The Relationship Between Recursive Multibody Dynamics and Discrete-Time Optimal Control

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Abstract—**A recursive algorithm, based on a Newton–Euler formulation, is developed for the solution of the simulation dynamics problem for a chain of rigid bodies. Arbitrary joint constraints are permitted, that is, joints may allow translational and/or rotational degrees of freedom. The recursive procedure is also shown to be identical to that encountered in a discrete-time optimal control problem. For each relevant quantity in the multibody dynamics problem, there exists an analog in the context of optimal control. The performance index that is minimized in the control problem is identified as Gibbs' function for the chain of bodies.**

I. Introduction

The motion of multibody systems has been a subject of longstanding concern in analytical dynamics. With the advent of robotic systems in terrestrial and space applications, its importance has transcended mere academic interest. The laws governing multibody dynamics have been known for centuries, but, perhaps, in no other field does there exist such a wide variety of approaches to the formulation and the solution of the equations of motion. The object of this paper is to present a recursive algorithm, founded on a Newton–Euler formulation, for the solution of the simulation dynamics problem for rigid-body chains. It is furthermore shown that this recursive procedure is identical to, and indeed derivable from, the recursive solution to the two-point boundary value problem associated with discrete-time optimal control.

In the field of multibody dynamics there are essentially two problems to consider: the *inverse dynamics* problem and the *simulation dynamics* problem. The former, on which a large body of literature already exists, is concerned with the solution of the control forces and torques necessary to achieve a desired trajectory. Simulation dynamics (also known as *forward dynamics*) deals with the converse, that is, solving for the motion of a multibody system given the control inputs. It is the latter that we shall address here.

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There exist myriad approaches to the formulation of the motion equations for multibody systems, each with its own peculiar features. Among the most popular approaches are the Lagrangian formulation [1], the Newton–Euler method [2], and the Gibbs–Appell–Kane equations [3]. (For the interested reader, Likins [4] has conducted a thorough study of several formulational alternatives for mechanical systems.) Although a sweeping statement on the superiority of any one approach would be folly, the Newton–Euler method is rather attractive from the standpoint of physical insight and computational efficiency. It is thus the Newton–Euler formulation that will form our basis.

The concept of recursion in multibody simulation dynamics was introduced by Armstrong [5]. Recursive methods mathematically exploit the topological nature of multibody systems. Unlike the usual "conventional" approach in which a system is considered in its entirety and the motion equations written accordingly, in a recursive procedure, the system is treated on a body-by-body basis. As a result, the computational effort required in a recursive algorithm grows linearly with the number of bodies. By contrast, "global" approaches, which require Gaussian elimination or the equivalent, typically obey a cubic relationship [6].

Armstrong devised a recursive procedure for an n-link rigid-body manipulator with spherical (three rotational degrees of freedom) joints, which could be modified to handle hinge (one rotational degree of freedom) joints as well. Featherstone [7] later proposed an alternative algorithm for rigid-body chains with hinge joints. The difference between Armstrong's and Featherstone's methods lies in the manner in which the constraint forces are eliminated.

Others have also investigated the application of recursive techniques to multibody dynamics. Golla [8] has developed an alternate recursive algorithm founded on Newton–Euler. Bae and Haug [9], [10] have forged a recursive procedure based on a variational approach and have considered closed kinematic loops as well. In addition, Book [1] has employed recursion in a Lagrangian formulation.

The present recursive algorithm may be regarded as a generalization of Featherstone's method in that the elimination of the constraint forces is done in a similar fashion. (Featherstone [11] has, in fact, also considered how his method can be extended to account for more general constraints at the joints.) As we shall see in due course, the algorithm can also be related directly to a comparable generalization of Armstrong's procedure [12].

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In its most general form, our method is applicable to topological trees of elastic bodies with arbitrary joint constraints. For explanatory purposes, though, only chains of rigid bodies are addressed in this paper. However, arbitrary translational and/or rotational interbody constraints, a key feature of the algorithm, are allowed; that is, each joint may permit at least one degree of freedom and at most six. But without loss in generality, we shall take the translational displacements to be small.

The paper, as its title implies, contains two main themes. One is the development of the recursive algorithm itself and the other is the relationship of the recursive algorithm to discrete-time optimal control. The latter was motivated by the work of Rodriquez [13], who recognized the correspondence between the equations of multibody dynamics and the those of optimal filtering (the Kalman filter) and smoothing (the Bryson–Frazier smoother). In this work, we demonstrate a one-to-one correspondence between the multibody simulation dynamics problem and the discrete-time optimal control problem—the dual of the estimation (smoothing) problem. The recursive feedback solution for the control in terms of the state is precisely that which yields the joint accelerations in terms of the (absolute) body accelerations. The analogy is moreover made complete by identifying the performance index, in the multibody analysis, as Gibbs' function [14].

II. Equations of Motion

Let us consider a chain of contiguous bodies $\mathscr{B}_0, \mathscr{B}_1, \cdots, \mathscr{B}_N$ as shown in Fig. 1. Interbody joints may permit arbitrary relative (rotational and/or translational) motion. Each joint, therefore, possesses at least one degree of freedom and at most six. For convenience, we shall assume interbody translations to be small; however, the extension to large translations can be incorporated into the present formulation. For additional details on the derivation of the equations of motion, the reader should consult Sincarsin and Hughes [15].

The motion of $\mathscr{B}_n$ is defined by the velocity $\mathbf{v}_n$ of $\mathcal{O}_n$ and the angular velocity $\boldsymbol{\omega}_n$ of $\mathscr{B}_n$. (See Fig. 2.) Both $\mathbf{v}_n$ and $\boldsymbol{\omega}_n$ are measured with respect to inertial space but are expressed in $\mathscr{F}_n$, a reference frame attached to $\mathscr{B}_n$. We shall define

$$\boldsymbol{v}_n \triangleq \begin{bmatrix} \mathbf{v}_n \\ \boldsymbol{\omega}_n \end{bmatrix} \tag{1}$$

as the *generalized velocity* (cf. twist velocity) of $\mathscr{B}_n$ at $\mathcal{O}_n$. We furthermore introduce the accompanying definition for a *generalized force* (cf. wrench) acting at $\mathcal{O}_n$:

$$\boldsymbol{f}_n^{n-1} \triangleq \begin{bmatrix} \mathbf{f}_n^{n-1} \\ \mathbf{g}_n^{n-1} \end{bmatrix} \tag{2}$$

where $\mathbf{f}_n^{n-1}$ and $\mathbf{g}_n^{n-1}$ are the reaction forces and torques on $\mathscr{B}_n$ due to $\mathscr{B}_{n-1}$ as expressed in $\mathscr{F}_n$.

The resulting equation of motion for $\mathscr{B}_n$ can be written as

$$\mathscr{M}_n \dot{\boldsymbol{v}}_n = \boldsymbol{f}_{nT} + \boldsymbol{f}_{nI} \tag{3}$$

where

$$\mathscr{M}_n \triangleq \begin{bmatrix} m_n \mathbf{1} & -\mathbf{c}_n^{\times} \\ \mathbf{c}_n^{\times} & \mathbf{J}_n \end{bmatrix}$$

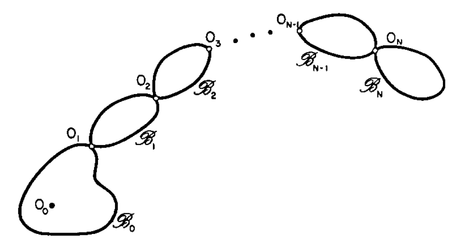

Fig. 1. A chain of rigid bodies.

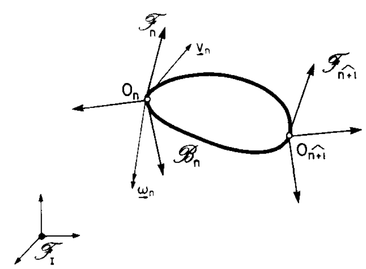

Fig. 2. Body $\mathscr{B}_n$.

is the (constant) mass matrix corresponding to $\mathscr{B}_n$, that is, m_n, $\mathbf{c}_n$ and $\mathbf{J}_n$ are the zeroeth (mass), first and second moments of inertia (about $\mathcal{O}_n$) of $\mathscr{B}_n$. Also, $\boldsymbol{f}_{nT}$ is the total external (generalized) force acting on $\mathscr{B}_n$, including interbody forces, and $\boldsymbol{f}_{nI}$, which accounts for the nonlinear inertial terms (owing to centrifugal and Coriolis effects), can be neatly written as

$$\boldsymbol{f}_{nI} = (\boldsymbol{v}_n^{\times})^T \mathscr{M}_n \boldsymbol{v}_n \tag{4}$$

where

$$\boldsymbol{v}_n^{\times} \triangleq \begin{bmatrix} \boldsymbol{\omega}_n^{\times} & \mathbf{v}_n^{\times} \\ \cdot & \boldsymbol{\omega}_n^{\times} \end{bmatrix}$$

and $(\cdot)^{\times}$ operating on a Cartesian (3×1) column matrix, such as $\mathbf{v}_n$, $\boldsymbol{\omega}_n$ or $\mathbf{c}_n$, is the matrix equivalent of the vector cross product. In a rate-linear model, one would set $\boldsymbol{f}_{nI} \equiv \mathbf{0}$.

A. Interbody Constraints

The set of equations (3) does not yet describe a chain of bodies since it does not take into consideration the interbody constraint imposed by the joints. To do so, we begin by observing that

$$\boldsymbol{v}_n = \mathscr{T}_{n,n-1} \boldsymbol{v}_{n-1} + \boldsymbol{v}_{n,\mathrm{int}} \tag{5}$$

which introduces the *relative* interbody generalized velocity $\boldsymbol{v}_{n,\mathrm{int}}$ of $\mathscr{B}_n$ with respect to $\mathscr{B}_{n-1}$. In addition

$$\mathscr{T}_{n,n-1} = \begin{bmatrix} \mathbf{C}_{n,n-1} & -\mathbf{C}_{n,n-1}\mathbf{r}_{n-1}^{n\,\times} \\ \cdot & \mathbf{C}_{n,n-1} \end{bmatrix}$$

is the *generalized transformation matrix* between $\mathscr{B}_{n-1}$ and $\mathscr{B}_n$; $\mathbf{C}_{n,n-1}$ is the rotation matrix from $\mathscr{F}_{n-1}$ to $\mathscr{F}_n$ and $\mathbf{r}_{n-1}^n$ is the position of $\mathcal{O}_n$ with respect to $\mathcal{O}_{n-1}$. The geometric constraints imposed by the joints can thus be expressed formally as

$$\boldsymbol{v}_{n,\mathrm{int}} = \mathscr{P}_n \boldsymbol{v}_{n\gamma} \tag{6}$$

where $\mathscr{P}_n$ is a projection matrix and $v_{n\gamma}$ is the column of free joint (rate) variables. The absolute velocities v_n can be obtained *recursively* from v_{n-1} and $v_{n\gamma}$.

We also note that

$$f_{nT} = \mathscr{T}^T_{n+1,n} f^n_{n+1} - f^{n-1}_n + f_{n,\text{ext}} \tag{7}$$

where $f_{n,\text{ext}}$ is due solely to external influences. Furthermore, the generalized interbody forces f^{n-1}_n can be expressed as a sum of control forces $f_{n,c}$ and constraint forces $f_{n,\square}$, i.e.,

$$f^{n-1}_n = -\mathscr{P}_n f_{n,c} - \mathscr{Q}_n f_{n,\square}. \tag{8}$$

The projection matrix $\mathscr{Q}_n$ is the complement of $\mathscr{P}_n$.

As stated at the outset, interbody translations have been assumed small. This is tantamount to assuming that the outboard origin $\mathscr{O}'_n$ on $\mathscr{B}_{n-1}$ is coincident with the inboard origin $\mathscr{O}_n$ on $\mathscr{B}_n$. However, the extension to large interbody translations is inherent in the present analysis since it can be shown [12] that the form of all the foregoing equations remains intact when the assumption is relaxed.

B. Projection Matrices

A few words are perhaps in order regarding the projection matrices. First, as a simple yet very important example, consider a joint with a single rotational degree of freedom about, say, the third axis of an appropriately chosen reference frame. The corresponding projection matrix $\mathscr{P}_n$ is

$$\mathscr{P}_n = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T.$$

We may also add that $v_{n\gamma} = \dot{\gamma}_3$, where γ_3 is the angle of rotation.

In general, $\mathscr{P}_n$ is not constant, as above, but rather is dependent on configuration. Contemplation of a universal joint will quickly reveal this fact. The columns of $\mathscr{P}_n$ are in general not orthonormal but

$$\mathscr{P}^T_n \mathscr{P}_n = \mathscr{I}_n \tag{9}$$

where $\mathscr{I}_n$ is nonsingular. The complementary projection matrix $\mathscr{Q}_n$ satisfies

$$\mathscr{P}^T_n \mathscr{Q}_n = \mathbf{O}. \tag{10}$$

Without loss in generality, the columns of $\mathscr{Q}_n$ can be taken as orthonormal.

C. Kinematical Equations

The kinematical equations accompanying the dynamical equations (3) can be summarized in terms of $\mathscr{T}_{n,n-1}$:

$$\dot{\mathscr{T}}_{n,n-1} = -v^\times_{n,\text{int}} \mathscr{T}_{n,n-1} \tag{11}$$

If we express $v_{n,\text{int}}$ as

$$v_{n,\text{int}} \triangleq \begin{bmatrix} \mathbf{v}_{n,\text{int}} \\ \omega_{n,\text{int}} \end{bmatrix} \tag{12}$$

we can extract from (11)

$$\dot{\mathbf{C}}_{n,n-1} = -\omega^\times_{n,\text{int}} \mathbf{C}_{n,n-1}. \tag{13}$$

For physical reasons, Euler angles make for the most convenient and expedient representation of rotational joint degrees of freedom. Interbody translation is given by the integration of $v_{n,\text{int}}$ and would be reflected in $\mathbf{r}^n_{n-1}$.

III. Simulation Dynamics

The recursive method presented here is a generalization of Featherstone's method applicable to rigid multibody chains with arbitrary interbody constraints. The development, in fact, runs parallel to a similar generalization of Armstrong's recursive method [12]. The essential difference is that the former is based on an affine relationship of the total interbody force of the absolute (generalized) body acceleration while the latter relates explicitly only the interbody constraint force. The generalized Featherstone approach is particularly appealing because of its direct analogy to the discrete-time optimal problem. As shall be demonstrated, however, a simple equivalence exists between the two schemes.

Let[1]

$$\dot{v}_n = a_n + a_{n,\text{non}} \tag{14}$$

such that

$$a_n = \mathscr{T}_{n,n-1} a_{n-1} + \mathscr{P}_n \dot{v}_{n\gamma}. \tag{15}$$

Differentiating (5) and inserting (14) reveals that we must have

$$a_{n,\text{non}} = \mathscr{T}_{n,n-1} a_{n-1,\text{non}} + \dot{\mathscr{T}}_{n,n-1} v_{n-1} + \dot{\mathscr{P}}_n v_{n\gamma} \tag{16}$$

for (15) to hold. In essence, the acceleration quantities a_n account for the rate-linear effects and $a_{n,\text{non}}$ for the nonlinear effects. Moreover, not only is a_n found recursively (outward) but $a_{n,\text{non}}$ as well.

Upon substitution of (14) into the motion equation (3), we have

$$\mathscr{M}_n a_n = f_{nT} + f_{nI} + f_{n,\text{non}} \tag{17}$$

where

$$f_{n,\text{non}} \triangleq -\mathscr{M}_n a_{n,\text{non}}.$$

In fact, we can write (17) as

$$\mathscr{M}_n a_n = \mathscr{T}^T_{n+1,n} f^n_{n+1} - f^{n-1}_n + f_{n,\text{net}} \tag{18}$$

where

$$f_{n,\text{net}} \triangleq f_{n,\text{ext}} + f_{nI} + f_{n,\text{non}}.$$

We now proceed to eliminate the interbody forces.

A. Recursion for f^{n-1}_n

We conjecture that the interbody forces f^{n-1}_n can be written as

$$-f^{n-1}_n = \mathbf{\Psi}_n a_n + \psi_n \tag{19}$$

which is a generalization of Featherstone's hypothesis. Note that $\mathbf{\Psi}_n$ is, in effect, a mass matrix and ψ_n is a generalized force quantity. The recursive algorithm is based on this result and the fact that $\mathbf{\Psi}_n$ and ψ_n can be determined recursively from $\mathscr{B}_N$ to $\mathscr{B}_0$.

[1] The authors are indebted to Dr. D. F. Golla for this insight.

The proof of (19) is by induction:

Step I: For $\mathscr{B}_N$, (18) becomes

$$\mathscr{M}_N a_N = -f_N^{N-1} + f_{N,\text{net}} \tag{20}$$

where it has been observed that

$$f_{N+1}^N \equiv \mathbf{0}$$

since $\mathscr{B}_N$ is the (free) terminal body. It is immediately obvious that if we set

$$\Psi_N = \mathscr{M}_N$$
$$\psi_N = -f_{N,\text{net}} \tag{21}$$

(19) is satisfied for $n = N$.

Step II: We assume that

$$-f_{n+1}^n = \Psi_{n+1} a_{n+1} + \psi_{n+1}. \tag{22}$$

Step III: Given (22), we shall show that (19) follows. Now

$$v_{n+1} = \mathscr{T}_{n+1,n} v_n + \mathscr{P}_{n+1} v_{n+1,\gamma} \tag{23}$$

and

$$a_{n+1} = \mathscr{T}_{n+1,n} a_n + \mathscr{P}_{n+1} \dot{v}_{n+1,\gamma}. \tag{24}$$

Substituting (24) and (8) in (22) and premultiplying by $\mathscr{P}_{n+1}^T$ gives

$$\mathscr{I}_{n+1} f_{n+1,c} = \Psi_{n+1,PP} \dot{v}_{n+1,\gamma} + \mathscr{P}_{n+1}^T \Psi_{n+1} \mathscr{T}_{n+1,n} a_n + \psi_{n+1,P} \tag{25}$$

where, in general,

$$\Psi_{nPP} \triangleq \mathscr{P}_n^T \Psi_n \mathscr{P}_n, \quad \psi_{nP} \triangleq \mathscr{P}_n^T \psi_n.$$

Solving for $\dot{v}_{n+1,\gamma}$ from (25), inserting back into (24), and using the result with (22) in (18) eventually leads to

$$\begin{aligned} -f_n^{n-1} = {} & \{\mathscr{M}_n + \mathscr{T}_{n+1,n}^T (\Psi_{n+1} \\ & - \Psi_{n+1} \mathscr{P}_{n+1} \Psi_{n+1,PP}^{-1} \mathscr{P}_{n+1}^T \Psi_{n+1}) \mathscr{T}_{n+1,n}\} a_n \\ & + \{\mathscr{T}_{n+1,n}^T [\Psi_{n+1} \mathscr{P}_{n+1} \Psi_{n+1,PP}^{-1} (\mathscr{I}_{n+1} f_{n+1,c} \\ & - \psi_{n+1,P}) + \psi_{n+1}] - f_{n,\text{net}}\}. \end{aligned} \tag{26}$$

Hence, we can identify

$$\begin{aligned} \Psi_n = {} & \mathscr{M}_n + \mathscr{T}_{n+1,n}^T (\Psi_{n+1} \\ & - \Psi_{n+1} \mathscr{P}_{n+1} \Psi_{n+1,PP}^{-1} \mathscr{P}_{n+1}^T \Psi_{n+1}) \mathscr{T}_{n+1,n} \\ \psi_n = {} & \mathscr{T}_{n+1,n}^T [\Psi_{n+1} \mathscr{P}_{n+1} \Psi_{n+1,PP}^{-1} (\mathscr{I}_{n+1} f_{n+1,c} \\ & - \psi_{n+1,P}) + \psi_{n+1}] - f_{n,\text{net}} \end{aligned} \tag{27}$$

Step IV: By induction, then, (19) is proven. ■

The matrix Ψ_n has an attractive physical interpretation. It is the mass matrix (about $\mathscr{O}_n$) of the part of the chain from $\mathscr{B}_n$ to $\mathscr{B}_N$ associated with the constrained degrees of freedom. Featherstone [7] would refer to Ψ_n as the *articulated-body inertia*. It should also be pointed out that Ψ_n, which is positive-definite, and ψ_n are configuration dependent.

B. Recursion for $\dot{v}_{n\gamma}$

By the inductive nature of the proof for (19), it has been shown that the matrices Ψ_n and ψ_n can be evaluated recursively inward, i.e., from $\mathscr{B}_N$ to $\mathscr{B}_0$. Having done so, one can then perform outward recursion, from $\mathscr{B}_0$ to $\mathscr{B}_N$, to solve for $\dot{v}_{n\gamma}$. This is evident from (25).

Rewriting (25) for $\mathscr{B}_n$ instead of $\mathscr{B}_{n+1}$ and solving explicitly for $\dot{v}_{n\gamma}$ yields

$$\dot{v}_{n\gamma} = \Psi_{nPP}^{-1} (\mathscr{I}_n f_{n,c} - \mathscr{P}_n^T \Psi_n \mathscr{T}_{n,n-1} a_{n-1} - \psi_{nP}). \tag{28}$$

Examining this result, we see that at $\mathscr{B}_n$ all the quantities on the right-hand side are known since a_{n-1} can be computed recursively from its inboard neighbor according to (24).

C. Relationship to Armstrong's Work

Before proceeding onward, it is worth pointing out that

$$\Psi_n - \Psi_n \mathscr{P}_n \Psi_{nPP}^{-1} \mathscr{P}_n^T \Psi_n = \mathscr{Q}_n \Phi_n \mathscr{Q}^T \tag{29}$$

where

$$\Phi_n = \Psi_{nQQ} - \Psi_{nPQ}^T \Psi_{nPP}^{-1} \Psi_{nPQ}$$

and

$$\Psi_{nPQ} \triangleq \mathscr{P}_n^T \Psi_n \mathscr{Q}_n$$
$$\Psi_{nQQ} \triangleq \mathscr{Q}_n^T \Psi_n \mathscr{Q}_n.$$

Showing (29) requires invoking the identity

$$\mathscr{P}_n \mathscr{I}_n^{-1} \mathscr{P}_n^T + \mathscr{Q}_n \mathscr{Q}_n^T = \mathbf{1}.$$

By virtue of (29), we can rewrite the first part of (27) as

$$\Psi_n = \mathscr{M}_n + \mathscr{T}_{n+1,}^T \mathscr{Q}_{n+1} \Phi_{n+1} \mathscr{Q}_{n+1}^T \mathscr{T}_{n+1,n} \tag{30}$$

which is a more streamlined expression.

The significance of Φ_n, however, lies in the fact that

$$f_{n,\square} = \Phi_n \mathscr{Q}_n^T a_n + \phi_n \tag{31}$$

where

$$\phi_n = \mathscr{Q}_n^T \psi_n + \Psi_{nPQ}^T \Psi_{nPP}^{-1} (\mathscr{I}_n f_{n,c} - \psi_{nP}).$$

This result is equivalent to the generalized version of Armstrong's method for rigid multibody chains with arbitrary joint constraints [12].

IV. A Discrete-Time Optimal Control Problem

Diverting our attention from multibody dynamics momentarily, let us consider the following optimal control problem: Minimize

$$\mathscr{J} = \sum_{k=0}^{N} \frac{1}{2} \mathbf{x}_k^T \mathbf{M}_k \mathbf{x}_k + \mathbf{x}_k^T \mathbf{h}_k - \mathbf{u}_{k-1}^T \mathbf{t}_{k-1} \tag{32}$$

subject to the linear state equation

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k, \quad \mathbf{x}_{-1} = \mathbf{0}. \tag{33}$$

Here, $\mathbf{M}_k$ is a sequence of positive-definite weighting matrices, and $\mathbf{h}_k$ and $\mathbf{t}_k$ are vector weighting sequences. Since $\mathbf{u}_N$ does not influence $\mathbf{x}_k$, $k \leq N$, we shall assume that $\mathbf{t}_N = \mathbf{0}$.

This problem is slightly different than the standard "linear-quadratic" version that one typically encounters. The cost functional in the present case is linear in the control variable.

Minimizing $\mathcal{J}$ subject to the state equation is a straightforward optimization problem. Introducing the Lagrange multiplier or adjoint variable $\boldsymbol{\lambda}_k$, we define the augmented performance index as follows:

$$\mathcal{J}' \triangleq \sum_{k=0}^{N} \frac{1}{2}\mathbf{x}_k^T\mathbf{M}_k\mathbf{x}_k + \mathbf{x}_k^T\mathbf{h}_k - \mathbf{u}_{k-1}^T\mathbf{t}_{k-1} + \boldsymbol{\lambda}_k^T(\mathbf{x}_k - \mathbf{A}_{k-1}\mathbf{x}_{k-1} - \mathbf{B}_{k-1}\mathbf{u}_{k-1}). \quad (34)$$

The necessary conditions for optimality

$$\frac{\partial \mathcal{J}'}{\partial \boldsymbol{\lambda}_{k+1}} = \frac{\partial \mathcal{J}'}{\partial \mathbf{x}_k} = \frac{\partial \mathcal{J}'}{\partial \mathbf{u}_k} = \mathbf{0}$$

produce the two-point boundary value problem (TPBVP)

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k, \quad \mathbf{x}_{-1} = \mathbf{0} \quad (35)$$

$$\boldsymbol{\lambda}_k = \mathbf{A}_k^T\boldsymbol{\lambda}_{k+1} - \mathbf{M}_k\mathbf{x}_k - \mathbf{h}_k, \quad \boldsymbol{\lambda}_{N+1} = \mathbf{0} \quad (36)$$

$$\mathbf{t}_k = -\mathbf{B}_k^T\boldsymbol{\lambda}_{k+1}. \quad (37)$$

We have taken $\boldsymbol{\lambda}_{N+1} = \mathbf{0}$, without loss in generality, since $\mathbf{t}_N = \mathbf{0}$. Hence, from (36), $\boldsymbol{\lambda}_N = -\mathbf{M}_N\mathbf{x}_N - \mathbf{h}_N$, which supplies the basis for the inhomogeneous Riccati transformation, sometimes called the sweep method

$$\boldsymbol{\lambda}_k = -\mathbf{S}_k\mathbf{x}_k - \mathbf{r}_k \quad (38)$$

with $\mathbf{S}_N = \mathbf{M}_N$ and $\mathbf{r}_N = \mathbf{h}_N$. Substituting (38) into the equation for $\mathbf{t}_k$ (37) and replacing $\mathbf{x}_{k+1}$ with the right side of (35) produces the feedback law

$$\mathbf{u}_k = -\mathbf{K}_k\mathbf{x}_k + \mathbf{R}_k^{-1}(\mathbf{t}_k - \mathbf{B}_k^T\mathbf{r}_{k+1}) \quad (39)$$

where

$$\mathbf{R}_k \triangleq \mathbf{B}_k^T\mathbf{S}_{k+1}\mathbf{B}_k$$

$$\mathbf{K}_k \triangleq \mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1}\mathbf{A}_k.$$

The matrix $\mathbf{R}_k$ will be invertible if $\mathbf{B}_k$ is monic and $\mathbf{S}_{k+1}$ is positive-definite. Substituting the sweep solution (38) for $\boldsymbol{\lambda}_k$ and $\boldsymbol{\lambda}_{k+1}$ and using (35) for $\mathbf{x}_{k+1}$ and (39) for $\mathbf{u}_k$ gives

$$[\mathbf{S}_k - \mathbf{A}_k^T(\mathbf{S}_{k+1} - \mathbf{S}_{k+1}\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1})\mathbf{A}_k - \mathbf{M}_k]\mathbf{x}_k = -\mathbf{r}_k + (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{r}_{k+1} + \mathbf{K}_k^T\mathbf{t}_k + \mathbf{h}_k.$$

Since this must hold for general $\mathbf{x}_k$, the coefficient of $\mathbf{x}_k$ must vanish as well as the right-hand side. Hence

$$\mathbf{S}_k = \mathbf{A}_k^T(\mathbf{S}_{k+1} - \mathbf{S}_{k+1}\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1})\mathbf{A}_k + \mathbf{M}_k \quad (40)$$

which is the discrete-time matrix Riccati equation and

$$\mathbf{r}_k = (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{r}_{k+1} + \mathbf{K}_k^T\mathbf{t}_k + \mathbf{h}_k. \quad (41)$$

We now return to the question of the invertibility of $\mathbf{R}_k$. The definitions of $\mathbf{K}_k$ and $\mathbf{R}_k$ reveal that $(\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{S}_{k+1}\mathbf{B}_k = \mathbf{O}$, which allows us to write the Riccati equation as

$$\mathbf{S}_k = (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{S}_{k+1}(\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k) + \mathbf{M}_k. \quad (42)$$

Since $\mathbf{S}_N = \mathbf{M}_N$ is symmetric and positive-definite, $\mathbf{S}_k$ is symmetric and positive-definite (using backward induction). Hence, $\mathbf{R}_k$ defined previously is positive-definite and is always invertible.

The optimal control policy can now be summarized as follows: One solves the Riccati equation (40) (or (42)) and the vector equation (41) backward from $k = N$ to $k = 0$ using the boundary conditions $\mathbf{S}_N = \mathbf{M}_N$ and $\mathbf{r}_N = \mathbf{h}_N$. The optimal control can then be calculated using (39) while propagating the state forward using the state equation (35).

V. Relation Between Optimal Control and Recursive Dynamics

The TPBVP generated by the previous optimal control problem (35)–(37) is identical in form to that of the multibody dynamics problem (18), (24), and

$$\mathcal{I}_n \boldsymbol{f}_{n,c} = -\mathcal{P}_n^T \boldsymbol{f}_n^{n-1} \quad (43)$$

which follows from premultiplying (8) by $\mathcal{P}_n^T$ while recognizing (9) and (10). Therefore, we make the following identifications:

$$\begin{array}{ll} \mathbf{x}_k \leftrightarrow \boldsymbol{a}_n & \boldsymbol{\lambda}_k \leftrightarrow \boldsymbol{f}_n^{n-1} \\ \mathbf{u}_k \leftrightarrow \dot{\boldsymbol{v}}_{n+1,\gamma} & \mathbf{h}_k \leftrightarrow -\boldsymbol{f}_{n,\mathrm{net}} \\ \mathbf{A}_k \leftrightarrow \mathcal{T}_{n+1,n} & \mathbf{M}_k \leftrightarrow \mathcal{M}_n \\ \mathbf{B}_k \leftrightarrow \mathcal{P}_{n+1} & \mathbf{t}_k \leftrightarrow \mathcal{I}_{n+1}\boldsymbol{f}_{n+1,c}. \end{array}$$

Hence, the accelerations $\boldsymbol{a}_n$ are analogous to the states, the interbody forces $\boldsymbol{f}_n^{n-1}$ are analogous to the adjoint states, the joint accelerations $\dot{\boldsymbol{v}}_{n\gamma}$ play the role of the control inputs, and the projection matrices $\mathcal{P}_{n+1}$ take the place of the input matrix $\mathbf{B}_k$. It can be shown that the interbody transformation matrices $\mathcal{T}_{n+1,n}$ possess the properties of the state transition matrix thus completing the analogy. Comparing the transformation (38) with the generalization of Featherstone's solution (19) allows us to identify

$$\mathbf{S}_k \leftrightarrow \boldsymbol{\Psi}_n$$

$$\mathbf{r}_k \leftrightarrow \boldsymbol{\psi}_n$$

$$\mathbf{R}_k \leftrightarrow \boldsymbol{\Psi}_{n+1,PP}.$$

We also emphasize the recursion in time (k) has been replaced by spatial recursion (n) at a given instant in time.

Using the above identifications, the performance index $\mathcal{J}$ can be written as

$$\mathcal{J} = \sum_{n=0}^{N} \frac{1}{2}\boldsymbol{a}_n^T\mathcal{M}_n\boldsymbol{a}_n - \boldsymbol{f}_{n,\mathrm{net}}^T\boldsymbol{a}_n - \boldsymbol{f}_{n,c}^T\mathcal{I}_n\dot{\boldsymbol{v}}_{n\gamma}$$

Hence, in the multibody dynamics problem one can minimize $\mathcal{J}$ subject to the kinematical constraint equation (15) to arrive at the defining equations. Compare this with Gibbs' formulation [14] of the dynamics of a system of N particles with masses m_n, coordinates x_n, y_n, z_n, are subjected to forces X_n, Y_n, Z_n: Minimize

$$\sum_{n=1}^{N} \frac{1}{2}m_n(\ddot{x}_n^2 + \ddot{y}_n + \ddot{z}_n^2) - X_n\ddot{x}_n - Y_n\ddot{y}_n - Z_n\ddot{z}_n$$

subject to the kinematical constraints. The minimization of a quadratic function of acceleration to obtain the equations of motion for a rigid multibody chain has been considered by Vukobratović and Potkonjak [16]. These authors refer to the method as Gauss' principle of least compulsion and attribute its use in manipulator dynamics to the Russian literature [17].

In his work, Rodriguez [13] points out the similarity between the equations describing a chain of hinged bodies and the TPBVP that arises in discrete-time, optimal estimation, and smoothing problems. In his formulation, the bodies in the chain are numbered inwardly (i.e., the tip body is $\mathscr{B}_0$ and the root body is $\mathscr{B}_N$). Here, the numbering of the bodies is outward (the root body is $\mathscr{B}_0$ and the tip body is $\mathscr{B}_N$). With this convention, the equations are rendered *dual* to those of Rodriguez. As such, the corresponding discrete-time problem is not one of estimation and smoothing but one of control. It is interesting to note the dual relationships inherent in Rodriguez's work. The role of the state is played by the interbody forces and the adjoint states are the link accelerations, which are an interchange of the results given above. The control torque at each joint plays the role of a measurement of the states whereas we have the joint accelerations acting as "control inputs."

VI. Summary of the Recursive Algorithm

We now summarize the procedures for determining the motion of the chain of bodies. The control forces $f_{n,c}(t)$ and external force distribution $f_{n,\text{ext}}(t)$ are prescribed on the time interval of interest. Beginning with $t = 0$, we proceed as follows:

Step 1: At time t, the relative velocities $v_{n\gamma}(t)$ and the rotation matrices $\mathbf{C}_{n,n-1}(t)$ are known.

Step 2: Outward recursion for the velocities v_n and determination of $f_{n,\text{net}}$:

Do $n = 0$ to N;

Generate $\mathscr{T}_{n,n-1}$ using $\mathbf{C}_{n,n-1}$.

$v_{n,\text{int}} = \mathscr{P}_n v_{n\gamma}$.

$\dot{\mathscr{T}}_{n,n-1} = -v_{n,\text{int}}^{\times}\mathscr{T}_{n,n-1}$.

$v_n = \mathscr{T}_{n,n-1}v_{n-1} + v_{n,\text{int}}$.

$a_{n,\text{non}} = \mathscr{T}_{n,n-1}\, a_{n-1,\text{non}} + \dot{\mathscr{T}}_{n,n-1}\, v_{n+1} + \dot{\mathscr{P}}_n v_{n\gamma}$.

$f_{nI} = (v_n^{\times})^T \mathscr{M}_n v_n,\ f_{n,\text{non}} = -\mathscr{M}_n a_{n,\text{non}}$.

$f_{n,\text{net}} = f_{n,\text{ext}} + f_{nI} + f_{n,\text{non}}$.

Next n.

Step 3: Inward recursion for $\mathbf{\Psi}_n$ and ψ_n:

Set $\mathbf{\Psi}_N = \mathscr{M}_N$ and $\psi_N = -f_{N,\text{net}}$.

Do $n = N - 1$ to 0;

$\mathbf{\Psi}_{n+1,PP} = \mathscr{P}_{n+1}^T\ \mathbf{\Psi}_{n+1}\mathscr{P}_{n+1}$,

$\psi_{n+1,P} = \mathscr{P}_{n+1}^T\psi_{n+1}$.

$\mathbf{K}_n = \mathbf{\Psi}_{n+1,PP}^{-1}\ \mathscr{P}_{n+1}^T\ \mathbf{\Psi}_{n+1}\mathscr{T}_{n+1,n}$,

$\mathbf{\Gamma}_{n+1,n} = \mathscr{T}_{n+1,n} - \mathscr{P}_{n+1}\mathbf{K}_n$.

$\mathbf{\Psi}_n = \mathbf{\Gamma}_{n+1,n}^T\mathbf{\Psi}_{n+1}\mathbf{\Gamma}_{n+1,n} + \mathscr{M}_n$.

$\psi_n = \mathbf{\Gamma}_{n+1,n}^T\psi_{n+1} + \mathbf{K}_n^T\mathscr{I}_{n+1}f_{n+1,c} - f_{n,\text{net}}$.

Next n.

If $\mathscr{P}_0 \neq \mathbf{O}$ ($\mathbf{B}_0$ is at least partially unconstrained), then

$\mathbf{\Psi}_{0PP} = \mathscr{P}_0^T\mathbf{\Psi}_0\mathscr{P}_0,\ \psi_{0P} = \mathscr{P}_0^T\psi_0$;

Otherwise, continue on to Step 4.

Step 4: Outward recursion for $\dot{v}_{n\gamma}$:

If $\boldsymbol{P}_0 = \mathbf{O}$ ($\mathscr{B}_0$ is constrained), then

$\dot{v}_{0\gamma} = a_0 = \mathbf{0}$;

Otherwise, $\dot{v}_{0\gamma} = \mathbf{\Psi}_{0PP}^{-1}(\mathscr{I}_0 f_{0,c} - \psi_{0P})$,

$a_0 = \mathscr{P}_0\dot{v}_{0\gamma}$.

Do $n = 1$ to N;

$\dot{v}_{n\gamma} = -\mathbf{K}_{n-1}a_{n-1} + \mathbf{\Psi}_{n,PP}^{-1}(\mathscr{I}_n f_{n,c} - \psi_{nP})$.

$\dot{\mathbf{C}}_{n,n-1} = -\omega_{n,\text{int}}^{\times}\mathbf{C}_{n,n-1}$.

$a_n = \mathscr{T}_{n,n-1}a_{n-1} + \mathscr{P}_n\dot{v}_{n\gamma}$.

Next n.

Step 5: Estimate $v_{n\gamma}(t + \Delta t)$, $\mathbf{C}_{n,n-1}(t + \Delta t)$ using some quadrature scheme. Go back to Step 1 and replace t with $t + \Delta t$.

This completes the summary of the recursive simulation procedure. Note that in a rate-linear simulation, one ignores the contributions of f_{nI} and $f_{n,\text{non}}$ to $f_{n,\text{net}}$ in Step 2. We have written the recursion for $\mathbf{\Psi}_n$ and ψ_n, in Step 3, in terms of the quantities $\mathbf{K}_n$ and $\mathbf{\Gamma}_{n+1,n}$ since this leads to the most compact and efficient expressions. The fourth step produces the joint accelerations $\dot{v}_{n\gamma}$, which can be integrated in conjunction with the kinematical relationships for the rotation matrices to produce the joint orientations/positions and velocities.

VII. Concluding Remarks

We have presented herein a recursive simulation dynamics procedure, based on a Newton–Euler formulation, which is applicable to chains of rigid bodies with arbitrary translational and/or rotational interbody constraints. Furthermore, it has been demonstrated that this procedure is derivable from the recursive scheme used in discrete-time optimal control. The underlying analogy that makes this possible yields great insight into the structure of the multibody dynamics problem.

Although the present analysis was restricted to chains of rigid bodies, the extension to topological rigid-body trees is reasonably straightforward. Also, the subject of structural flexibility in multibody systems is very current. It can be shown that the structure of the resulting motion equations can be left unaltered by elasticity [12]. Indeed, there exists a one-to-one correspondence between the rigid and the flexible case. With this duality in hand, we have been able to extend the present algorithm to the problem of elastic multibody chains [18].

Recursive methods are attractive because of their computational efficiency, yet a comparison of Featherstone's recursive method to Walker and Orin's most efficient "global" method (for rigid-link chains with hinge joints) indicates that recursion does not become computationally favorable until the length of the chain reaches 12 links [7]. While this result may be somewhat disappointing to the practically minded, it should be pointed out that designs for Space Station manipulators include systems that may have as many as 21 links. But more important, recursive methods become increasingly more attractive as the number of joint and elastic degrees of freedom increase. A recent study [19] investigated the performance of two recursive methods and one global method having identical capabilities. It shows that the breakeven point for the recursive schemes occurs at about four bodies when each model with three elastic degrees of freedom. Recursive simulation procedures, therefore, seem exceptionally well suited for elastic-body systems.

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