers $\boldsymbol{G}_2$ and $\boldsymbol{G}_3$ to control the robot motion, and then less and less of controller $\boldsymbol{G}_2$ and more and more of $\boldsymbol{G}_3$ control the motion as $t \to t_3$. Also, although we have specified that s_1, s_2, and s_3 should exactly equal 1 when time equals t_1, t_2, and t_3, respectively, we could let the scheduling signals take on different values, or let the optimization process choose optimal values. Doing so would in fact simplify the optimization process (because there would be less constraints), but it is more intuitive to constrain s_1, s_2, and s_3 to be exactly equal to 1 at times t_1, t_2, and t_3, respectively.

Note that $t_3 \neq T$, where T is the total robot simulation time. We will be concerned with any remaining vibrations in the robot structure after motion has stopped at set point 3, between t_3 and T. However, after t_3 has passed, controller $\boldsymbol{G}_3$ will be the only controller active. Thus, $s_3 = 1$ and $s_i = 0$ for $i \neq 3$ for $t_3 \leq t \leq T$. Also note that all the above scheduling signals defined over $t \in [0, T]$ and are in L_{2e} and L_∞, as required.

We will have the optimizer determine the exact shape of the scheduling signals given a particular objective function. Thus, the design variables for the optimization problem will be the coefficients of the scheduling signal polynomials

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$$\mathbf{x} = [a_{10} \;\; a_{11} \;\; a_{12} \;\; a_{13} \;\; a_{14} \;\; a_{20} \;\; a_{21} \;\; a_{22} \;\; a_{23} \;\; a_{24} \;\; a_{30} \;\; a_{31} \;\; a_{32} \;\; a_{33} \;\; a_{34}]$$

The optimization objective will be to minimize the weighted end-effector tracking error, rate, and joint torque

$$\mathcal{J} = \eta_1 \|\mathbf{e}\|_{2T}^2 + \eta_2 \|\dot{\mathbf{e}}\|_{2T}^2 + \eta_3 \|\boldsymbol{\tau}\|_{2T}^2$$

where

$$\mathbf{e} = \boldsymbol{\rho}_{\mu=1} - \boldsymbol{\rho}_d, \quad \dot{\mathbf{e}} = \dot{\boldsymbol{\rho}}_{\mu=1} - \dot{\boldsymbol{\rho}}_d$$

and $\eta_1 > 0$, $\eta_2 > 0$, and $\eta_3 > 0$ are weights. From the optimal control theory, the need to penalize the position error, rate error, and control effort is well known, else infeasible results will ensue (such as infinite control effort).

Given the above gain-scheduled controller form and parameterization, the numerical optimization problem is posed as follows.

Design variables:

$$\mathbf{x} = [a_{10} \;\; a_{11} \;\; a_{12} \;\; a_{13} \;\; a_{14} \;\; a_{20} \;\; a_{21} \;\; a_{22} \;\; a_{23} \;\; a_{24} \;\; a_{30} \;\; a_{31} \;\; a_{32} \;\; a_{33} \;\; a_{34}]$$

Constraints:

$$s_1(t) = \begin{cases} 1 & t = t_1 \\ 0 & t \geq t_2 \end{cases}$$

$$s_2(t) = \begin{cases} 0 & t = t_1 \\ 1 & t = t_2 \\ 0 & t \geq t_3 \end{cases}$$

$$s_3(t) = \begin{cases} 0 & t \leq t_2 \\ 1 & t \geq t_3 \end{cases}$$

Objective function:

$$\mathcal{J} = \eta_1 \|\mathbf{e}\|_{2T}^2 + \eta_2 \|\dot{\mathbf{e}}\|_{2T}^2 + \eta_3 \|\boldsymbol{\tau}\|_{2T}^2$$

A SQP algorithm utilizing finite differencing for gradient calculation will be used to solve the above optimization problem. Note that the optimal basis controllers $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_3$, and the scheduling signals s_1, s_2, and s_3 are designed offline, given a desired trajectory.

7 Gain-Scheduling Optimization Results

The three set points used to design the three SPR controllers to be scheduled are $\boldsymbol{\theta}_1 = [22.5 \text{ deg}, 45 \text{ deg}]^T$, $\boldsymbol{\theta}_2 = [0 \text{ deg}, 67.5 \text{ deg}]^T$, and $\boldsymbol{\theta}_3 = [22.5 \text{ deg}, -22.5 \text{ deg}]^T$. The set point times will be $t_1 = 0$ s, $t_2 = 1$ s, and $t_3 = 3$ s. The first motion set between $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is to be completed in 1 s ($t_2 - t_1 = 1$ s), and the second motion set between $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ is to be completed in 2 s ($t_3 - t_2 = 2$ s). Past t_3, the desired joint angle will be constant and equal to $\boldsymbol{\theta}_3$, and the simulation will run for 6 s total ($T = 6$ s). The two-link manipulator properties are described in Table 1.

The gradient based, numerical optimization procedure was successful; the objective function was minimized to a value of $\mathcal{J} = 1.4019$ when the weights within the objective function were set to $\eta_1 = \eta_2 = 10$ and $\eta_3 = 1$. The convergence tolerance was set to 1×10^{-3}. The optimized scheduling signals are shown in Fig. 3. Figure 4 shows the simulated system response, along with the response of the same manipulator controlled via unscheduled control, i.e., controller $\boldsymbol{G}_3$ controlling the manipulator throughout the entire maneuver. Using the same performance index, the system was able to attain $\mathcal{J} = 1.7015$ when controlled just by $\boldsymbol{G}_3$ (which is optimal about set point three). Figure 5 shows the tip position and velocity error $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$, respectively, for both scheduled and unscheduled control. The gain-scheduling controller increases the system performance by 21.37%. Note that the scheduling signals take on values greater than 1, and therefore increase the overall gain of the controller they are scheduling.

When $\eta_1, \eta_2 > \eta_3$, a greater emphasis is on the minimization of the end-effector tracking error rather than the minimization of the control effort. Consider next the following weights: $\eta_1 = \eta_2 = \eta_3 = 1$. With these weights, the minimization of the position and rate tracking error is equally as important as the minimization of the control effort. The resultant optimal scheduling signals are shown in Fig. 6, the manipulator system response in Fig. 7, and tip position and velocity errors in Fig. 8. The optimizer converged to an objective function value of $\mathcal{J} = 1.4501$, which is still a 14.78% increase in performance as compared with unscheduled control. It is very interesting to see that scheduling signals s_1 and s_2 in Fig. 6 are, for all intents and purposes, linear. Historically, scheduling signals have arbitrarily been chosen to be linear. These results show that for a particular objective function, linear scheduling signals are indeed optimal.

As previously mentioned, the optimal scheduling signals presented in Fig. 3 are greater than 1, and increase the gain of the controller. Recall that the passivity theorem permits high gain very strictly passive controllers. It is well known that high gain feedback generally outperforms gain limited feedback. Therefore, it is not surprising that the system, as controlled by the scheduling signals of Fig. 3, has less tracking error as compared with the system as controlled by the scheduling signals of Fig. 6 because the gain of the gain-scheduled very strictly passive controller is greater in the first case.

Table 1 Two-link manipulator physical properties

Length	L_1, L_2	0.5 m
Link mass	m_1, m_2	0.3375 kg
Modulus of elasticity	E_1, E_2	70×10^9 Pa
Link height	h_1, h_2	50 mm
Link base width	w_1, w_2	4 mm
Link second moment of area	$I_1 = I_2 = \frac{1}{12} h w^3$	5.2083×10^{-10} m^4
Link 1 payload mass (motor 2)	$m_{\text{tip},1}$	0.5 kg
Link 1 payload inertia	$J_{\text{tip},1}$	5×10^{-4} kg m^2
Link 2 payload mass	$m_{\text{tip},2}$	2.5 kg
Link 2 payload inertia	$J_{\text{tip},2}$	2.5×10^{-3} kg m^2

8 Concluding Remarks

In this paper, the optimization of SPR controllers and a time-varying gain-scheduled controller composed of SPR controllers was investigated. The gain-scheduled controller composed of SPR controllers was shown to be very strictly passive. While controlling a two-link flexible manipulator with the gain-scheduled SPR controller, the closed-loop tracking performance was improved as compared with control via a single linear (yet optimal in the region of the final set point) SPR controller. Stability is guaranteed via the passivity theorem.

We first investigated how to optimize a set of linear SPR controllers to be used as the basis for the gain-scheduled controller. In prescribing the objective function to be the minimization of the closed-loop $\mathcal{H}_2$-norm, and by parameterizing the controller in a

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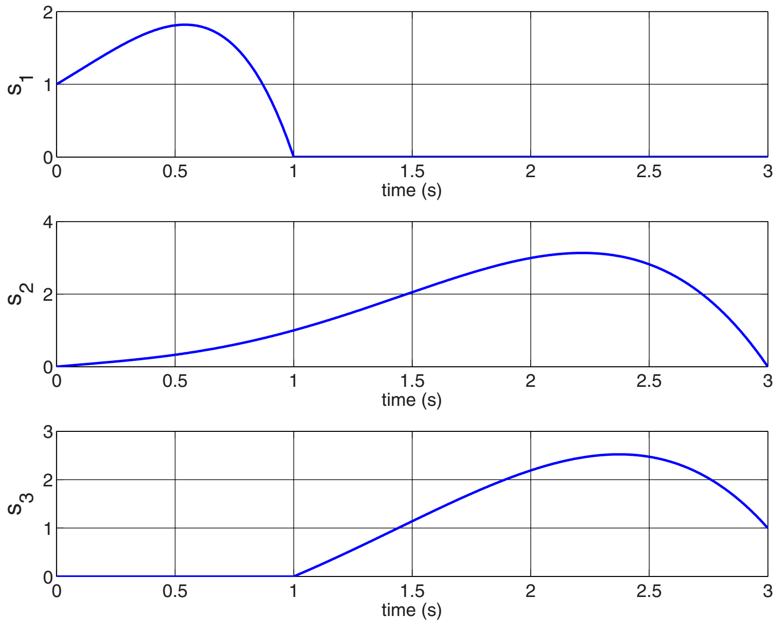

Fig. 3 Optimal scheduling signals with $\eta_1=\eta_2=10$, $\eta_3=1$

simple way, we were able to optimize each basis controller about a specific set point. Next, considering the optimization of a gain-scheduled controller, we parameterized the scheduling signals as polynomials in time, yielding a set of simple design variables. Having no analytical tools available to interpret system performance for a nonlinear system controlled via a time-varying controller, the optimization objective function was specified to be the weighted sum of the squared L_{2T}-norms of the position error, rate error, and joint torque. The optimization of a gain-scheduled SPR controller was successful.

An interesting result is that depending on the weighting of the optimization objective function, very different scheduling signals may be considered optimal. High performance tracking requires scheduling signals that are highly nonlinear. However, for equal weighting of the tracking error and control torque, linear scheduling signals are close to optimal, as traditionally would be assumed.

It should be noted that the scheduling signals are not constrained to be less than 1 at all times. In fact, with the weights η_1, η_2, and η_3 biased toward minimization of the tracking error, the optimal scheduling signals produced by the optimizer were greater than 1 for the majority of their active scheduling time. This leads to an increase in individual controller gain. Originally, the SPR controllers $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_3$ were individually optimized using the method presented in Sec. 4 with no knowledge possible gain increase in the control input and output, $\mathbf{u}_c$ and $\mathbf{y}_c$. Therefore, the basis SPR controllers may not be considered "optimal," given the original objective function used to optimize the controllers about their unique operating points. Future work may be to incorporate both the optimization of the controllers and the scheduling signals together, rather than first optimizing the individual controllers, and then optimizing the scheduling signals in a sequential manner.

Unlike the work in Refs. [1,2], we have not specified (a priori) that the scheduling signals should vary slowly and capture the plant's nonlinearities. We have simply constrained points that the

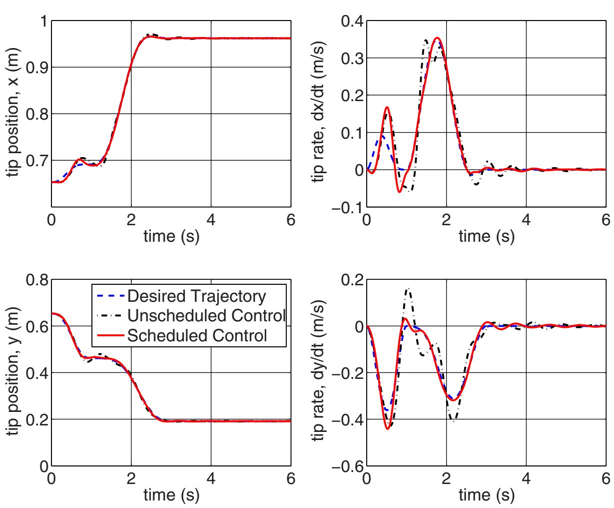

Fig. 4 Optimal gain-scheduling control system response with $\eta_1=\eta_2=10$, $\eta_3=1$

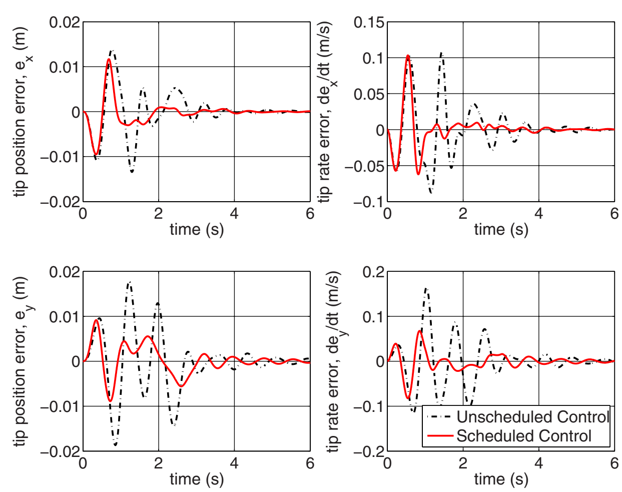

Fig. 5 Optimal gain-scheduling control system response error with $\eta_1=\eta_2=10$, $\eta_3=1$

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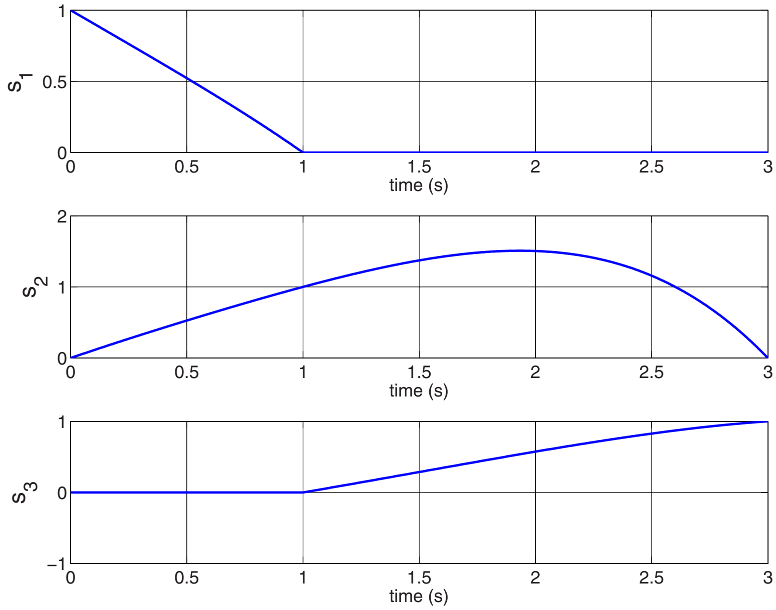

Fig. 6 Optimal scheduling signals with $\eta_1=\eta_2=\eta_3=1$

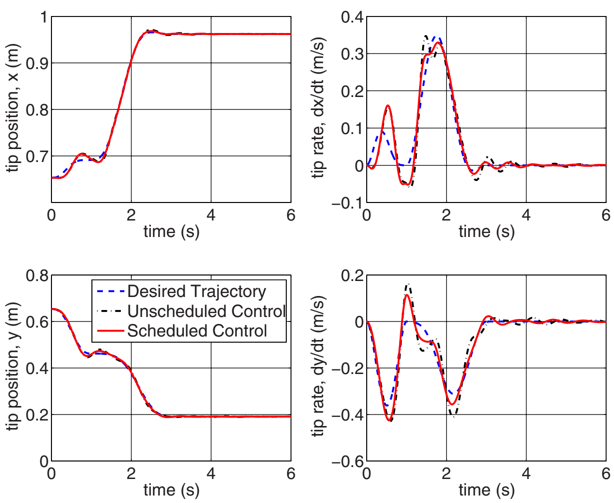

Fig. 7 Optimal gain-scheduling control system response with $\eta_1=\eta_2=\eta_3=1$

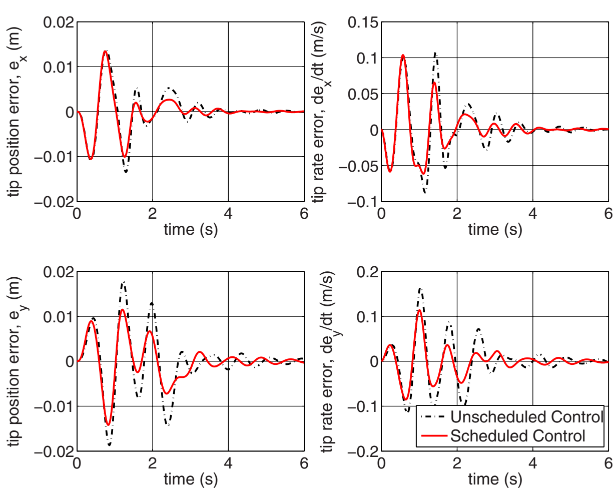

Fig. 8 Optimal gain-scheduling control system response error with $\eta_1=\eta_2=\eta_3=1$

scheduling signal must pass through at some time during the robot motion. From our results in Sec. 7, we can see that the optimal scheduling signals do vary slowly with time, but it is the optimizer, which determines their shape with respect to time. Our formulation is much less restrictive in that the optimizer is free to determine what scheduling signal is best, rather than heuristically imposing conservative constraints on the scheduling signals.

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