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A population game approach for dynamic resource allocation problems

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ABSTRACT

We consider a water distribution system as an example of resource allocation, and investigate the use of a population game for its control. We use a game-theoretic approach based on two evolutionary dynamics, the Brown–von Neumann–Nash and the Smith dynamics. We show that the closed-loop feedback interconnection of the water distribution system and the game-theoretic-based controller has a Nash equilibrium as an asymptotically stable equilibrium point. The stability analysis is performed based on passivity concepts and the Lyapunov stability theorem. An additional control subsystem is considered for disturbance rejection. We verify the effectiveness of the method by simulations under different scenarios.

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Population game; control; stability; resource allocation

1. Introduction

In this work, we consider the distribution of resources in allocation problems by using a population game-theoretic approach. This class of allocation problem that can include distribution of labour, water, fuel, energy, etc. has become a challenging problem for engineers. As an example of resource distribution problems, we consider a water distribution system (WDS), since management of water distribution networks is one of the most important issues of concern in big cities.

We use a game-theoretic approach, based on modelling the WDS control as a population game. A population game is typically modelled as a memoryless mapping from the population state to a set of static payoff functions (Sandholm, 2010). To specify the process of a game play, a revision protocol describes how and when strategy choices are switched. Specific protocols lead to specific mean dynamics, each with their own properties. The corresponding mean or evolutionary dynamic can be analysed to predict the evolution of the game, and possible convergence to Nash equilibria. Inspired by Fox and Shamma (2013), in this work, we consider a more general class of games played over dynamical systems. This is an extension in that the evolutionary dynamics act on dynamically modified payoffs instead of static payoff functions. This modification can be interpreted as a coupling between a set of evolutionary dynamics and a population game with dynamic dependencies.

We consider a WDS modeled as a population game with dynamic dependencies, as a typical resource allocation application (Ramírez-Llanos & Quijano, 2010). A WDS consists of a number of storage tanks that need to supply water such that customer demands are satisfied while a certain pressure at the tanks output is maintained. We consider a water flow model, similar to the nonatomic game case (Roughgarden, 2007), where there are a very large number of players, each controlling a negligible portion of the overall water flow. Thus, as in population games, Sandholm (2010) assume that the tanks are the possible action choices for the water flow. What counts is the fraction of the population (hence of the water flow) that selects a particular tank or strategy. When a tank is selected, a small portion of the resource is dedicated to fill that tank. We assign a payoff to each strategy or tank and design different revision protocols. Using these protocols, we can model the evolution in time of the game as an interconnection between the corresponding evolutionary dynamics and the dynamical system modelling the tanks.

We utilise a game-theoretic approach along with a convex-optimisation-based controller. Unlike Ramírez-Llanos and Quijano (2010) where only the replicator dynamics (RD) is used to control the game, we consider different evolutionary dynamics with better properties. Specifically, it is well known that by using the replicator dynamics, a potential problem of non-Nash convergence can occur (Hofbauer & Sandholm, 2009; Sandholm, 2005). To address this issue, we consider the Brown–von Neumann–Nash (BNN) and the Smith dynamics, which discard non-Nash equilibrium points. Furthermore, we add the total input flow controller to enhance the

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behaviour of the system. We characterise convergence to a Nash equilibrium as stability of the closed-loop system. We present sufficient conditions that ensure stability of the feedback interconnection between the corresponding evolutionary dynamics and the dynamical system modelling the tanks. These are based on results for interconnection of passive and strictly passive dynamical systems and Lyapunov stability theorem. We show that when the total input flow is constant and can be tuned externally, a controller designed using either the BNN or the Smith dynamics achieves globally asymptotic stability.

Furthermore, we provide an improvement to controllers that achieves a degree of robustness for the system, by rejecting a certain type of disturbances. Moreover, we consider a generalisation to a class of distribution problems that maintains the same stability properties as for the WDS application. The effectiveness of our approach is verified by simulations.

1.1 Literature review

Game theory has become a widely used tool to tackle different problems in many different research fields. An area of interest in recent years is the design of control strategies for resource allocation problems. For instance, in wireless communication networks, some approaches are to minimise the power consumption while customers are satisfied with service quality, usually defined based on signal-to-noise ratio (SNR) (Alpcan, Başar, Srikant, & Altman, 2002; Altman, Boulogne, El-Azouzi, Jiménez, & Wynter, 2006; Yaïche, Mazumdar, & Rosenberg, 2000; Yates, 1995). It is assumed that the power is correlated with the SNR such that by increasing the power, the data transmission range will be extended but the noise level will also be higher (Johari & Tsitsiklis, 2004; Pavel, 2006). The problem of power distribution is solved via convex optimisation (Boyd & Vandenberghe, 2004), linear and nonlinear programming (Arrow & Hurwicz, 1958; Bertsekas, 1999), by defining cost functions with respect to the power and SNR (Fan, Alpcan, Arcak, Wen, & Başar, 2006; Feijer & Paganini, 2010). However, usually not all dynamics of the systems are considered, but only partial dynamics or functions linked to power and service quality. Other recent game theoretical approaches for resource allocation problems are channel assignment in wireless networks (Bacci, Luise, Poor, & Tulino, 2007; Miao, Himayat, Li, & Talwar, 2011; Wang, Scutari, & Palomar, 2011; Zappone, Sanguinetti, Bacci, Jorswieck, and Debbah, 2016); resource allocation in cloud computing services, routing (Duarte, Fadlullah, Vasilakos, & Kato, 2012; Khan, Tembine, & Vasilakos, 2012; Niyato, Vasilakos, & Kun, 2011; Rahimi, Venkatasubramanian, Mehrotra, & Vasilakos, 2012; Wei, Vasilakos, Zheng, & Xiong, 2010; Zeng, Xiang, Li, & Vasilakos, 2013), or security problems (Shan & Zhuang, 2013, 2014). On the other hand, in Eduardo and Nicanor (2010) and Ramírez-Llanos and Quijano (2010), the entire dynamics is considered for the control of a water distribution network (Bao & Lee, 2007) via an evolutionary dynamics approach. The RD, first introduced by Taylor in Taylor and Jonker (1978) as one of the classes of evolutionary dynamics, is employed as a controller to stabilise and make the WDS converge to desirable equilibrium points. Prior to these studies, cost minimisation of materials such as pipes, vales, and tanks (Broad, Dandy, and Maier, 2005; Zecchin, Simpson, Maier, & Nixon, 2005) or water production costs (Araujo, Ramos, & Coelho, 2006; Camarinha-Matos & Martinelli, 1999) have been considered, although, no rigorous mathematical proof has been provided.

1.2 Contributions

In this paper, we show that the RD, under some conditions, converges to undesirable equilibrium points (Sandholm, 2010) and makes the WDS not function properly. In order to solve this problem, we propose other classes of well-behaved evolutionary dynamics, namely the BNN and Smith dynamics (Fox & Shamma, 2012, 2013; Hofbauer & Sandholm, 2009), that discard the undesirable equilibria. For these classes of evolutionary dynamics, we extend the existing convergence results in Ramírez-Llanos and Quijano (2010) to global asymptotic convergence. Moreover, by means of the ideas in Fox and Shamma (2013), we provide a generalisation to a class of resource allocation problems to which the WDS is an example. In terms of stability analysis of the evolutionary dynamics, we follow similar methods utilised in Ramírez-Llanos and Quijano (2010), Sandholm (2010), and Fox and Shamma (2013) where passivity concepts and Lyapunov stability theorem (Khalil, 2002) are employed. A preliminary version of this work appeared in Pashaie, Pavel, and ChrisDamaren (2015). Here we also include a section that proposes an additional dynamics for disturbance rejection, as well as a detailed numerical study.

The paper is organised as follows. In Section 2, we introduce some preliminary background material. In Section 3, we present the problem statement, solution approach as well as stability analysis. In Section 4, a generalisation to the system under study is provided. Section 5 presents an additional dynamics for disturbance rejection, followed by simulation results in Sections 6 and conclusions in Section 7.

2. Background

2.1 Stability and passivity

Consider a dynamical system represented by

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x, u), \end{aligned} \tag{1}$$

where $x \in \mathcal{X}$ denotes the state and $u \in \mathcal{U}$ the input. We assume that $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is locally Lipschitz and $h : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^p$ is continuous, and that $f(0, 0) = h(0, 0) = 0$. Let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^p$ denote the state-space and the class of piecewise continuous inputs, respectively. The definition and the theorems below review passivity and Lyapunov stability.

Definition 2.1 (Khalil, 2002, Definition 6.3, p. 236)**:** Consider the system of form (1). If there exists a continuously differentiable nonnegative function $\mathcal{V} : \mathcal{X} \to \mathbb{R}$, such that $\forall (x, u) \in \mathcal{X} \times \mathcal{U}$

- $u^T y = \dot{\mathcal{V}} = \frac{\partial \mathcal{V}}{\partial x} f(x, u)$, then the system is lossless.
- $u^T y \geqslant \dot{\mathcal{V}}$, then the system is called passive.
- $u^T y \geqslant \dot{\mathcal{V}} + \varphi(x)$, where $\varphi(x) > 0 \ \forall x \neq 0$, then the system is strictly passive.

The function $\mathcal{V}$ is called the storage function.

Lemma 2.1 (Khalil, 2002, Theorem 4.1 and 4.2, p. 114, 124)**:** *Let $x = 0$ be an equilibrium point for the system in the form of (1) and let $\mathcal{X} \subset \mathbb{R}^n$ be a domain containing $x = 0$. If there exists a continuously differentiable positive definite candidate Lyapunov function $L : \mathcal{X} \to \mathbb{R}$ such that*

$$\begin{aligned} L(0) = 0 \quad \text{and} \quad L(x) &> 0, \quad \forall x \in \mathcal{X} - \{0\}, \\ \dot{L}(x) \quad &\leqslant 0, \quad \forall x \in \mathcal{X}, \end{aligned}$$

then, $x = 0$ is stable. If in addition

$$\dot{L}(x) < 0, \quad \forall x \in \mathcal{X} - \{0\},$$

then, $x = 0$ is asymptotically stable. Furthermore, if $\mathcal{X} = \mathbb{R}^n$ and $L(x)$ is radially unbounded, i.e.

$$\begin{aligned} \|x\| \to \infty &\implies L(x) \to \infty, \\ \dot{L}(x) < 0, &\qquad \forall x \neq 0, \end{aligned}$$

then, $x = 0$ is globally asymptotically stable.

Lemma 2.2 (Khalil, 2002, Theorem 6.1, p. 247)**:** *Consider the feedback interconnection of two dynamical systems of form (1), as in Figure 1. The closed-loop system is strictly passive if Σ_1 and Σ_2 are strictly passive with positive definite storage functions $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Moreover, for $u = 0$, the origin of the negative feedback interconnected system is asymptotically stable.*

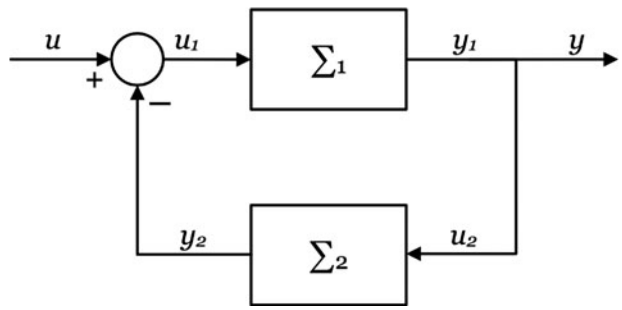


Figure 1. Negative feedback interconnection of two systems.

Note that this result extends to more than two systems.

2.2 Population games and evolutionary dynamics

Consider a large population of *players/agents* that are playing a game by selecting among a number of choices called *strategies*, denoted by the set $S = \{1, \dots, N\}$. The *simplex* or the *strategy distribution set* of the population is defined as $\Delta = \{p = [p_1, \dots, p_i, \dots, p_N]^T \in \mathbb{R}^N_+ : \sum_{i \in S} p_i = 1\}$ where p as the vector of players proportions, p_i's, is called the *population state*.

A game is generally identified by a function that describes payoffs. Let $f : \Delta \to \mathbb{R}^N$ denote the *payoff function*, which assigns a vector of payoffs to each population state. Consider that f is a continuous mapping. We denote the vector of payoffs by $f(p) = [f_1(p), \dots, f_N(p)]^T$, where $f_i(p)$ is a continuous payoff function of playing the i-th strategy. The *Nash equilibrium state* of a population game is a state where every utilised strategy earns the maximum payoff (Sandholm, 2010), i.e.

$$NE(f) = \left\{ p \in \Delta | \ p_i > 0 \Rightarrow f_i(p) \geqslant f_j(p) \ \ \forall j \in S \right\}. \tag{2}$$

Since at an NE, $p \in NE(f)$ every utilised strategy receives the maximum payoff, we can write

$$\forall p_i > 0, \quad f_i(p) = \max_{j \in S} f_j(p). \tag{3}$$

At a Nash equilibrium, a player achieves the maximum of his utility, with respect to its own strategy choice, assuming the others' strategies are fixed. To specify the process of a game play, we consider how decisions are made. In a population game, once in a while, agents are given opportunities to change their choices, using a *revision protocol* to switch to a better strategy. Revision protocols describe how and when strategies are switched, hence account for selection and mutation processes. The definition of a revision protocol $\rho(f, p)$ is presented in the following:

Definition 2.2 (Sandholm, 2010, p. 121)**:** Given the payoff vector f and the population state p, a revision protocol ρ, $\rho : \mathbb{R}^N \times \Delta \to \mathbb{R}_+^{N\times N}$, describes the switch from a strategy to another, through the scalar *conditional switch rate*, ρ_{ij}, from strategy $i \in S$ to strategy $j \in S$.

Under the assumption of a large population, the stochastic process of strategy selection is well approximated in the limit as the population tends to infinity by a set of ordinary differential equations called the *mean dynamic* (Sandholm, 2010). The mean dynamic of a population game is defined next.

Definition 2.3 (Sandholm, 2010, p. 124)**:** The mean dynamic of the population game f is a set of ordinary differential equations that describe the evolution in time of the population state p under a given revision protocol ρ,

$$\dot{p} = V(f, p), \tag{4}$$

where

$$V_i(f, p) = \sum_{j\in S} p_j \rho_{ji}(f, p) - p_i \sum_{j\in S} \rho_{ij}(f, p), \tag{5}$$

and, in general, is called the evolutionary dynamic.

The *deterministic mean (evolutionary) dynamics* describes the process under the revision protocol: the first term captures switches from other strategies to strategy i (inflow), while the second captures switches from strategy i to others (outflow).

Notice that under the condition that $V : \mathbb{R}^N \times \Delta \to \mathbb{R}^N$ is Lipschitz continuous, existence and uniqueness of the solution to Equation (4) follows from standard results (Khalil, 2002; Sandholm, 2010). Therefore, we ensure the Lipschitz continuity of $V(f, p)$ by assuming that f and ρ are Lipschitz continuous. That being the case, by considering the Definitions 2.2 and 2.3, one can conclude that the evolutionary dynamic is a mapping from the payoff function f to the population state of the game under which the mean dynamic is generated.

Considering population games as mappings from the population state to the payoffs, we can view the connection of the evolutionary dynamic with the corresponding population game as a closed-loop system, as shown in Figure 2.

As explained in detail in Fox and Shamma (2012, 2013), this closed loop can be modelled as a feedback interconnection between two input–output systems

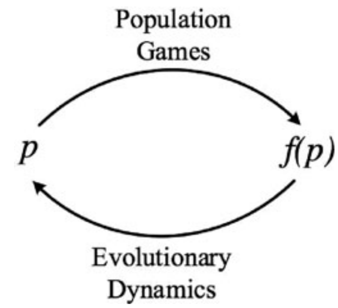


Figure 2. Connection of population game and mean dynamic.

shown in Figure 1, where Σ_1 and Σ_2 are the input–output models of the population game and the evolutionary dynamic, respectively. Next, an extension to this interconnection is provided.

2.3 Games with dynamic dependencies

Traditionally, population games are considered as memoryless mappings from the population state p to the payoffs $f(p)$ where the payoffs are static functions. In Fox and Shamma (2012, 2013), the authors extend these results to a more general class of games. As an extension from static games, they consider games that have dynamics, which means that the games are being played through dynamical systems. In this setup, evolutionary dynamics act on dynamically modified payoffs instead of static payoff functions. This modification can be interpreted as a coupling between a set of evolutionary dynamics and a population game with dynamic dependencies. In this configuration, dynamically modified games are mappings from *strategy trajectories* $p(t)$ to *payoff trajectories* $f(t)$ (Fox & Shamma, 2013). Figure 3 demonstrates this configuration, in which $x(t)$ is the state vector and $f(x(t))$ is the payoff vector of the dynamically modified game, respectively. In other words, modified games can be written as

$$\dot{x} = F(x, p), \tag{6}$$

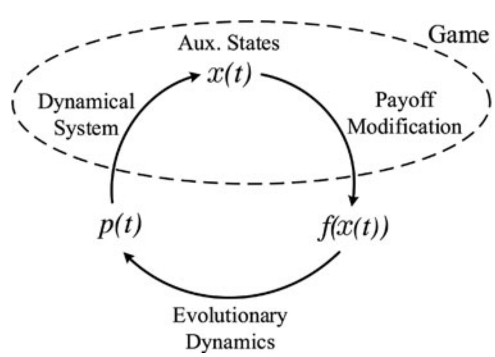


Figure 3. Connection of dynamic population game with evolutionary dynamics (Fox & Shamma, 2013).

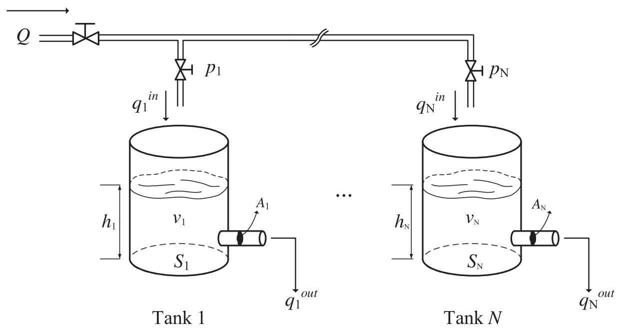


Figure 4. Water distribution system.

where the Lipschitz continuous function $F : \mathbb{R}^N \times \Delta \to \mathbb{R}^N$ describes the dynamical model of the modified game. As can be seen, the payoffs are not necessarily functions of the population state only, but also of some auxiliary states of the dynamical system as a solution of Equation (6). As a consequence, these new payoff functions correspond to the dynamics of the game. This is exactly the case for the dynamic WDS which we present in the next section.

3. WDS – modelling/control problem

3.1 Water distribution system

We consider a WDS as a typical example application, and model it as a population game with dynamic dependencies. As in Ramírez-Llanos and Quijano (2010), the WDS consists of a number of storage tanks and is employed to supply water to its customers who require a specific amount of water stored in their tanks and a certain pressure at the tanks' output. We assume that there exists a total input flow of water as the resource to fill the tanks with water. Thus, if we assign a certain portion of the total input flow to each tank, we can control the amount of water in the tanks.

We consider a water flow model, similar to the nonatomic game case (Roughgarden, 2007), where there are a very large number of agents, each controlling a negligible portion of the overall water flow. Thus, as in population games, Sandholm (2010) assume that the tanks are the possible action choices for the water flow. What counts is the fraction of the agent population (hence of the water flow) that selects a particular tank or strategy. When a tank is selected, a small portion of the resource is dedicated to fill that tank. We assign a payoff to each strategy or tank and design different revision protocols. Using these protocols, we can model the evolution in time of the game as an interconnection between the corresponding evolutionary dynamics and the dynamical system modelling the tanks.

A WDS, illustrated in Figure 4, is used to supply water to all of its consumers and meet their demands. Assume that N different tanks need to deliver water to N consumers by allocating a total available flow of water, Q. Let the volume, input and output flow of the i-th tank be denoted by v_i, q_i^{in} and q_i^{out}, respectively. The change with respect to time of the water volume in each tank is described by Ramírez-Llanos and Quijano (2010),

$$\frac{dv_i}{dt} = q_i^{in} - q_i^{\text{out}}, \tag{7}$$

for $i = 1, \ldots, N$. We formulate the WDS as a population game with dynamic dependencies and design a controller based on the mean dynamic of the corresponding game under a given revision protocol. We assume that each tank is equipped with a control valve at its input pipe to limit the water flow. Since p_i's, which are the fractions of the flow or of the population selecting the i-th tank, are proportions between 0 and 1, they can be used as gates of the control valves to determine the percentage of how open or close the valves are to let water in the tanks. Therefore, the share of the i-th tank will be

$$q_i^{in} = Qp_i. \tag{8}$$

To model the output flow, we assume that the tanks are being drained over the atmospheric pressure (gravity tank system; Bao & Lee, 2007; Ramírez-Llanos & Quijano, 2010), so that the output flow, q_i^{out}, depends on the water height in the tank, h_i, and on the pipe transversal area, A_i. Therefore, if we assume that the transversal area of each tank, denoted by S_i, is constant and independent of h_i, we can model the output flow of the i-th tank as follows (Ramírez-Llanos & Quijano, 2010):

$$q_i^{\text{out}} = A_i C_i \sqrt{2gh_i} = A_i C_i \sqrt{\frac{2gv_i}{S_i}}, \tag{9}$$

where g is the acceleration due to the gravity, C_i is the loss coefficient and $v_i = h_i S_i$. Therefore, by using Equations (7)–(9), differential equations of the WDS dynamical model are given by

$$\dot{v}_i = Qp_i - c_i\sqrt{v_i}\,. \tag{10}$$

The parameter $c_i = A_i C_i \sqrt{2g/S_i}$ for each tank depends on its output pipe and the transversal area.

3.2 Replicator dynamics as system controller

In Ramírez-Llanos and Quijano (2010), the RD is considered as an evolutionary dynamic to control the WDS. For this, first, a payoff function should be associated with each tank. The authors in Ramírez-Llanos and Quijano (2010) considered the logistic-type function (see Britton, 2003) of form

$$f_i = -\frac{r_i}{\upsilon_{m_i}}\upsilon_i + r_i, \qquad (11)$$

for $i = 1, \dots, N$. For each tank i, υ_{m_i} and r_i denote the maximum possible level of water and the maximum payoff. In this configuration, the payoff function decreases as the volume of each tank gets closer to its maximum level, and therefore empty tanks are more attractive.

The RD, in the class of *pairwise proportional imitation dynamics*, is generated by the following revision protocol prototype (Sandholm, 2010; Taylor & Jonker, 1978):

$$\rho_{ij} = p_j\left[f_j(p) - f_i(p)\right]_+, \qquad (12)$$

where $[\cdot]_+$ denotes the projection to positive numbers. Substituting this revision protocol in (5) leads to the RD having the form

$$\dot{p}_i = p_i\left(-\frac{r_i}{\upsilon_{m_i}}\upsilon_i + r_i - \sum_{j=1}^{N} p_j\left(-\frac{r_j}{\upsilon_{m_j}}\upsilon_j + r_j\right)\right). \qquad (13)$$

Consider the feedback interconnection between the WDS and the RD as in Figure 3, or as in the block diagram of Figure 1. The components Σ_1 and Σ_2 are identified as the WDS tank system and the RD. Σ_1 and Σ_2 are characterised by the state variables $\upsilon = [\upsilon_1, \dots, \upsilon_N]^T$, the volume of water in the tanks, and $p = [p_1, \dots, p_N]^T$, the population state of the RD, respectively. The outputs are $y_1 = [\upsilon_1, \dots, \upsilon_N]^T$ and $y_2 = -[p_1, \dots, p_N]^T$ and the input u is zero. For simplicity, it is assumed that the maximum levels and the maximum payoffs of all tanks are equal, i.e. $\upsilon_{m_i} = \upsilon_m$ and $r_i = r$. Therefore, by (10) and (13), the differential equations for the negative feedback interconnection become

$$\Sigma_1 : \begin{cases} \dot{\upsilon}_i = Qp_i - c_i\sqrt{\upsilon_i} \\ y_{1i} = \upsilon_i, \qquad u_{1i} = p_i \end{cases},$$
$$\Sigma_2 : \begin{cases} \dot{p}_i = -\frac{r}{\upsilon_m}p_i\left(\upsilon_i - \sum_{j=1}^{N} p_j\upsilon_j\right) \\ y_{2i} = -p_i, \qquad u_{2i} = \upsilon_i \end{cases}, \qquad (14)$$

where $i = 1, \dots, N$. The equilibrium points υ_i^*'s and p_i^*'s of the zero-input interconnected system satisfy

$$-\frac{r}{\upsilon_m}p_i^*\left(\upsilon_i^* - \sum_{j=1}^{N} p_j^*\upsilon_j^*\right) = 0, \qquad (15)$$

$$Qp_i^* - c_i\sqrt{\upsilon_i^*} = 0. \qquad (16)$$

In Equation (15), we can simply see that υ_i^*'s are equal since

$$\upsilon_i^* - \sum_{j=1}^{N}\upsilon_j^* p_j^* = 0 \;\Rightarrow\; \upsilon_i^* = \sum_{j=1}^{N}\upsilon_j^* p_j^* = \upsilon^*. \qquad (17)$$

By summing the left-hand side of (16) over i and using $C = \sum_{i=1}^{N} c_i$, we get

$$Q\sum_{i=1}^{N} p_i^* - \sqrt{\upsilon^*}\sum_{i=1}^{N} c_i = 0 \;\Rightarrow\; \upsilon^* = \left(\frac{Q}{C}\right)^2, \qquad (18)$$

which, substituted in (16), yields

$$p_i^* = \frac{c_i}{C}. \qquad (19)$$

The interconnected system (14) that uses the RD was analysed in Ramírez-Llanos and Quijano (2010), based on the Lyapunov stability theorem. The RD model of the evolutionary dynamic is one of the most known dynamics. However, it has a disadvantage which could cause some undesirable behaviours for the system. The problem occurs because the RD admits non-Nash equilibrium points on the boundary of the simplex (Sandholm, 2010). For instance, consider

$$\begin{cases} p_j^* = 0 \;\Rightarrow\; \upsilon_j^* = 0 \\ p_i^* = \dfrac{c_i}{\sum_{k\neq j} c_k} \Rightarrow \upsilon_i^* = \left(\dfrac{Q}{\sum_{k\neq j} c_k}\right)^2 \end{cases}, \quad i \neq j, \qquad (20)$$

This is an equilibrium point but it is not desired because the j-th tank is empty and, by the definition, is not a Nash equilibrium. As discussed in Sandholm (2010), this is due to the fact that the RD does not satisfy the property of *Nash Stationary*, namely that the *NE*s of a population game coincide with the equilibria of the corresponding evolutionary dynamic (Sandholm, 2005). Therefore, the RD can admit non-Nash equilibrium points. In addition, as shown in Ramírez-Llanos and Quijano (2010), the RD is lossless, and hence Theorem 2.2 is not applicable and asymptotic stability of the desired equilibrium is not guaranteed.

3.3 BNN and Smith dynamics as system controller

In this paper, we consider new evolutionary dynamics with better properties that discard the undesirable equilibrium points and are convergent only to the *NE*s. Moreover, we show that unlike the RD, the new set of dynamics are strictly passive.

In the following discussion, adapted from Sandholm (2005, 2010), two revision protocols that lead to these different evolutionary dynamics are introduced.

3.3.1 Excess payoff target dynamic (EPT)

In this class of dynamics, in switching strategies, only the difference between a randomly selected strategy's payoff and the population's average payoff is considered. A prototype of this switching logic is the *BNN* dynamic with

$$\rho_{ij}(f,p) = [\hat{f}_j(f,p)]_+ = [f_j(p) - \sum_{k=1}^{N} p_k f_k]_+,$$

where $\hat{f}_j$ is called the excess payoff function. By substituting this type of revision protocol in Equation (5), the BNN dynamic will be of the form

$$\dot{p}_i = \left[\hat{f}_i(f,p)\right]_+ - p_i \sum_{j=1}^{N} \left[\hat{f}_j(f,p)\right]_+. \quad (21)$$

3.3.2 Pairwise comparison dynamic

In this class, the difference between the payoffs of the current strategy and a randomly selected strategy is important. An example of this class is called the *Smith* dynamic:

$$\dot{p}_i = \sum_{j=1}^{N} p_j \left[f_i(p) - f_j(p)\right]_+ - p_i \sum_{j=1}^{N} \left[f_j(p) - f_i(p)\right]_+, \quad (22)$$

where

$$\rho_{ij}(f,p) = \rho_{ij}(f_j - f_i) = \left[f_j(p) - f_i(p)\right]_+,$$

is the revision protocol prototype.

Clearly, in the BNN and the Smith dynamics, $p_j = 0$ is not an equilibrium point. Moreover, the *NE*s of the corresponding population games are the only equilibria of these dynamics where $f_i = f_j = f_{\max}$ for a $p \in NE(f)$.

3.4 Stability analysis with the BNN and Smith dynamics

In order to analyse the stability of the WDS with the evolutionary dynamic as in (10), (21) and (22), we utilise a new approach introduced in Fox and Shamma (2013), based on extended systems. In general, for the system of form (1), the *extended system* is defined as (Fox & Shamma, 2013)

$$\begin{aligned} \dot{u} &= u_e, \\ \dot{x} &= f(x,u), \\ y_e &= \dot{y} = \nabla_x h(x,u)\dot{x} + \nabla_u h(x,u)\dot{u}, \end{aligned} \quad (23)$$

where the input and output of u_e and y_e are the time derivatives of the original input and output, respectively. We write (10), (21) and (22) as extended systems and, for convenience, we write the equations in error coordinates a standard notation. To write the equations in the standard form, let

$$\begin{aligned} \upsilon &= [\upsilon_1, \ldots, \upsilon_N]^T = [x_{11}, \ldots, x_{1N}]^T = x_1, \\ p &= [p_1, \ldots, p_N]^T = [x_{21}, \ldots, x_{2N}]^T = x_2, \\ \upsilon^* &= x_1^*, \qquad p^* = x_2^*, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \Delta x_1 &= x_1 - x_1^* \mathbf{1} = \upsilon - \upsilon^* \mathbf{1}, \\ \Delta x_2 &= x_2 - x_2^* \quad = p - p^*, \end{aligned}$$

where $\mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^N$. Then, as in (23), we construct the extended game dynamic from the WDS and the extended evolutionary dynamic as

$$\text{Extended Game Dynamic } \Sigma_1 : \begin{cases} \Delta\dot{x}_{1i} = Q(\Delta x_{2i} + x_{2i}^*) - c_i\sqrt{\Delta x_{1i} + x_1^*} \\ \dot{u}_{1i} = \Delta\dot{x}_{2i} = u_{e_{1i}} \\ y_{e_{1i}} = \dot{y}_{1i} = \Delta\dot{x}_{1i} \end{cases}, \quad (25)$$

$$\text{Extended BNN Dynamic } \Sigma_2 : \begin{cases} \Delta\dot{x}_{2i} = \left[\hat{f}_i\right]_+ - (\Delta x_{2i} + x_{2i}^*)\sum_{j=1}^{N}\left[\hat{f}_j\right]_+ \\ \dot{u}_{2i} = \Delta\dot{x}_{1i} = u_{e_{2i}} \\ y_{e_{2i}} = \dot{y}_{2i} = -\Delta\dot{x}_{2i} \end{cases}, \quad (26)$$

$$\text{Extended Smith Dynamic } \Sigma_2 : \begin{cases} \Delta\dot{x}_{2i} = \sum_{j=1}^{N}(\Delta x_{2j} + x_{2j}^*)\left[f_i - f_j\right]_+ \\ \qquad - (\Delta x_{2i} + x_{2i}^*)\sum_{j=1}^{N}\left[f_j - f_i\right]_+ \\ \dot{u}_{2i} = \Delta\dot{x}_{1i} = u_{e_{2i}} \\ y_{e_{2i}} = \dot{y}_{2i} = -\Delta\dot{x}_{2i} \end{cases}. \quad (27)$$

Moreover, we impose the following domain:

$$\mathcal{X} = \Big\{\Delta x = [\Delta x_1^T, \Delta x_2^T]^T \in \mathbb{R}^{2N} : \Delta x_{1i} > -x_1^*, \; \Delta x_2 + x_2^* \in \Delta, \forall i \in \{1, \ldots, N\}\Big\}, \quad (28)$$

to ensure that the system equations remain Lipschitz continuous throughout the following analysis. The following theorem shows the passivity properties of the extended WDS system.

Theorem 3.1: *The extended water distribution system as the extended game dynamic, shown in Equation (25), is strictly passive with the following positive definite storage function:*

$$\mathcal{V}_1 = \frac{1}{2Q}\sum_{j=1}^{N}\left(\Delta\dot{x}_{1j}\right)^2. \tag{29}$$

Proof: (see Appendix). ■

The passivity properties of the extended BNN and Smith dynamics are presented in Lemmas 3.1 and 3.2, adapted from Sandholm (2010), Fox and Shamma (2013), and Hofbauer and Sandholm (2009).

Lemma 3.1: *The extended BNN evolutionary dynamic, shown in Equation (26), is strictly passive with the following positive definite storage function:*

$$\mathcal{V}_2 = \frac{\upsilon_m}{r}\sum_{j=1}^{N}\int_0^{\hat{f}_j}[s]_+\,ds, \tag{30}$$

where $\hat{f}_j = \frac{r}{\upsilon_m}\left(\sum_{k=1}^{N}\Delta x_{1k}(\Delta x_{2k}+x^*_{2k}) - \Delta x_{1j}\right)$.

Proof: See Appendix. ■

Lemma 3.2 (Fox & Shamma, 2013; Hofbauer & Sandholm, 2009)**:** *The extended Smith evolutionary dynamic, shown in Equation (27), is strictly passive with the following positive definite storage function:*

$$\mathcal{V}_3 = \frac{\upsilon_m}{r}\sum_{i=1}^{N}\left(\Delta x_{2i}+x^*_{2i}\right)\sum_{j=1}^{N}\int_0^{f_j-f_i}[s]_+\,ds. \tag{31}$$

Now consider for Σ_1 and Σ_2 a new storage function of form

$$\mathcal{V}_{\text{new}} = \mathcal{V}_1 + \mathcal{V}_2, \tag{32}$$

or

$$\mathcal{V}_{\text{new}} = \mathcal{V}_1 + \mathcal{V}_3, \tag{33}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are defined in Equations (29)–(31). By Lemma 2.2, one can conclude that the negative feedback interconnection of the extended game dynamic Σ_1 and the extended evolutionary dynamic Σ_2 (as in Figure 1) is strictly passive. Also, since the storage function $\mathcal{V}_1$ is radially unbounded, by Lemma 2.1, the equilibrium point is globally asymptotically stable for zero input. Thus, $\mathcal{V}_{new}^{-1}(0)$ that coincides with the *NE* of the underlying population game (Sandholm, 2010) is globally asymptotically stable.

4. Generalisation to a class of affine games

In this section, a generalisation to a class of population games with dynamic dependencies is presented. This class of games has the property of strict passivity, and hence, by Lemma 2.2, its negative feedback interconnection with a set of strictly passive evolutionary dynamic, shown in Figure 1, converges to the Nash equilibrium points of the game. As a result, we can alternatively show the convergence for WDS by using the stability properties of this generalisation.

Consider a dynamically modified population game, presented in Figure 3 and described by Equation (6):

$$\dot{x} = F(x, p), \tag{34}$$

that is being controlled in a feedback interconnection with a set of strictly passive evolutionary dynamics, given by

$$\dot{p} = V(p, f),$$

where $V(p, f)$ can be either the extended BNN or Smith dynamics. In this configuration, $x \in \mathbb{R}^N$ is the state vector of the game and $p \in \Delta \subset \mathbb{R}^N$ is the population state. Since in a dynamically modified game, payoff functions are defined with respect to the game state vector (Fox & Shamma, 2013) (i.e. $f(x)$), one can reconsider the notation of the evolutionary dynamic as follows:

$$\dot{p} = V(p, f(x)) = V(p, x). \tag{35}$$

If we model the dynamically modified game and the evolutionary dynamic, shown in Equations (34) and (35), as the extended input–output systems of form (23), we arrive at

$$\begin{array}{c}\text{Extended}\\ \text{Modified Game}\end{array}: \begin{cases}\dot{x} = F(x, p)\\ u_1 = \dot{p}, \quad y_1 = \dot{x}\end{cases}, \tag{36}$$

$$\begin{array}{c}\text{Extended}\\ \text{Mean Dynamic}\end{array}: \begin{cases}\dot{p} = V(p, x)\\ u_2 = \dot{x}, \quad y_2 = -\dot{p}\end{cases}. \tag{37}$$

Thus, we can utilise the configuration of Figure 3 and model the connection with a negative feedback interconnection as in Figure 1. Theorem 4.1 sums up the stability analysis of the feedback interconnection mentioned above.

Theorem 4.1: *Suppose that a dynamically modified population game in the form of an extended input–output* *dynamical system is described by Equation (36), where F: $\mathbb{R}^N \times \Delta \to \mathbb{R}^N$ is Lipschitz continuous. The negative feedback interconnection of the modified population game with a set of strictly passive evolutionary dynamic described by Equation (37) is strictly passive if*

- *F is strictly decreasing in x, and*
- *F is linear and strictly increasing in p.*

Furthermore, for zero input, the origin of the feedback interconnected system is asymptotically stable.

Proof: See Appendix. ■

Theorem 4.1 is adapted from Fox and Shamma (2013) where the *smoothed payoffs* are considered as the dynamically modified payoffs. Fox and Shamma (2013) introduced a dynamical system to model and present the modified payoffs as a result of payoff smoothing. This system is defined as

$$\dot{x} = \lambda(Ap + b - x), \tag{38}$$

where $\lambda, b > 0$ and $A < 0$. It is shown that the system in the form of (38) is a strictly passive mapping from strategies to smoothed modified payoffs.

However, in our system, dynamically modified payoffs are derived by defining the payoffs with respect to the existing dynamical model of the WDS in Equation (11). We have also reformulated and generalised this class of payoff modification in Equation (A1) where $g(x)$ could be any decreasing function of the strategy trajectories. It is easy to show that the WDS could be described in this setup.

5. Disturbance rejection

In this section, we assume that the input u, shown in Figure 1, plays the role of a disturbance to the system. This disturbance could be considered as an extra usage of water, a change in water consumption pattern or type of the demands or a problem in piping system which has to deliver water to the consumers. It could cause a difference between the actual and the desired volume in the tanks, so the main task is to reject the disturbance. In this problem, it is assumed that the disturbance is a step-like (constant) function that is applied to the system for a limited time interval.

In addition, we consider a model to dynamically control the total input flow of the system, Q, based on the consumers' demands and the desirable level of water in the tanks. We assume that instant overloads of Q are allowed in the total input flow system but persistent over-usages

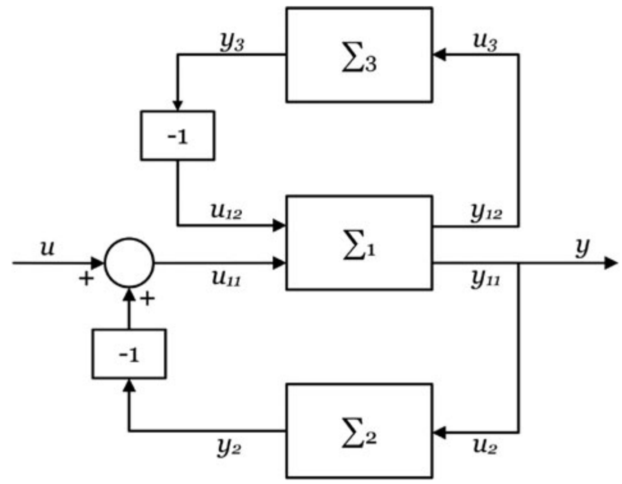


Figure 5. Interconnected system with gradient controller.

may cause damage to the system, and let $\hat{Q}$ be the maximum possible flow of water as the resource. Consider a quadratic cost J

$$\mathcal{J} = Q\left(\gamma - \alpha + \beta \sum_{j=1}^{N} \upsilon_j\right) + \frac{\alpha}{2\hat{Q}} Q^2, \tag{39}$$

which penalises higher volume of water in the tanks; α and γ are parameters chosen such that $\alpha > \gamma$. We consider an optimisation-based controller, as gradient-descent of the cost J,

$$\dot{Q} = -k\frac{d\mathcal{J}}{dQ} = k\left(\alpha\left(1 - \frac{Q}{\hat{Q}}\right) - \gamma - \beta \sum_{j=1}^{N} \upsilon_j\right), \tag{40}$$

where $k > 0$. In error coordinates, (40) is written as Σ_3,

$$\Sigma_3 : \begin{cases} \Delta\dot{x}_3 &= -\dfrac{k\alpha}{\hat{Q}}\Delta x_3 - k\beta u_3 \\ y_3 &= -\Delta x_3 \end{cases}, \tag{41}$$

and the interconnected system is as in Figure 5. To this we add an integrator Σ_4, which can ensure asymptotic disturbance rejection to step-like signals. Since the model is in feedback interconnection, we just add Σ_4 with gain $k_1 > 0$ to the feedback loop of the gradient controller (Figure 6).

$$\Sigma_4 : \begin{cases} \Delta\dot{x}_4 = u_4 \\ y_4 = k_1 \Delta x_4 \end{cases}. \tag{42}$$

We consider an extra feedback loop with gain $k_2 > 0$, in order to be able to move the pole from the origin to the open-left half-plane as needed for passivity (see Figures 6 and 7).

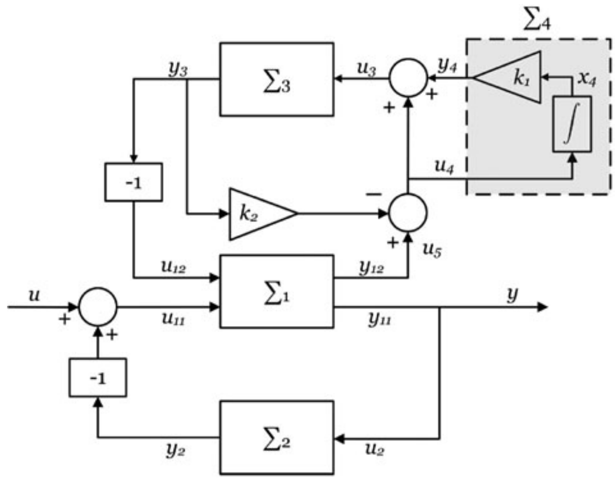


Figure 6. Interconnected system with added integrator.

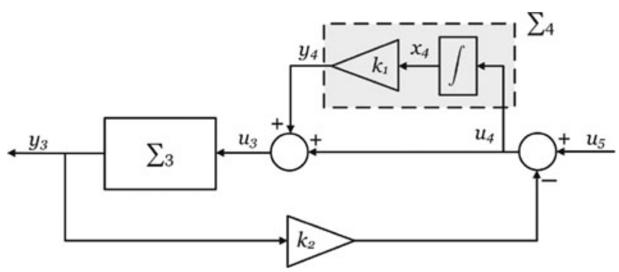


Figure 7. Integrator and the gradient controller, Σ_3'.

In the following, we discuss the passivity properties of the new system. Consider the upper feedback loop of the system, that consists of the gradient controller and the integrator as shown in Figure 7. We concatenate these two subsystems, Σ_3 and Σ_4, and call the augmented subsystem Σ_3'. Then, the interconnected system is as in Figure 5, but with Σ_3' instead of Σ_3.

Theorem 5.1 characterises the passivity of Σ_3'.

Theorem 5.1: *The system Σ_3' in Figure 7 is passive from input u_5 to output y_3 if $k_1 \leqslant \frac{k\alpha}{\hat{Q}} + k\beta k_2$. Furthermore, if $k_1 < \frac{k\alpha}{\hat{Q}} + k\beta k_2$, then Σ_3' is strictly passive.*

Proof: See Appendix. ∎

This section presented an augmented subsystem to reject external disturbances to the system. The stability analysis follows the same argument, based on passivity of the interconnected system and deriving a condition under which the system remains strictly passive. Specifically, since Σ_3' is strictly passive by Theorem 5.1, it follows that all components, Σ_1, Σ_2 and Σ_3', of the feedback interconnected system shown in Figure 6 are strictly passive. Then, by the same argument as in Lemma 2.1, it follows that interconnected system is asymptotically stable for zero inputs.

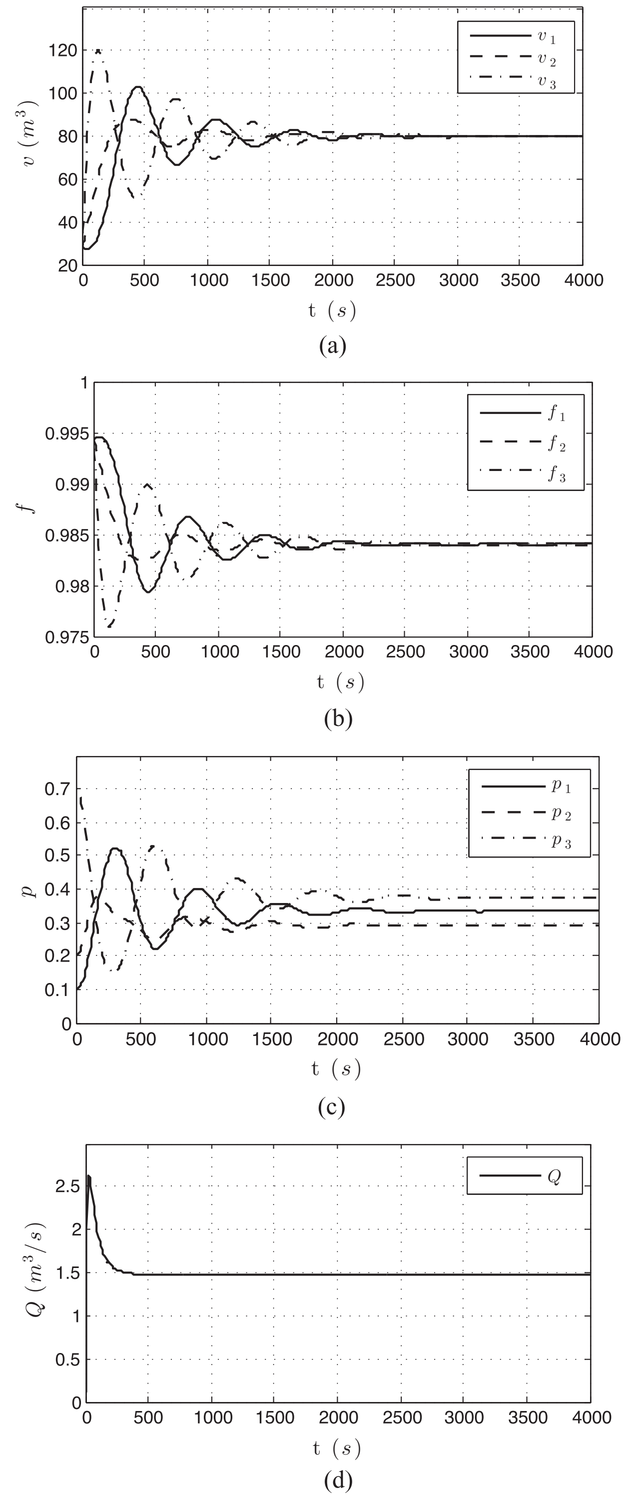


Figure 8. Tank system with replicator dynamic (initial condition 1). (a) Tank volumes vs. time; (b) payoffs vs. time; (c) population state vs. time; (d) total input flow vs. time.

6. Simulation results

In this section, we present simulation results for a $N = 3$ tank system where the output flows are under the atmospheric pressure. In order to compare our results with those similar in the literature, for the tank system we use the same parameters as in Ramírez-Llanos and Quijano (2010), summarised in Table A1 (Appendix). We begin with a comparison between the proposed BNN and Smith dynamics versus the RD as used in Ramírez-Llanos and Quijano (2010). Then, we consider disturbance rejection scenarios.

6.1 Convergence to the interior NE

First, we show tank simulation results with the RD (Ramírez-Llanos & Quijano, 2010), and with the BNN and Smith dynamics and the total flow controller as in Figure 5 (no integrator). The initial conditions are

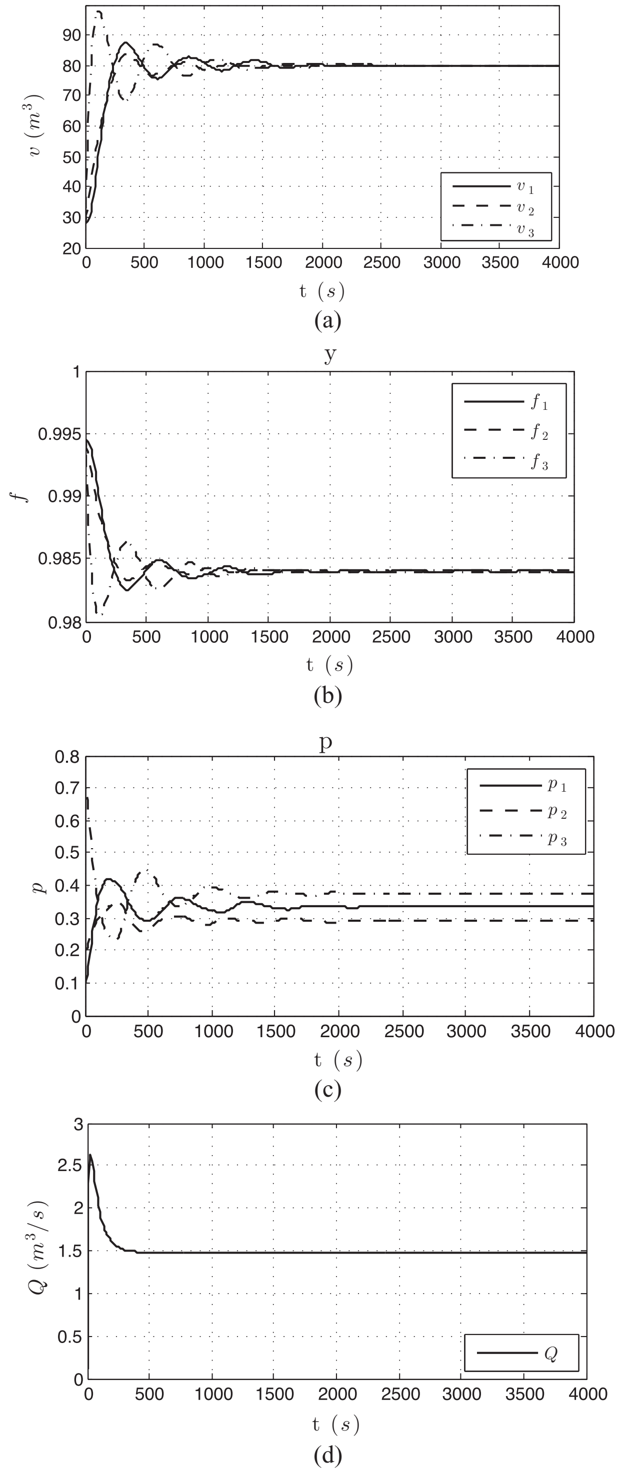


Figure 9. Tank system with BNN dynamic (initial condition 1). (a) Tank volumes vs. time; (b) payoffs vs. time; (c) population state vs. time; (d) total input flow vs. time

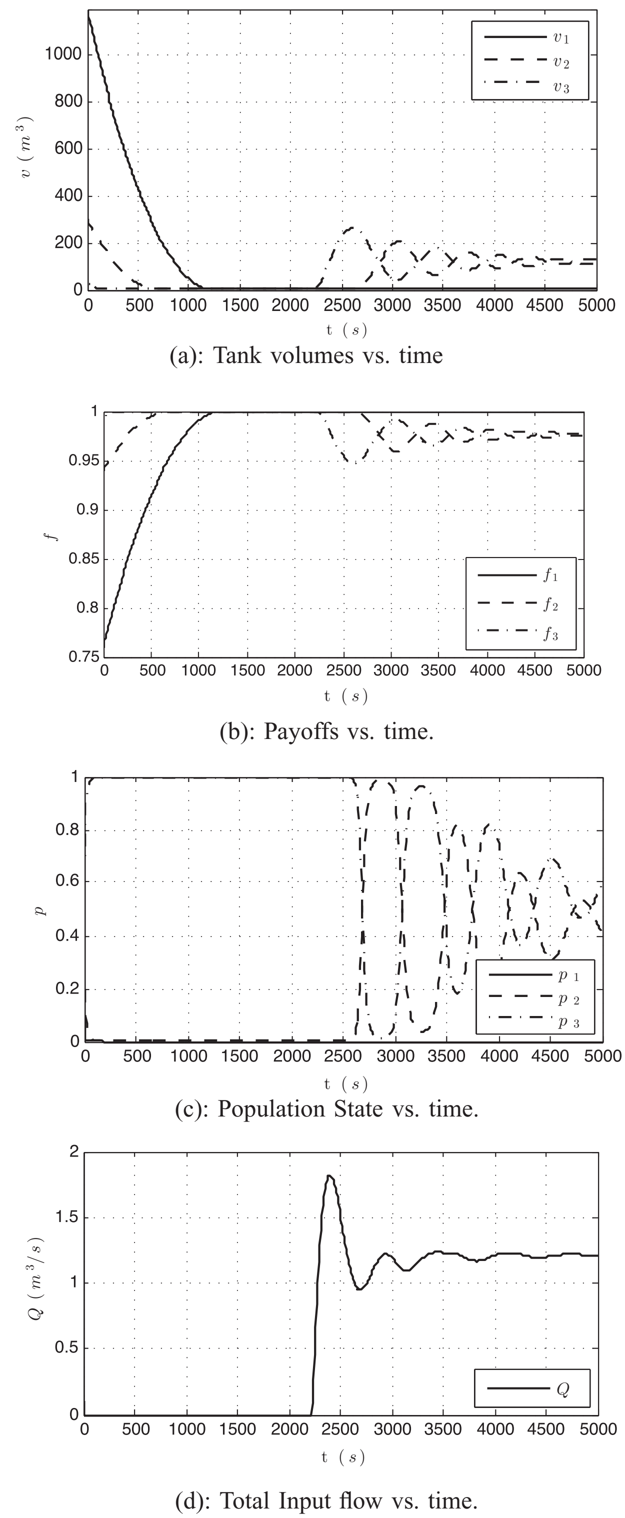
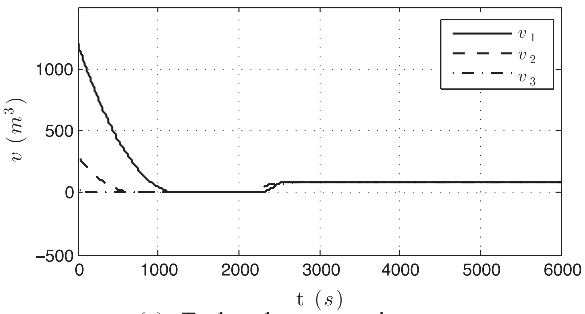
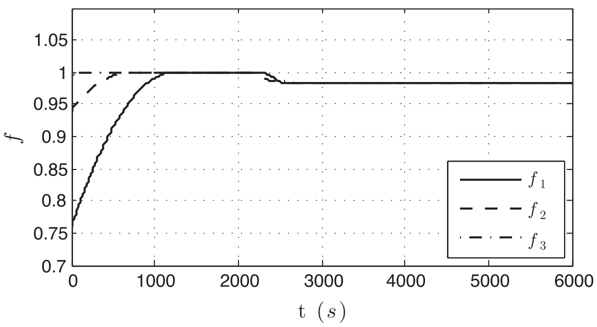


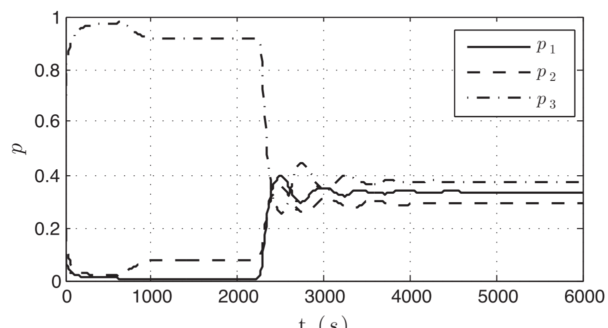
Figure 10. Tank system with replicator dynamic (initial condition 2).



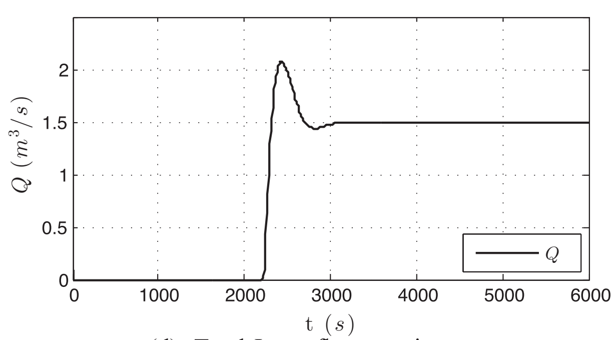
(a): Tank volumes vs. time



(b): Payoffs vs. time.



(c): Population State vs. time.



(d): Total Input flow vs. time.

Figure 11. Tank system with BNN dynamic (initial condition 2).

chosen as $v(0) = [30, 30, 30]^T$ m^3, $p(0) = 0.1, 0.2, 0.7]^T$, $Q(0) = 0.1$ m^3/s (called system initial condition 1). Controller parameters are as in Table A1 (Appendix). Figures 8 and 9 show the results for the tank system with the RD and with the BNN dynamic, respectively.

Figure 8(a) shows the evolution in time of the volume of water when the RD is used, Figure 8(b) shows the payoffs vs. time, while Figure 8(c,d) shows the evolution of the fraction of the water flow (controller or population state), and total input flow, respectively.

Similarly, Figure 9(a) shows the evolution in time of the volume of water when the BNN dynamic is used, Figure 9(b) shows the payoffs vs. time, while Figure 9(c,d) shows the evolution of the fraction of the water flow (controller or population state), and total input flow, Q, respectively.

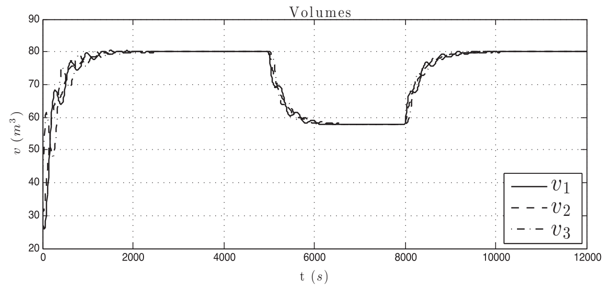
It can be seen that in both cases, the RD controllers and the BNN dynamic controllers drive the system toward the interior *NE*. The final volume of the tanks is the same and payoffs are the maximum available. As can be seen from the plots, the tank system in connection with the BNN dynamic is slightly faster than the RD in converging to the *NE*. The Smith dynamics has a similar performance as the BNN.

Next, we change initial conditions to $v(0) = [1200, 300, 30]^T\ m^3$ (initial condition 2). We plot the results of the WDS system interconnected with the RD in Figure 10 and with the BNN dynamics in Figure 11, respectively.

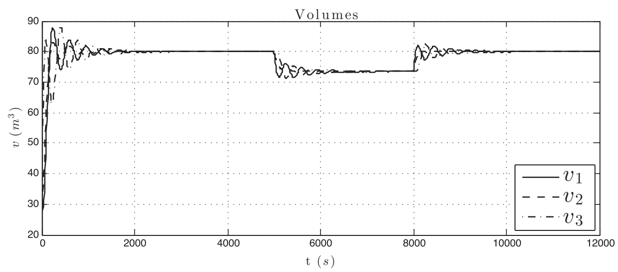
Figure 10(a) shows the evolution in time of the volume of water when the RD is used, Figure 10(b) shows the payoffs vs. time, while Figure 10(c,d) shows the evolution of the fraction of the water flow (controller or population state), and total input flow, Q, respectively. Similarly, Figure 11(a) shows the evolution in time of the volume of water when the BNN dynamic is used, Figure 11(b) shows the payoffs vs. time, while Figure 11(c,d) shows the evolution of the fraction of the water flow (controller or population state), and total input flow, Q, respectively.

In this case, it can be seen (see Figure 10(a)) that for the RD controllers, the volume of tank 1 converges to 0, and the corresponding fraction $p_1 = 0$ (see Figure 10(c)). This shows convergence to boundary rest points which are non-Nash equilibria, hence not the desired *NE*.

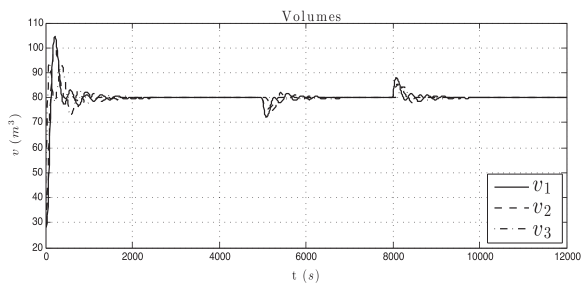
Indeed, it is clear from Figure 10(a) that the first tank remains empty while the total input flow is being distributed between tanks 2 and 3. Furthermore, due to the payoff configuration, empty tanks have higher payoffs; therefore, it is desirable to switch to tank 1 as strategy. However, since $p_1 = 0$ is one of the equilibria, the RD will remain at this point, indicating convergence to a boundary-fixed point (non-Nash). However, having an empty tank is not desired, hence the RD is not a suitable controller. On the other hand, the results with the BNN dynamic controller in Figure 11 show an improved behaviour. From Figure 11(a,c), it can be seen that neither of the tanks remains empty and all tanks converge to the *NE* (Equations (18) and (19)). Figure 11(b) shows that the payoffs are the same and maximum available.



(a): Tank volumes with static Q.



(b): Tank volumes with dynamic Q.



(c): Tank volumes with dynamic Q and the integrator.

Figure 12. Tank system and the Smith dynamic.

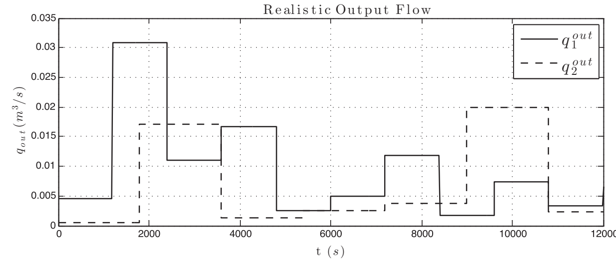


Figure 13. Stochastic rectangular disturbance pulses.

Similar results are obtained with the Smith dynamics and are not shown. Both these dynamics are known to have the Nash-stationarity property (Sandholm, 2010). Thus, the BNN dynamics and the Smith dynamics are effective alternatives to control the WDS, since convergence to non-Nash equilibrium points (on the boundary) is avoided.

6.2 Simulation results: disturbance rejection

In this section, the effects of the total input flow controller with respect to disturbance rejection will be discussed. We consider u as a step-like disturbance, for example, as an extra water consumption or as some malfunctioning in the water supply system.

We show the results (see Figure 12) for three cases of the total input flow controller: (a) Q is static, (b) Q is the dynamic but with no integrator component, and (c) Q is dynamic with an integrator component.

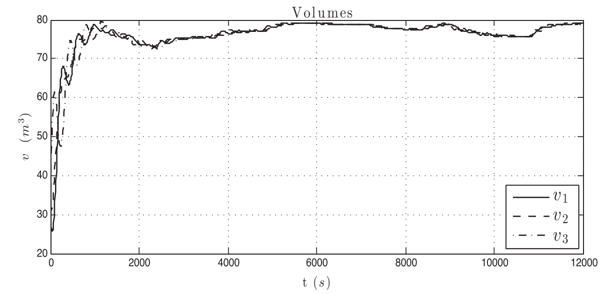
A step-like disturbance is applied to the first and second tank systems for a period of 3000 s, but leave the third tank being drained under the atmospheric pressure to see the effects of the disturbance on non-disturbed tanks while others are disturbed.

The sub-figures in Figure 12 show the response of the tank system with the Smith dynamics in the presence of disturbance, for each of these three cases. In Figure 12(a), Q is a constant parameter, while in Figure 12(b,c), Q has dynamics; in Figure 12(c), the integrator component (Figure 6) is also added. Each of the figures shows the evolution in time of the volume of water in each of the three tanks. Comparing the plots shown in these figures indicates that the response of the system in converging to the *NE* in the presence of disturbance has been dramatically improved. In Figure 12(a), the system cannot reject the disturbance. With Q dynamic, it can be seen that a smaller residual offset is present in Figure 12(b) when compared to Figure 12(a). This offset is completely removed due to the introduction of the integrator component, as seen in Figure 12(c), indicating a perfect asymptotic disturbance rejection. Similar results are obtained for the BNN dynamics and are not shown.

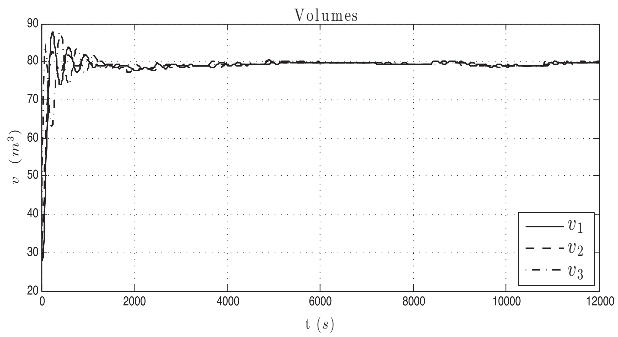
Next, we present a more realistic output flow of the tanks, as due to changes in the pattern of the consumers' demands. This could be represented by the following stochastic rectangular pulses as shown in Figure 13.

The response of the WDS system with the Smith dynamic controller in the presence of this type of disturbance is presented in Figure 14. In Figure 14(a), Q is a constant parameter while in Figure 14(b,c), Q has dynamics. The integrator component is also considered in Figure 14(c)).

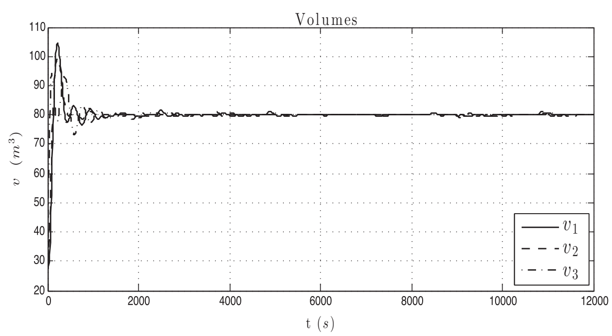
A comparison of these results shows robustness of the WDS system with the Smith dynamics and Q controller with integrator (Figure 14(c)). The numerical results in Section 6.1 show the benefits of the proposed schemes when compared to the RD (Ramírez-Llanos & Quijano (2010)); while the RD can lead to convergence to boundary equilibrium points, the BNN and Smith dynamics do not have this problem and converge to interior NEs. The



(a): Tank volumes with static Q.



(b): Tank volumes with dynamic Q.



(c): Tank volumes with dynamic Q and the integrator.

Figure 14. Tank system with the Smith dynamic.

results in Section 6.2 verify the robustness of the proposed schemes in the presence of demand disturbance.

7. Conclusions

In this paper, we investigated the WDS as a straightforward example of a resource allocation problem. We utilised a novel population game-theoretic approach to control and stabilise the WDS based on its consumers' demands, by using BNN and Smith evolutionary dynamics. We showed that the feedback interconnection of the WDS and the evolutionary dynamic controller converged to a globally asymptotically stable equilibrium point. Numerical results showed the benefits when compared to the RD, since convergence to non-Nash equilibrium points (on the boundary) is avoided. Furthermore, we introduced an additional dynamics for disturbance rejection and verified the proposed scheme robustness to demand variation.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix

Proof of Theorem 3.1: By taking the derivative of $\mathcal{V}_1$ along (25), we obtain

$$\dot{\mathcal{V}}_1 = \frac{1}{Q}\sum_{j=1}^{N} Q\Delta\dot{x}_{2j}\Delta\dot{x}_{1j} + \frac{-1}{2Q}\sum_{j=1}^{N} c_j \frac{(\Delta\dot{x}_{1j})^2}{\sqrt{\Delta x_{1j} + x_1^*}}.$$

It is clear that the second term is strictly negative, so

$$\dot{\mathcal{V}}_1 = u_{e_1}^T y_{e_1} + \left(\frac{-1}{2Q}\sum_{j=1}^{N} c_j \frac{(\Delta\dot{x}_{1j})^2}{\sqrt{\Delta x_{1j} + x_1^*}}\right) < u_{e_1}^T y_{e_1},$$

which shows that the extended WDS is strictly passive.

Table A1. Tank system parameters and total input flow controller parameters.

Parameter	Value	Parameter	Value
v_m	5000 m^3	C_1	1
r	1	C_2	1
S_1	1024 m^2	C_3	1
S_2	729 m^2	c_1	0.0555
S_3	441 m^2	c_2	0.0483
A_1	0.3973 m^2	c_3	0.0622
A_2	0.2919 m^2	g	9.8 m/s^2
A_3	0.2919 m^2	N	3
α	4.1350	k	0.1
β	0.01	k_1	0.01
γ	0.2	k_2	0
$\hat{Q}$	4 m^3/s		

Proof of Lemma 3.1: By taking the derivative along (25) and (26), we get

$$\begin{aligned}\dot{\mathcal{V}}_2 &= \sum_{j=1}^{N}\left[(\Delta x_{2j}+x_{2j}^*)\sum_{k=1}^{N}[\hat{f}_k]_+ - [\hat{f}_j]_+\right]\Delta\dot{x}_{1j}\\ &\quad+\sum_{k=1}^{N}[\hat{f}_k]_+\sum_{j=1}^{N}\Delta x_{1j}\Delta\dot{x}_{2j}\\ &= -\,\Delta\dot{x}_2^T\Delta\dot{x}_1+\left(\frac{-v_m}{r}f^T\Delta\dot{x}_2\sum_{k=1}^{N}[\hat{f}_k]_+\right).\end{aligned}$$

Due to the non-negativity of $[f_j]_+$ and the *positive correlation* property of the BNN dynamic – $\dot{p}\neq 0$ implies $f^T\dot{p}>0$ (See Sandholm (2005) for a thorough discussion) – the second term on the right-hand side of the above equation is negative definite. So

$$\dot{\mathcal{V}}_2 < u_{e_2}^T y_{e_2},$$

which proves the system of (26) is strictly passive.

Proof of Theorem 4.1: The first condition of being strictly decreasing means that $\nabla_x F<0$ and the second condition says that there exists a constant symmetric positive definite matrix $\mathcal{M}$ such that $\nabla_p F=\mathcal{M}>0$. Thus, we can represent this type of dynamically modified population game as

$$F(x,p)=\mathcal{M}p+g(x), \tag{43}$$

where $g(x)$ is Lipschitz continuous and $\nabla_x g<0$. Therefore, if we consider the following storage function (Fox & Shamma, 2013):

$$L(x,p)=\frac{1}{2}\dot{x}^T\mathcal{M}^{-1}\dot{x}=F(x,p)^T\mathcal{M}^{-1}\big(\mathcal{M}p+g(x)\big), \tag{44}$$

by taking the derivative along the trajectories we obtain

$$\begin{aligned}\dot{L} &= \dot{x}^T\,\mathcal{M}^{-1}\,\nabla_x F\,\dot{x}+\dot{x}^T\,\mathcal{M}^{-1}\,\nabla_p F\,\dot{p}\\ &= \dot{x}^T\,\mathcal{M}^{-1}\,\nabla_x g\,\dot{x}+\dot{x}^T\,\mathcal{M}^{-1}\,\mathcal{M}\,\dot{p}\\ &< \dot{x}^T\dot{p}=u_1^Ty_1.\end{aligned}$$

The last inequality is due to the fact that $\nabla_x g=\nabla_x F<0$ and $\mathcal{M}^{-1}>0$. This proves that the dynamically modified game is strictly passive. Suppose we have a set of strictly passive evolutionary dynamic of form (37) such as the extended BNN or Smith dynamics (Fox & Shamma, 2013). By Lemma 2.2, one can conclude that the negative feedback interconnection of this type of modified game and the corresponding evolutionary dynamic is strictly passive and for zero input, the origin is asymptotically stable. Moreover, if $g(x)$ is radially unbounded in x, the storage function L is also unbounded and the system will be globally asymptotically stable.

Proof of Theorem 5.1: Using (41) and (42) and the interconnections in Figure 7, Σ_3' is described by

$$\Sigma_3':\begin{cases}\Delta\dot{x}_3=-\left(\frac{k\alpha}{\hat{Q}}+k\beta k_2\right)\Delta x_3-k\beta k_1\Delta x_4-k\beta u_5,\\ \Delta\dot{x}_4=k_2\Delta x_3+u_5,\\ y_3=-\Delta x_3\end{cases},$$

hence is linear time-invariant. The transfer function of Σ_3' can be calculated to be

$$G(s)=\frac{k\beta\,(s+k_1)}{s^2+s\big(\frac{k\alpha}{\hat{Q}}+k\beta k_2\big)+k\beta k_1k_2}.$$

Clearly, since all of the parameters are positive, poles of Σ_3' are in the open left-half plane. Let $s=j\omega$, and

$$G(j\omega)=\frac{k\beta\,\big(j\omega+k_1\big)}{(j\omega)^2+j\omega\big(\frac{k\alpha}{\hat{Q}}+k\beta k_2\big)+k\beta k_1k_2}$$

It can be easily verified that $k_1<\frac{k\alpha}{\hat{Q}}+k\beta k_2$ implies that $Re[G(j\omega)]>0$, which means that $G(s)$ is strictly positive real and by Lemma 6.4 in Khalil (2002), Σ_3' is strictly passive.