Synthesis of Optimal Finite-Frequency Controllers Able to Accommodate Passivity Violations

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***Abstract*— In this paper, we explore the relationship between the hybrid passivity/finite-gain systems framework and the generalized Kalman–Yakubovich–Popov (GKYP) lemma. In particular, we investigate how to optimally design finite-frequency (FF) controllers that possess strictly positive real (SPR) properties over a low-frequency range and bounded real (BR) properties over a high-frequency range. Such FF SPR/BR controllers will be used to control systems that have experienced a passivity violation. We first review the hybrid passive/finite-gain systems framework and how linear time-invariant hybrid passive/finite-gain systems relate to systems with low-frequency FF positive real (PR) or SPR properties, and high-frequency FF BR properties as characterized by the GKYP lemma. Optimal design of FF SPR/BR controllers is considered next. A convex optimization problem constrained by a set of linear matrix inequalities is posed where constraints are imposed using various forms of the GKYP lemma, yielding optimal FF SPR/BR controllers. The FF SPR/BR controllers are optimal in that they approximate the traditional $\mathcal{H}_2$ control solution. Finally, FF SPR/BR controllers are used within a gain-scheduling architecture to control a two-link flexible manipulator. Experimental results successfully demonstrate closed-loop stability and good closed-loop performance.**

***Index Terms*— Finite frequency controllers, linear matrix inequality (LMI) controller synthesis, passivity violations, two-link manipulator, vibration control.**

I. Introduction

THE PASSIVITY theorem states that a passive plant can be stabilized by a very strictly passive (VSP) compensator [1], [2]. Passivity-based control has been successfully used to control various plants including resistor–inductor–capacitor circuits, rigid and flexible robots, and spacecraft in linear, nonlinear, and adaptive contexts [3]–[9]. Passivity-based control is robust when perturbations do not destroy the passive nature of the plant. For example, flexible robotic manipulators that have colocated (rate) sensors and actuators are robust to mass and stiffness modeling errors because such errors do not destroy the passive nature of the structure being controlled.

In practice, plant inputs and outputs are applied by actuators and measured by sensors that are dynamic. Additionally, filters are often used to filter high-frequency signal noise or to estimate unmeasured parameters. Actuators, sensors, and filters that have unity gain and zero phase lag over all frequencies do not exist. Such dynamics unfortunately do destroy the passive input–output (IO) map of the plant being controlled; passivity is violated, and the traditional passivity theorem cannot be used to guarantee closed-loop stability. In a linear sense, passivity violations generally occur at high frequencies where actuator, sensor, and filter dynamics induce phase lag and have gain that rolls off. For instance, a passive plant augmented with sensor dynamics tends to maintain a passive IO map at low frequency, and, although the IO map at high frequency is not passive, its gain is finite because the gain of both the plant and sensor dynamics naturally subsides.

Various authors have independently investigated describing and controlling systems that have an IO map that is not purely passive, but rather a mixture, blend, or combination of a passive IO map and a finite-gain IO map. Linear time-invariant (LTI) systems that have this property are positive real (PR) within a bandwidth and bounded real (BR), i.e., have gain that is finite, outside the PR bandwidth. Examples include [10] and [11], in which "mixed" systems are defined; [12] and [13] in which, finite-frequency PR and BR systems are defined; and [14], in which "hybrid" passive and finite-gain (passive/finite gain) systems are defined.

The hybrid passive/finite-gain systems theory was originally created specifically to overcome passivity violations [14]. It was developed with [10] and [11] as the starting points. The hybrid passive/finite-gain systems theory is quite general, being applicable to linear single-input single-output as well as nonlinear multi-input multi-output (MIMO) systems. Recent work has focused on the numerical optimization of LTI controllers using frequency domain inequalities to constrain the controllers to be strictly positive real (SPR) within a bandwidth and BR past a critical frequency [15], [16]. Additionally, [16] shows that a set of hybrid VSP/finite-gain subcontrollers gain-scheduled appropriately yields a gain-scheduled controller that is also hybrid VSP/finite-gain. Experimental results have shown that the hybrid passive/finite-gain systems framework as well as the controller synthesis procedures considered perform well in practice. In [15], control of a single-link flexible manipulator experiment is demonstrated, and in [16] a two-link flexible manipulator experiment system is controlled. Both these systems are nominally passive, but passivity is destroyed when rate information is acquired by a derivative filter.

The generalized Kalman–Yakubovich–Popov (GKYP) lemma developed by Iwasaki *et al.* [12], [13], [17]–[22] can be used to characterize LTI systems that have PR or BR characteristics over a finite bandwidth in the frequency

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domain. As mentioned above, Iwasaki and Hara call these systems finite-frequency (FF) PR and BR systems (FF PR and FF BR) because the PR and BR properties hold over a finite-frequency range rather than an infinite-frequency range. Their work aligns harmoniously with the LTI form of the hybrid passive/finite-gain systems framework. An LTI system that is passive at low frequency, no longer passive but still possesses a gain that is finite at high frequency, is not only a hybrid passive/finite-gain system but is also an FF PR/BR system as expressed by the low- and high-frequency PR and BR forms of the GKYP lemma. The hybrid passive/finite-gain systems framework and the GKYP lemma form an excellent match; the hybrid passive/finite-gain systems framework provides closed-loop stability criteria, while the GKYP lemma provides an elegant means to determine whether a system is PR or BR over a finite-frequency range. Noting and exploiting this relationship is one of the contributions of this paper.

Although the controller optimization formulations presented in [15] and [16] are effective, the formulations are nonconvex and essentially enforce FF SPR/BR constraints in a brute-force manner. One of the objectives of this paper is to formulate a controller synthesis procedure that is convex by defining a convex objective function and expressing the FF SPR/BR constraints in terms of linear matrix inequalities (LMIs) via the GYKP lemma. Convex optimization problems are simple and efficient to solve numerically [23], [24]. The formulation we present here will make extensive use of the PR, SPR, and BR forms of the GKYP lemma in low- and high-frequency ranges. The GKYP Lemma has been used to design FF proportional-integral-derivative controllers [21] as well as FF $\mathcal{H}_\infty$ filters [25]. In the interest of optimality, the convex objective function we define will, in essence, have the controller mimic a standard $\mathcal{H}_2$ controller. Our controller (or, to be specific, our FF SPR/BR controller) will mimic an $\mathcal{H}_2$ controller so that the closed-loop performance is as close to optimal as possible while simultaneously being robust to passivity violations. The controller synthesis method developed in this paper will be used to control a two-link flexible manipulator that has experienced a passivity violation, and experimental results will be included. Other systems that are nominally passive and have experienced a passivity violation can also be controlled using the results of this paper and our companion papers [14]–[16].

This paper is organized as follows. We will first review the hybrid passivity/finite-gain systems framework, as well as the low- and high-frequency PR, SPR, and BR forms of the GKYP lemma. In particular, we will highlight how LTI hybrid passivity/finite-gain and LTI hybrid VSP/finite-gain systems are related to FF PR/BR and FF SPR/BR systems as characterized by the GKYP lemma. We then pose our controller optimization problem in terms of a convex objective function and a set of LMIs that force the controller to be FF SPR/BR. Control of a two-link flexible manipulator will be considered. The manipulator dynamics along with how passivity is violated will be briefly discussed, clearly motivating the definition of a hybrid passive/finite-gain system. Controller synthesis results will be presented, along with experimental results. In particular, the two-link manipulator will be controlled by gain-scheduling two FF SPR/BR controllers, each optimally designed about two different set points. We will close with some final remarks.

II. Hybrid Passive/Finite-Gain Systems and the GKYP Lemma

A. Hybrid Passive/Finite-Gain Systems Theory

In this section we will briefly review hybrid passive/finite-gain systems and the hybrid passivity/finite-gain stability theorem originally developed in [14]. To start, recall that $\mathbf{y} \in L_2$ if $\|\mathbf{y}\|_2 = \langle \mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} = \sqrt{\int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)dt} < \infty$, and $\mathbf{y} \in L_{2e}$ if $\|\mathbf{y}\|_{2T} = \sqrt{\int_0^\infty \mathbf{y}_T^\mathsf{T}(t)\mathbf{y}_T(t)dt} < \infty$, $0 \le T < \infty$, where $\mathbf{y}_T(t) = \mathbf{y}(t)$, $0 \le t \le T$, and $\mathbf{y}_T(t) = \mathbf{0}$, $t > T$ [1], [2]. An inner product such as $\langle \mathbf{y}, \mathbf{e}\rangle_T = \int_0^\infty \mathbf{y}_T^\mathsf{T}(t)\mathbf{e}_T(t)dt$ can equivalently be written as $\langle \mathbf{y}, \mathbf{e}\rangle_T = (1/2\pi)\,\mathrm{Re}\int_{-\infty}^\infty \mathbf{y}_T^\mathsf{H}(j\omega)\mathbf{e}_T(j\omega)d\omega$ via Parseval's theorem, where $\mathbf{e}(j\omega)$ is the Fourier transform of $\mathbf{e}(t)$, and $\mathbf{e}^\mathsf{H}(j\omega) = \mathbf{e}^\mathsf{T}(-j\omega)$ is the complex-conjugate transpose of $\mathbf{e}(j\omega)$.

Hybrid passive/finite-gain systems theory can be thought of as an extension or generalization of the dissipative systems framework [26], and was originally motivated by the mixed systems framework presented in [10] and [11]. Consider a general MIMO system, linear or nonlinear, $\mathbf{y}(t) = (\boldsymbol{\mathcal{G}}\mathbf{e})(t)$, where the operator $\boldsymbol{\mathcal{G}} : L_{2e} \to L_{2e}$ maps the input $\mathbf{e} \in L_{2e}$ to the output $\mathbf{y} \in L_{2e}$. The system is a hybrid passive/finite-gain system if

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} &\mathbf{y}_T^\mathsf{T}(-j\omega)\mathbf{Q}(\omega)\mathbf{y}_T(j\omega)d\omega\\ &+\frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{\infty}\mathbf{y}_T^\mathsf{T}(-j\omega)\mathbf{S}(\omega)\mathbf{e}_T(j\omega)d\omega\\ &+\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{e}_T^\mathsf{T}(-j\omega)\mathbf{R}(\omega)\mathbf{e}_T(j\omega)d\omega \ge 0 \qquad (1)\end{aligned}$$

holds, where

$$\begin{aligned}\mathbf{Q}(\omega) &= -\left[\epsilon\alpha(\omega) + \gamma^{-1}(1-\alpha(\omega))\right]\mathbf{1}\\ \mathbf{S}(\omega) &= \tfrac{1}{2}\alpha(\omega)\mathbf{1}\\ \mathbf{R}(\omega) &= \left[\gamma(1-\alpha(\omega)) - \delta\alpha(\omega)\right]\mathbf{1}. \qquad (2)\end{aligned}$$

The constant parameters $0 \le \delta < \infty$ and $0 \le \epsilon < \infty$ depend on the passive nature of the system, and $0 < \gamma < \infty$ depends on the finite-gain nature of the system when passivity has been violated. Notice that the units of $\mathbf{Q}(\cdot)$ and $\mathbf{R}(\cdot)$ are consistent; ϵ and γ^{-1} have units of one over gain, while δ and γ have units of gain. The variable α can be 0 or 1, and it is used to distinguish between passive system characteristics and nonpassive but still finite-gain system characteristics. When the system in question possesses a passive IO map, $\alpha(\omega) = 1$. When the system fails to possess a passive IO map, i.e., the system has experienced a passivity violation, but the map has finite gain, $\alpha(\omega) = 0$. The divide occurs at a critical frequency $\omega_c \in [0, \infty]$, which is used to define α

$$\alpha(\omega) = \begin{cases} 1 & \forall\omega \in \Omega_l \\ 0 & \forall\omega \in \Omega_h \end{cases}$$

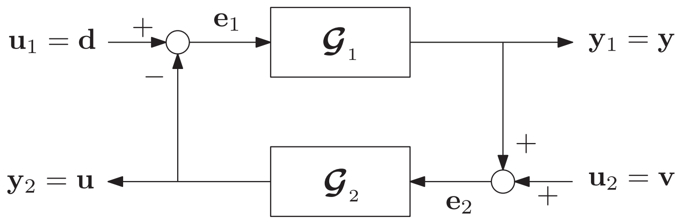

Fig. 1. General negative feedback interconnection of systems $\mathcal{G}_1$ and $\mathcal{G}_2$.

where

$$\Omega_l = \{\omega \in \mathbb{R} : |\omega| < \omega_c\}$$
$$\Omega_h = \{\omega \in \mathbb{R} \cup \{\infty\} : |\omega| \geq \omega_c\}.$$

We can intuitively think of α as an ideal low-pass filter filtering the signals into two parts: a passive part and a finite-gain part.

Sufficient conditions for (1) to hold are

$$\frac{1}{2\pi}\mathrm{Re}\int_{-\omega_c}^{\omega_c} \mathbf{y}_T^{\mathsf{H}}(j\omega)\mathbf{e}_T(j\omega)d\omega \geq \frac{\delta}{2\pi}\int_{-\omega_c}^{\omega_c} \mathbf{e}_T^{\mathsf{H}}(j\omega)\mathbf{e}_T(j\omega)d\omega + \frac{\epsilon}{2\pi}\int_{-\omega_c}^{\omega_c} \mathbf{y}_T^{\mathsf{H}}(j\omega)\mathbf{y}_T(j\omega)d\omega \quad (3)$$

and

$$\frac{1}{\pi\gamma}\int_{\omega_c}^{\infty} \mathbf{y}_T^{\mathsf{H}}(j\omega)\mathbf{y}_T(j\omega)d\omega \leq \frac{\gamma}{\pi}\int_{\omega_c}^{\infty} \mathbf{e}_T^{\mathsf{H}}(j\omega)\mathbf{e}_T(j\omega)d\omega. \quad (4)$$

When $\alpha(\omega) = 1$ in (1), the IO map is said to be:

1) VSP when $0 < \delta < \infty$ and $0 < \epsilon < \infty$;
2) input strictly passive (ISP) when $0 < \delta < \infty$ and $\epsilon = 0$;
3) output strictly passive (OSP) when $\delta = 0$ and $0 < \epsilon < \infty$;
4) passive when $\delta = \epsilon = 0$.

If the system is hybrid ISP and has finite gain when $\alpha(\omega) = 1$, then the system is hybrid VSP when $\alpha(\omega) = 1$ [15], [16]. The gain, $0 < \kappa < \infty$, when $\alpha(\omega) = 1$ satisfies

$$\frac{1}{2\pi\kappa}\int_{-\omega_c}^{\omega_c} \mathbf{y}_T^{\mathsf{T}}(-j\omega)\mathbf{y}_T(j\omega)d\omega \leq \frac{\kappa}{2\pi}\int_{-\omega_c}^{\omega_c} \mathbf{e}_T^{\mathsf{T}}(-j\omega)\mathbf{e}_T(j\omega)d\omega. \quad (5)$$

The parameter κ is called the passive system gain. Upon violation of passivity $\alpha(\omega) = 0$, the IO map is no longer passive, and (1) is said to be a finite-gain IO map.

Notice that as $\omega_c \to \infty$, (1) and (2) reduce to the traditional definition of a passive system. Similarly, as $\omega_c \to 0$, the traditional definition of a finite-gain system is recovered. Also, notice that the hybrid passive/finite-gain parameters δ, ϵ (or κ), and γ are not defined globally, but rather in terms of specific IO mappings. Using the GKYP lemma to ensure that an LTI system is hybrid passive/finite-gain or hybrid VSP/finite-gain will be explored in Section II-B.

Consider the negative feedback interconnection of two systems $\mathcal{G}_1 : L_{2e} \to L_{2e}$ and $\mathcal{G}_2 : L_{2e} \to L_{2e}$, presented in Fig. 1. The critical frequency ω_c is assumed to be known, and the hybrid passivity/finite-gain parameters associated with each system are δ_1, ϵ_1, and γ_1 and δ_2, ϵ_2, and γ_2, respectively. The hybrid passivity/finite-gain stability theorem states that the negative feedback interconnection presented in Fig. 1 is L_2-stable if the variables δ_1, ϵ_1, γ_1, δ_2, ϵ_2, and γ_2 satisfy $\epsilon_1 + \delta_2 > 0$, $\epsilon_2 + \delta_1 > 0$, and $\gamma_1\gamma_2 < 1$ [14].

The hybrid passivity/finite-gain stability theorem is a combination or amalgamation of the traditional passivity and small-gain theorems. Its development is motivated by systems that are nominally passive but have had their passive IO map partially destroyed in some way. The theorem allows high-gain feedback to be partially reintroduced where the traditional small-gain theorem would be overly conservative and the traditional passivity theorem alone would not guarantee closed-loop stability.

A specific form of the hybrid passivity/finite-gain stability theorem that we will make use of is the negative feedback interconnection of a hybrid passive/finite-gain plant and a hybrid VSP/finite-gain controller. A hybrid passive/finite-gain plant will have $\delta_1 = \epsilon_1 = 0$ ($\kappa_1 = \infty$), and $0 < \gamma_1 < \infty$. A hybrid VSP/finite-gain controller will have $0 < \delta_2 < \infty$, $0 < \epsilon_2 < \infty$ ($0 < \kappa_2 < \infty$), and $0 < \gamma_2 < \infty$. In this particular situation, if $\gamma_1\gamma_2 < 1$, the closed-loop system will be stable. This form of the theorem is particularly useful for stabilizing plants that are nominally passive in the traditional sense (i.e., in an LTI context with PR over all frequencies) but have their passive IO map destroyed in some way.

B. State-Space Representation of Hybrid Passive/Finite-Gain Systems Using the GKYP Lemma

To use the hybrid passivity/finite-gain stability theorem, we must know or be able to estimate the δ, ϵ, and γ parameters associated with the plant being controlled and design a controller, which, together with the plant, satisfies the hybrid passivity/finite-gain stability theorem. Given a general MIMO system $\mathbf{y}(t) = (\mathcal{G}\mathbf{e})(t)$, we can approximate the hybrid passivity/finite-gain parameters using a linearization of the system, $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{e}(s)$, where $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is the system transfer matrix, and $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal state-space realization. We assume that $\mathbf{G}(s)$ is composed of real rational transfer functions. If the original system is LTI, then $\mathbf{G}(s)$ exactly represents the original system. If the original system is nonlinear, $\mathbf{G}(s)$ represents the linearized system. We will assume that the linearized system captures the hybrid passive/finite-gain properties of the nonlinear system reasonably well.

We will begin our discussion with LTI systems that are passive within a low-frequency bandwidth, i.e., those that satisfy (3) with $\delta = 0$ and $\epsilon = 0$. An LTI system that is passive $\forall\omega \in \Omega_l$ is PR $\forall\omega \in \Omega_l$, also called FF PR. a transfer matrix $\mathbf{G}(s) \in \mathbb{C}^{n\times n}$ is PR $\forall\omega \in \Omega_l$ if [12], [19]

$$\begin{bmatrix}\mathbf{G}(s)\\ \mathbf{1}\end{bmatrix}^{\mathsf{H}} \mathbf{\Pi}_p \begin{bmatrix}\mathbf{G}(s)\\ \mathbf{1}\end{bmatrix} \leq 0 \quad \forall\omega \in \Big\{\omega \in \mathbb{R} : \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0, |\omega| \leq (\omega_c - \bar{\omega}_c)\Big\} \quad (6)$$

where

$$\mathbf{\Pi}_p = \begin{bmatrix}\mathbf{0} & -\mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{bmatrix}$$

and $\bar{\omega}_c$ is a trivially small number that effectively transforms $|\omega| \leq (\omega_c - \bar{\omega}_c)$ into the strict inequality $|\omega| < \omega_c$. This condition can also be written in terms of an LMI using the GKYP lemma [12], [19]. The system $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}+$

$\mathbf{D}$ is PR $\forall\omega \in \{\omega \in \mathbb{R} : \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0, |\omega| \leq (\omega_c - \bar{\omega}_c)\}$ if there $\exists$ $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ where $\mathbf{P} = \mathbf{P}^{\mathsf{T}}$ and $\mathbf{Q} = \mathbf{Q}^{\mathsf{T}} \geq 0$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \\ \mathbf{P} & (\omega_c - \bar{\omega}_c)^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathsf{T}} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \leq 0. \quad (7)$$

To be clear, both $\mathbf{P}$ and $\mathbf{Q}$ are symmetric, and although $\mathbf{Q}$ must be positive semidefinite, $\mathbf{P}$ could be negative semidefinite, positive semidefinite, or indefinite. It is also worth noting that the FF PR definitions given in (6) and (7) permit poles on the imaginary axis, including the origin [12]. Additionally, if $\omega_c \to \infty$, then $\mathbf{P} = \mathbf{P}^{\mathsf{T}} > 0$, $\mathbf{Q} = \mathbf{0}$, and the traditional PR lemma is recovered [3].

Next, we will consider systems that are ISP and have finite gain within a low-frequency bandwidth. Such a system must satisfy (3); that is, there must exist $0 < \delta < \infty$ and $0 < \epsilon < \infty$. In particular, if there exists $0 < \delta < \infty$ and $0 < \kappa < \infty$ where κ satisfies (5), then (3) will be satisfied [15], [16]. In terms of LTI systems, a system that is ISP and finite gain $\forall\omega \in \mathbf{\Omega}_l$ is SPR $\forall\omega \in \mathbf{\Omega}_l$, or called FF SPR. A transfer matrix $\mathbf{G}(s) \in \mathbb{C}^{n\times n}$ is SPR $\forall\omega \in \mathbf{\Omega}_l$ if all the poles of $\mathbf{G}(s)$ are in the open left-half plane and [17]

$$\begin{bmatrix} \mathbf{G}(s) \\ \mathbf{1} \end{bmatrix}^{\mathsf{H}} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{G}(s) \\ \mathbf{1} \end{bmatrix} < 0 \qquad \forall\omega \in \{\omega \in \mathbb{R} : |\omega| \leq (\omega_c - \bar{\omega}_c)\}.$$

This strict inequality can be written as an LMI using the GKYP lemma [13], [17], [18]. The system $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is SPR $\forall\omega \in \mathbf{\Omega}_l$ if there $\exists$ $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$, where $\mathbf{P} = \mathbf{P}^{\mathsf{T}}$ and $\mathbf{Q} = \mathbf{Q}^{\mathsf{T}} > 0$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \\ \mathbf{P} & (\omega_c - \bar{\omega}_c)^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathsf{T}} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \quad (8)$$

Let us now move on to discuss the properties of LTI systems at high frequency. A system that has finite gain above ω_c is BR above ω_c. Such a system satisfies (4) with $0 < \gamma < \infty$. A transfer matrix $\mathbf{G}(s) \in \mathbb{C}^{n\times n}$ is BR $\forall\omega \in \mathbf{\Omega}_h$ with gain $0 < \gamma < \infty$ if all the poles of $\mathbf{G}(s)$ are in the open left-half plane and [20]

$$\begin{bmatrix} \mathbf{G}(s) \\ \mathbf{1} \end{bmatrix}^{\mathsf{H}} \mathbf{\Pi}_b \begin{bmatrix} \mathbf{G}(s) \\ \mathbf{1} \end{bmatrix} \leq 0 \qquad \forall\omega \in \mathbf{\Omega}_h$$

where

$$\mathbf{\Pi}_b = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\gamma^2\mathbf{1} \end{bmatrix}.$$

This inequality can be written as an LMI using the GKYP lemma. The system $\mathbf{G}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is BR $\forall\omega \in \mathbf{\Omega}_h$ with gain $0 < \gamma < \infty$ if there $\exists$ $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$, where $\mathbf{P} = \mathbf{P}^{\mathsf{T}}$ and $\mathbf{Q} = \mathbf{Q}^{\mathsf{T}} \geq 0$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbf{Q} & \mathbf{P} \\ \mathbf{P} & -\omega_c^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathsf{T}} \mathbf{\Pi}_b \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \leq 0. \quad (9)$$

Of interest to us are systems that are hybrid, possessing passive or ISP and finite-gain properties below ω_c (i.e., at low frequency), and finite-gain properties above ω_c (i.e., at high frequency). In particular, an LTI system $\mathbf{G}(s)$ that is passive below ω_c and finite-gain above ω_c is a hybrid passive/finite-gain system, but also an FF PR/BR system. FF PR/BR systems generally describe plants that have experienced a passivity violation. An LTI system $\mathbf{G}(s)$ that is ISP and has finite gain below ω_c and has a finite-gain mapping above ω_c is a hybrid VSP/finite-gain system, but also an FF SPR/BR system. FF SPR/BR systems will be used as controllers to control FF PR/BR systems where stability will be guaranteed via the hybrid passivity/finite-gain stability theorem. Note that, although an FF SPR/BR system has finite gain over all frequencies, the gain at low frequency is different from that at high frequency.

III. Controller Design

This paper pertains to systems that are ideally PR over all frequencies, but have their PR nature destroyed and are rendered hybrid having PR properties below ω_c and BR properties above ω_c, i.e., FF PR/BR. In terms of the passivity and finite gain parameters, as just discussed in Section II-B, an LTI plant $\mathbf{G}_1(s)$ that is FF PR/BR will have $\delta_1 = 0$, $\epsilon = 0$, and $0 < \gamma_1 < \infty$. In order to stabilize such a system via the hybrid passivity/finite-gain stability theorem, a controller $\mathbf{G}_2(s)$ must be synthesized so that $0 < \delta_2 < \infty$, $0 < \epsilon_2 < \infty$ (i.e., $0 < \delta_2 < \infty$ and $0 < \kappa_2 < \infty$), and $0 < \gamma_2 < \infty$ where $\gamma_1\gamma_2 < 1$. As discussed in Section II-B, such a controller takes the form of an FF SPR/BR transfer matrix. We assume that we do not know ω_c and γ_1 exactly because we do not know the exact effect of unmodeled actuator, sensor, and filter dynamics. However, we are confident that these unmodeled dynamics render the plant hybrid passive/finite-gain (FF PR/BR) and that we are able to estimate both ω_c and γ_1.

The purpose of this section is to formulate a convex optimization problem that yields an FF SPR/BR system, given estimates of ω_c and γ_1, to act as a controller. In particular, our approach will be to mimic a classic $\mathcal{H}_2$ controller as closely as possible to ensure that the FF SPR/BR controller is optimal in some sense. We will first review the standard $\mathcal{H}_2$ formulation [27]. The nominal system (i.e., one that ignores sensors, actuators, etc., which induce passivity violations) to be controlled is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u} \\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, $\mathbf{u} \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{y} \in \mathbb{R}^{n_y}$ is the measurement, $\mathbf{z} \in \mathbb{R}^{n_z}$ is the regulated output, the disturbances/noise are $\mathbf{w} = \begin{bmatrix} \mathbf{d}^{\mathsf{T}} & \mathbf{v}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n_w}$, and all matrices are dimensioned appropriately. It is assumed that:

1) $(\mathbf{A}, \mathbf{B}_1)$ is controllable and $(\mathbf{C}_1, \mathbf{A})$ is observable;
2) $(\mathbf{A}, \mathbf{B}_2)$ is controllable and $(\mathbf{C}_2, \mathbf{A})$ is observable;
3) $\mathbf{D}_{12}^{\mathsf{T}}\mathbf{C}_1 = \mathbf{0}$ and $\mathbf{D}_{12}^{\mathsf{T}}\mathbf{D}_{12} > 0$;
4) $\mathbf{D}_{21}\mathbf{B}_1^{\mathsf{T}} = \mathbf{0}$ and $\mathbf{D}_{21}\mathbf{D}_{21}^{\mathsf{T}} > 0$.

The $\mathcal{H}_2$ optimal controller takes the following form:

$$\left.\begin{aligned}\dot{\mathbf{x}}_c &= \overbrace{(\mathbf{A}-\mathbf{B}_2\mathbf{K}_c-\mathbf{K}_e\mathbf{C}_2)}^{\mathbf{A}_c}\mathbf{x}_c+\mathbf{K}_e\mathbf{y}\\ -\mathbf{u} &= \mathbf{K}_c\mathbf{x}_c\end{aligned}\right\}$$
$$\Leftrightarrow -\mathbf{u}(s) = \mathbf{G}_2^{\star}(s)\mathbf{y}(s) = \mathbf{K}_c(s\mathbf{1}-\mathbf{A}_c)^{-1}\mathbf{K}_e\mathbf{y}(s) \quad (10)$$

where $\mathbf{K}_c$ is the optimal state-feedback gain matrix, and $\mathbf{K}_e$ is the optimal estimator gain matrix. Both $\mathbf{K}_c$ and $\mathbf{K}_e$ are found by solving two different Riccati equations.

Our approach to designing an optimal FF SPR/BR controller is to keep the controller dynamic matrix $\mathbf{A}_c$ and input matrix $\mathbf{K}_e$ the same as the standard $\mathcal{H}_2$ controller, and then to find a state-feedback gain matrix $\mathbf{K}_o$ that renders $\mathbf{G}_2(s) = \mathbf{K}_o(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$ FF SPR/BR. The state-feedback gain matrix $\mathbf{K}_o$ will ultimately render our controller FF SPR/BR. This approach is similar to the approaches proposed in [15] and [28], but here we will form a convex optimization problem using LMIs, much like in [29]–[31].

Let us first discuss our controller constraints. As previously mentioned, given ω_c and γ_1, we must design our controller $\mathbf{G}_2(s)$ to be FF SPR/BR. We assume that $\mathbf{G}_2(s) = \mathbf{K}_o(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$ is Hurwitz for any $\mathbf{K}_o$ by assuming that the nominal $\mathcal{H}_2$ solution renders $\mathbf{A}_c$ Hurwitz. By using (8), the controller $\mathbf{G}_2(s)$ will be SPR $\forall\omega \in \Omega_l$ if there $\exists\ \mathbf{P}_p, \mathbf{Q}_p \in \mathbb{R}^{n\times n}$ where $\mathbf{P}_p = \mathbf{P}_p^{\mathsf{T}}$ and $\mathbf{Q}_p = \mathbf{Q}_p^{\mathsf{T}} > 0$ such that

$$\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}-\mathbf{Q}_p & \mathbf{P}_p\\ \mathbf{P}_p & (\omega_c-\bar{\omega}_c)^2\mathbf{Q}_p\end{bmatrix}\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & -\mathbf{K}_o^{\mathsf{T}}\\ -\mathbf{K}_o & \mathbf{0}\end{bmatrix} < 0. \quad (11)$$

Notice that this is an LMI in $\mathbf{K}_o$, $\mathbf{P}_p$, and $\mathbf{Q}_p$ (because $\mathbf{A}_c$ is fixed). Next, using (9) our controller will be BR $\forall\omega \in \Omega_h$ with gain $\gamma_2 < 1/\gamma_1$ if there $\exists\ \mathbf{P}_b, \mathbf{Q}_b \in \mathbb{R}^{n\times n}$ where $\mathbf{P}_b = \mathbf{P}_b^{\mathsf{T}}$ and $\mathbf{Q}_b = \mathbf{Q}_b^{\mathsf{T}} \geq 0$ such that

$$\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\mathbf{Q}_b & \mathbf{P}_b\\ \mathbf{P}_b & -\omega_c^2\mathbf{Q}_b\end{bmatrix}\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{K}_o^{\mathsf{T}}\mathbf{K}_o & \mathbf{0}\\ \mathbf{0} & -\gamma_2^2\mathbf{1}\end{bmatrix} \leq 0.$$

This matrix inequality is linear in $\mathbf{P}_b$ and $\mathbf{Q}_b$, but not linear in $\mathbf{K}_o$. By using the Schur complement [23], we can transform it into an LMI as

$$\begin{bmatrix}\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\mathbf{Q}_b & \mathbf{P}_b\\ \mathbf{P}_b & -\omega_c^2\mathbf{Q}_b\end{bmatrix}\begin{bmatrix}\mathbf{A}_c & \mathbf{K}_e\\ \mathbf{1} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\gamma_2^2\mathbf{1}\end{bmatrix} & \begin{bmatrix}\mathbf{K}_o^{\mathsf{T}}\\ \mathbf{0}\end{bmatrix}\\ \begin{bmatrix}\mathbf{K}_o & \mathbf{0}\end{bmatrix} & -\mathbf{1}\end{bmatrix} \leq 0. \quad (12)$$

This is now an LMI in terms of $\mathbf{K}_o$, $\mathbf{P}_b$, and $\mathbf{Q}_b$. Therefore, the controller $\mathbf{G}_2(s) = \mathbf{K}_o(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$ will be FF SPR/BR if there $\exists\ \mathbf{P}_p, \mathbf{Q}_p, \mathbf{P}_b, \mathbf{Q}_b \in \mathbb{R}^{n\times n}$, where $\mathbf{P}_p = \mathbf{P}_p^{\mathsf{T}}$, $\mathbf{Q}_p = \mathbf{Q}_p^{\mathsf{T}} > 0$, $\mathbf{P}_b = \mathbf{P}_b^{\mathsf{T}}$, and $\mathbf{Q}_b = \mathbf{Q}_b^{\mathsf{T}} \geq 0$ such that both (11) and (12) are satisfied.

Although any $\mathbf{G}_2(s)$ that is FF SPR/BR will stabilize an FF PR/BR plant via the hybrid passivity/finite-gain stability theorem, we want a $\mathbf{G}_2(s)$ that is optimal in some sense. As mentioned at the beginning of this section, we will formulate our convex optimization problem so that $\mathbf{G}_2(s)$ mimics $\mathbf{G}_2^{\star}(s)$ as best as it can while simultaneously satisfying the FF SPR/BR constraints [i.e., the LMIs in (11) and (12)]. As such, consider the following objective function to be minimized:

$$\mathcal{J} = \operatorname{tr}\left[(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}\right]. \quad (13)$$

The "closer" $\mathbf{K}_o$ is to $\mathbf{K}_c$, the closer the FF SPR/BR controller is to the nominal $\mathcal{H}_2$ controller used as the basis for controller design. It can also be shown that minimizing the difference between $\mathbf{K}_o$ and $\mathbf{K}_c$ minimizes an approximate upper bound on the difference between the optimal and optimized sensitivity functions, $\boldsymbol{\mathcal{S}}(s) = [\mathbf{1} + \mathbf{G}_1(s)\mathbf{G}_2(s)]^{-1}$ and $\boldsymbol{\mathcal{S}}^{\star}(s) = [\mathbf{1} + \mathbf{G}_1(s)\mathbf{G}_2^{\star}(s)]^{-1}$, as discussed next. Noting that $\boldsymbol{\mathcal{S}}(s) = [\mathbf{1} + \mathbf{G}_1(s)\mathbf{G}_2^{\star}(s) + \mathbf{G}_1(s)(\mathbf{G}_2(s) - \mathbf{G}_2^{\star}(s))]^{-1}$, then to first order in $\mathbf{G}_2(s) - \mathbf{G}_2^{\star}(s) = (\mathbf{K}_o - \mathbf{K}_c)(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$, we have

$$\boldsymbol{\mathcal{S}}(s) = \boldsymbol{\mathcal{S}}^{\star}(s) - \boldsymbol{\mathcal{S}}^{\star}(s)\mathbf{G}_1(s)[\mathbf{G}_2(s) - \mathbf{G}_2^{\star}(s)]\boldsymbol{\mathcal{S}}^{\star}(s).$$

Following the arguments in [32], in general we can write $||\boldsymbol{\mathcal{A}}(s)\boldsymbol{\mathcal{B}}(s)\boldsymbol{\mathcal{C}}(s)||_2 \leq ||\boldsymbol{\mathcal{A}}(s)||_\infty||\boldsymbol{\mathcal{B}}(s)||_2||\boldsymbol{\mathcal{C}}(s)||_\infty$, where $||(\cdot)||_2$ is the usual $\mathcal{H}_2$ norm and $||(\cdot)||_\infty$ is the usual $\mathcal{H}_\infty$ norm. Therefore

$$||\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^{\star}||_2 \leq ||\boldsymbol{\mathcal{S}}^{\star}\mathbf{G}_1||_\infty||\mathbf{G}_2 - \mathbf{G}_2^{\star}||_2||\boldsymbol{\mathcal{S}}^{\star}||_\infty. \quad (14)$$

The usual calculation of the $\mathcal{H}_2$ norm yields

$$\begin{aligned}||\mathbf{G}_2 - \mathbf{G}_2^{\star}||_2^2 &= \operatorname{tr}[(\mathbf{K}_o - \mathbf{K}_c)\mathbf{P}_c(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}]\\ &= \operatorname{tr}[\mathbf{P}_c(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}(\mathbf{K}_o - \mathbf{K}_c)] \quad (15)\end{aligned}$$

where $\mathbf{P}_c$ is the positive-definite solution of the Lyapunov equation $\mathbf{A}_c\mathbf{P}_c + \mathbf{P}_c\mathbf{A}_c^{\mathsf{T}} = -\mathbf{K}_e\mathbf{K}_e^{\mathsf{T}}$. From [33], we have the identify $|\operatorname{tr}(\mathbf{PQ})| \leq \bar{\sigma}(\mathbf{P})\operatorname{tr}\mathbf{Q}$ if $\mathbf{P}$ and $\mathbf{Q}$ are positive-semidefinite matrices and $\bar{\sigma}(\cdot)$ denotes the maximum singular value. Applying this identity to (15) and using the result with (14) yields

$$\begin{aligned}||\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^{\star}||_2 &\leq ||\boldsymbol{\mathcal{S}}^{\star}\mathbf{G}_1||_\infty||\boldsymbol{\mathcal{S}}^{\star}||_\infty\\ &\quad\times\sqrt{\bar{\sigma}(\mathbf{P}_c)\cdot\operatorname{tr}[(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}]}. \quad (16)\end{aligned}$$

Hence, minimizing $\mathcal{J}$ minimizes an estimated upper bound on the difference between the optimal and optimized sensitivity functions where it has been assumed that $\mathbf{K}_o$ and $\mathbf{K}_c$ remain close after performing the constrained optimization. Similar arguments can be used to estimate bounds on other closed-loop transfer functions (complementary sensitivity function, control sensitivity function, etc.).

We will now rewrite the objective function given in (13). By using the "slack" variable $g \in \mathbb{R}^+$ and the symmetric positive-semidefinite "slack" matrix $\mathbf{Z} \in \mathbb{R}^{n_u\times n_u}$, the objective function given in (13) can be equivalently written as

$$\mathcal{J} = g \quad (17)$$

subject to

$$\operatorname{tr}[\mathbf{Z}] \leq g \quad (18a)$$
$$(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}} \leq \mathbf{Z}. \quad (18b)$$

As g is minimized, $\mathbf{Z}$ is minimized, and as $\mathbf{Z}$ is minimized, $(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}$ is minimized. By using the Schur complement, the constraint $(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^\mathsf{T} \leq \mathbf{Z}$ can be equivalently written as

$$\begin{bmatrix} \mathbf{Z} & (\mathbf{K}_o - \mathbf{K}_c) \\ (\mathbf{K}_o - \mathbf{K}_c)^\mathsf{T} & \mathbf{1} \end{bmatrix} \geq 0. \tag{19}$$

If the objective function in (17) is minimized subject to the constraints given in (11), (12), (18a), and (19), the resultant controller will be FF SPR/BR, but also in effect will mimic an unconstrained $\mathcal{H}_2$ controller.

Our optimization problem can be summarized as follows:

$$\begin{aligned} \text{minimize} \quad & \mathcal{J}(\mathbf{K}_o, \mathbf{P}_p, \mathbf{Q}_p, \mathbf{P}_b, \mathbf{Q}_b, g, \mathbf{Z}) = g \\ \text{with respect to} \quad & \mathbf{K}_o,\ \mathbf{P}_p,\ \mathbf{Q}_p,\ \mathbf{P}_b,\ \mathbf{Q}_b,\ g,\ \mathbf{Z} \\ \text{s.t.} \quad & \mathbf{Q}_p = \mathbf{Q}_p^\mathsf{T} > 0,\ (11),\ \mathbf{Q}_b = \mathbf{Q}_b^\mathsf{T} \geq 0 \\ & (12),\ \mathbf{Z} = \mathbf{Z}^\mathsf{T} \geq 0,\ (18a),\ \ (19). \end{aligned}$$

This optimization problem is convex; in fact, this optimization problem is a semidefinite program easily solved by a numerical algorithm such as an interior point method [23], [24]. In particular, we will use the MATLAB interface YALMIP [34] and the solver SeDuMi [35].

Ideally, we would like to be able to minimize the closed-loop $\mathcal{H}_2$ norm directly while simultaneously constraining the controller to be FF SPR/BR. Unfortunately, doing so does not yield a convex optimization problem. As such, to pose a convex optimization problem we deliberately parameterize in terms of $\mathbf{K}_o$ the controller state-feedback gain, and attempt to mimic an unconstrained $\mathcal{H}_2$ by minimizing the difference between $\mathbf{K}_c$ (the nominal state-feedback gain) and $\mathbf{K}_o$. Parameterizing in terms of $\mathbf{K}_o$ alone is indeed restrictive, as is simply mimicking an unconstrained $\mathcal{H}_2$. However, we do so in order to pose a tractable convex optimization problem constrained by LMIs.

IV. Flexible Robotic Manipulators: Nominal Passivity and Violation Thereof

The FF SPR/BR controller synthesis method of Section III will be used to control a two-link flexible manipulator. This system is nominally passive; however, as we will show, a simple filter dynamics destroys the passive IO properties of the system, rendering it hybrid passive/finite-gain in nature.

A. Two-Link Flexible Manipulator Dynamics and I/O Map

Consider the two-link flexible manipulator in Fig. 2(a). The first link is 210.00-mm long, 1.27-mm thick, and 76.20-mm high. The second link is 210.00-mm long, 0.89-mm thick, and 38.1-mm high. Each link is made of steel and has a strain gauge at its base. The manipulator is manufactured by Quanser Consulting Inc. Additional information can be found in [36]. The dynamics of the system is described by the following second-order nonlinear matrix differential equation [5], [6]:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \hat{\mathbf{B}}\boldsymbol{\tau} + \mathbf{f}_n(\mathbf{q}, \dot{\mathbf{q}}) \tag{20}$$

where $\mathbf{M} = \mathbf{M}^\mathsf{T} > 0$ is the mass matrix, $\mathbf{D} = \mathbf{D}^\mathsf{T} \geq 0$ is the damping matrix, $\mathbf{K} = \mathbf{K}^\mathsf{T} \geq 0$ is the stiffness matrix, and

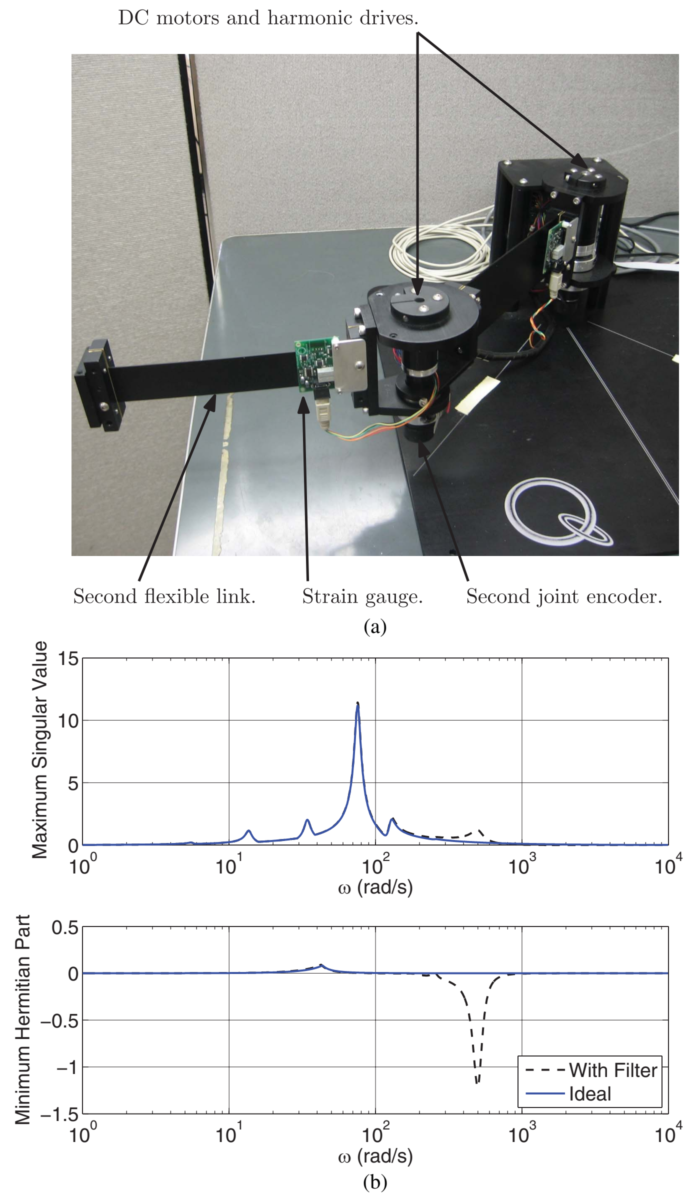

Fig. 2. Two-link experimental apparatus, ideal frequency response, and realistic frequency response. (a) Two-link flexible manipulator apparatus. (b) Frequency response of the ideal two-link system and the two-link system that uses $\mathbf{F}(s)$ to estimate rates.

$\hat{\mathbf{B}} = [\mathbf{1}\ \mathbf{0}]^\mathsf{T}$. The column matrix $\mathbf{q} = [\boldsymbol{\theta}^\mathsf{T}\ \mathbf{q}_e^\mathsf{T}]^\mathsf{T}$ is composed of the joint angles $\boldsymbol{\theta} = [\theta_1\ \theta_2]^\mathsf{T}$ and the elastic coordinates associated with the discretization of each link, $\mathbf{q}_e$. The term $\mathbf{f}_n$ stems from nonlinear inertial forces. Joint torques $\boldsymbol{\tau} = [\tau_1\ \tau_2]^\mathsf{T}$ are applied by motors at the base of each link. The apparatus in Fig. 2(a) is equipped with two encoders, one affixed to each joint [36]. The joint encoders provide θ_1 and θ_2. As such, proportional control can be implemented easily.

For rate control to be implemented, $\dot{\boldsymbol{\theta}}$ must be made available. Unfortunately, our apparatus is not equipped with any sort of rate sensor, and as such $\dot{\boldsymbol{\theta}}$ will be acquired through some sort of differentiation. If perfect differentiation were possible, then the mapping $\boldsymbol{\tau} \rightarrow \mathbf{y}$ where $\mathbf{y}(s) = s\boldsymbol{\theta}(s)$ would be passive. In practice, perfect differentiation is not possible and can only be approximated. In particular, we will use the following derivative filter:

$$\mathbf{y}(s) = \mathbf{F}(s)\boldsymbol{\theta}(s) = \operatorname*{diag}_{i=1,2}\{f_i(s)\}\,\boldsymbol{\theta}(s)$$
$$f_i(s) = \frac{\omega_{f,i}^2 s}{s^2 + 2\zeta_{f,i}\omega_{f,i}s + \omega_{f,i}^2}. \tag{21}$$

The derivative filter $\mathbf{F}(s)$ destroys the nominal passive IO map of the plant because each $f_i(s)$ within $\mathbf{F}(s)$ has a gain that rolls off and a phase that lags at high frequency. The addition of $\mathbf{F}(s)$ induces a passivity violation.

Knowing that $\mathbf{F}(s)$ destroys passivity, we cannot rely on the traditional passivity theorem for control. Therefore, we will control the system via the hybrid passivity/finite-gain stability theorem. To do so, we must be able to estimate the nature of the passivity violation. In particular, we will linearize the two-link system and investigate the FF PR/BR nature of the linearized system.

Consider the manipulator dynamics presented in (20) augmented with proportional control and linearized about a specific joint configuration $\boldsymbol{\theta}_d$. The proportional control gain used is $\mathbf{K}_p = \operatorname{diag}\{40, 40\}$ N · m. The linearized unforced/undamped augmented system can be written as

$$\mathbf{M}\delta\ddot{\mathbf{q}} + \mathbf{K}_a\delta\mathbf{q} = \mathbf{0}$$

where $\mathbf{K}_a = \mathbf{K}_a^{\mathsf{T}} > 0$ is the augmented stiffness matrix, and $\delta\mathbf{q} = \mathbf{q} - \mathbf{q}_d$ where $\mathbf{q}_d = [\boldsymbol{\theta}_d^{\mathsf{T}}\ \mathbf{0}]^{\mathsf{T}}$. By solving the eigenproblem associated with this simplified system, we can define a set of modal coordinates $\delta\mathbf{q} = \mathbf{Q}_e\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ are the modal coordinates and $\mathbf{Q}_e = \operatorname{row}\{\mathbf{q}_\alpha\}$ where $\mathbf{q}_\alpha$ are the eigenvectors normalized with respect to the mass matrix (i.e., $\mathbf{q}_\alpha^{\mathsf{T}}\mathbf{M}\mathbf{q}_\beta = \delta_{\alpha\beta}$). Additionally, we can define $\boldsymbol{\Omega} = \operatorname{diag}\{\omega_\alpha\}$, where ω_α are the natural frequencies of each mode corresponding to the eigenvalues associated with the original eigenproblem.

We can now write the linearized equations in the following first-order state-space form:

$$\dot{\mathbf{x}} = \overbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{\Omega} \\ -\boldsymbol{\Omega} & -2\bar{\mathbf{Z}}\boldsymbol{\Omega} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \boldsymbol{\Omega}\boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{Q}_e^{\mathsf{T}}\hat{\mathbf{B}} \end{bmatrix}}^{\mathbf{B}} \boldsymbol{\tau} \tag{22}$$

where $\bar{\mathbf{Z}} = \operatorname{diag}\{\zeta_\alpha\}$ and ζ_α is the damping ratio associated with each mode. Note that we are deliberately writing the linearized motion equations in this form because numerical computations tend to be much more stable (as we found out when actually computing controllers using the method in Section III). Given the above state-space form, the relation between $\mathbf{x}$, $\boldsymbol{\theta}$, and $\dot{\boldsymbol{\theta}}$ is

$$\boldsymbol{\theta} = \underbrace{\begin{bmatrix} \hat{\mathbf{B}}^{\mathsf{T}}\mathbf{Q}_e\boldsymbol{\Omega}^{-1} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}_p}\mathbf{x}, \quad \dot{\boldsymbol{\theta}} = \underbrace{\begin{bmatrix} \mathbf{0} & \hat{\mathbf{B}}^{\mathsf{T}}\mathbf{Q}_e \end{bmatrix}}_{\mathbf{C}}\mathbf{x}. \tag{23}$$

Let $\boldsymbol{\theta}(s) = \mathbf{G}_p(s)\boldsymbol{\tau}(s) = \mathbf{C}_p(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}\boldsymbol{\tau}(s)$, and $\dot{\boldsymbol{\theta}}(s) = \mathbf{G}(s)\boldsymbol{\tau}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B}\boldsymbol{\tau}(s)$, where $\mathbf{G}(s) := s\mathbf{G}_p(s)$.

The frequency response of the ideal (linearized) system $\mathbf{G}(s)$ is shown in Fig. 2(b). The linearization is performed about $\boldsymbol{\theta}_d = [-\pi/4\ \ 0]^{\mathsf{T}}$. Within Fig. 2(b) is plotted the maximum singular value of $\mathbf{G}(s)$ and the minimum Hermitian part as a function of frequency. The maximum singular value of $\mathbf{G}(j\omega)$ is $\bar{\sigma}(\mathbf{G}(j\omega)) = \sqrt{\bar{\lambda}\left[\mathbf{G}^{\mathsf{H}}(j\omega)\mathbf{G}(j\omega)\right]}$, while the minimum Hermitian part is $(1/2)\underline{\lambda}\left[\mathbf{G}(j\omega) + \mathbf{G}^{\mathsf{H}}(j\omega)\right]$. Clearly, the linearized system is PR over all frequencies owing to the fact the Hermitian part is positive over all frequencies. This result is expected.

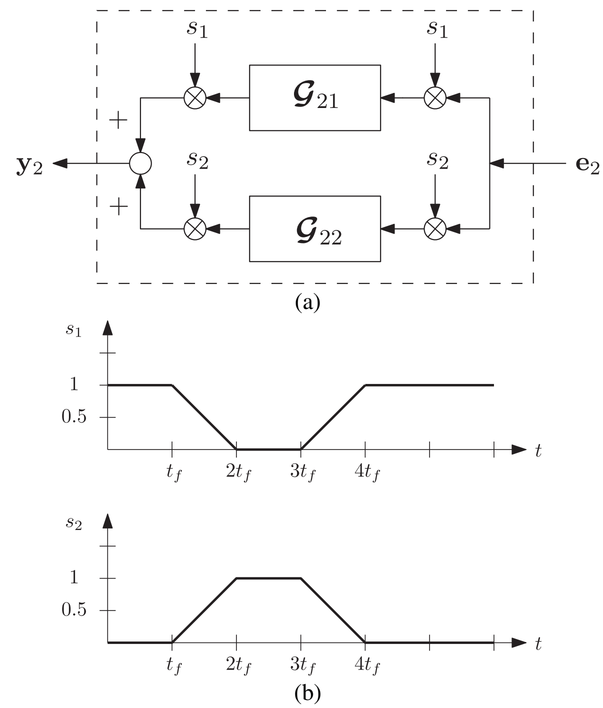

Fig. 3. Scheduling architecture and scheduling signals. (a) Scheduling architecture. (b) Time-dependent scheduling signals.

Now consider the IO mapping where $\dot{\boldsymbol{\theta}}$ is not directly measured, but acquired via differentiation using $\mathbf{F}(s)$, i.e., $\mathbf{y}(s) = \mathbf{G}_1(s)\boldsymbol{\tau}(s)$ where $\mathbf{G}_1(s) = \mathbf{F}(s)\mathbf{G}_p(s)$. The frequency response of this transfer matrix is also plotted in Fig. 2(b). The system $\mathbf{G}_1(s)$ is PR over a specific frequency range; below approximately 100 rad/s the transfer matrix has a Hermitian part that is positive, and hence PR. Above 100 rad/s, the system is no longer PR (i.e., the Hermitian part is negative) but is BR. The system is clearly hybrid passive/finite-gain possessing a frequency response that is FF PR/BR. Assuming that our linearized model accurately approximates the nonlinear system, by using the hybrid passivity/finite-gain stability theorem this system can be stabilized by an FF SPR/BR controller.

V. Controller Synthesis and Experimental Results

Rather than using one FF SPR/BR controller, we will synthesize and use two FF SPR/BR controllers within a scheduling architecture, i.e., we will gain-schedule two FF SPR/BR controllers. Consider the scheduling architecture shown in Fig. 3(a). Notice that the two scheduling signals s_1 and s_2 each influence the input and the output of the FF SPR/BR rate controllers they schedule (while the proportional control is not scheduled). As discussed in [16], this particular scheduling architecture ensures that the overall gain-scheduling controller maintains a hybrid VSP/finite-gain character. A similar scheduling architectures can be found in

[37] and [38] (although these works only consider the control of passive systems that have not experienced a passivity violation).

The scheduling signals may be a function of θ_2 or time. We elect to specify the scheduling signals to be an explicit function of time only. The scheduling signal profiles are shown in Fig. 3(b), where $t_f = 2.5$ s. This form of scheduling is simple to implement, and essentially is scheduling the controllers based on the assumed position of the manipulator. It should be noted, however, that scheduling can be a function of any other variable an engineer wishes to choose.

A. Controller Synthesis Results

The two controllers within the scheduling algorithm, $\mathbf{G}_{21}(s)$ and $\mathbf{G}_{22}(s)$, will each be designed about a specific linearization point: $\mathbf{G}_{21}(s)$ about set point 1, and $\mathbf{G}_{22}(s)$ about set point 2. Set point 1 corresponds to $[-\pi/4\ 0]^{\mathsf{T}}$ rad, while set point 2 corresponds to $[(\pi/4)\ (\pi/3)]^{\mathsf{T}}$ rad. The weights used for controller synthesis are

$$\begin{aligned}\mathbf{B}_1 &= 10\begin{bmatrix}\mathbf{B} & \mathbf{0}\end{bmatrix}\\ \mathbf{C}_1 &= \begin{bmatrix}100\mathbf{C}_p\\ 2\mathbf{C}\\ \mathbf{0}\end{bmatrix}\\ \mathbf{D}_{12} &= \begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\\ \mathbf{D}_{21} &= \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}\end{aligned} \tag{24}$$

where $\mathbf{B}$, $\mathbf{C}_p$, and $\mathbf{C}$ are given in (22) and (23).

To design the FF SPR/BR controller, both ω_c and γ_1 must be estimated. From Fig. 2(b), $\omega_c = 100$ rad/s while $\gamma_1 = 1.25$ rad/(N · m · s). Note that these values are estimated based on a linearization and an assumed filter $\mathbf{F}(s)$; the true nonlinear high-frequency gain may not be the γ_1 we have chosen. However, with no way to calculate a true nonlinear gain, we resort to estimating the high-frequency gain in this way.

The frequency response of the FF SPR/BR controllers synthesized about set points 1 and 2 using the scheme presented in Section III are shown in Fig. 4(a) and (b). The frequency responses of the $\mathcal{H}_2$ controllers used as the basis controllers for the the FF SPR/BR controllers are also shown in Fig. 4(a) and (b) as well. The singular value and Hermitian part profiles of each FF SPR/BR controller mimic the $\mathcal{H}_2$ controller frequency responses as close as possible without violating the low-frequency FF SPR and high-frequency FF BR constraints. In particular, notice that the Hermitian part of the FF SPR/BR controllers trace the Hermitian part of the $\mathcal{H}_2$ controllers, but each always remains positive in the frequency range below ω_c, thus adhering to the low-frequency FF SPR constraint. The gain profile of each FF SPR/BR controller is reduced below that of the $\mathcal{H}_2$ controller so that the constraint $\gamma_1\gamma_2 < 1$ is satisfied as well. Interestingly, above ω_c, the Hermitian part of $\mathbf{G}_{22}(s)$ dips below zero slightly. This is permitted, as the controller is not constrained to be SPR at high frequency, but only BR, such that $\gamma_1\gamma_2 < 1$ holds.

The value of the closed-loop $\mathcal{H}_2$ norm using the $\mathcal{H}_2$ and FF SPR/BR controllers about set point 1 is 178.45 and 201.59,

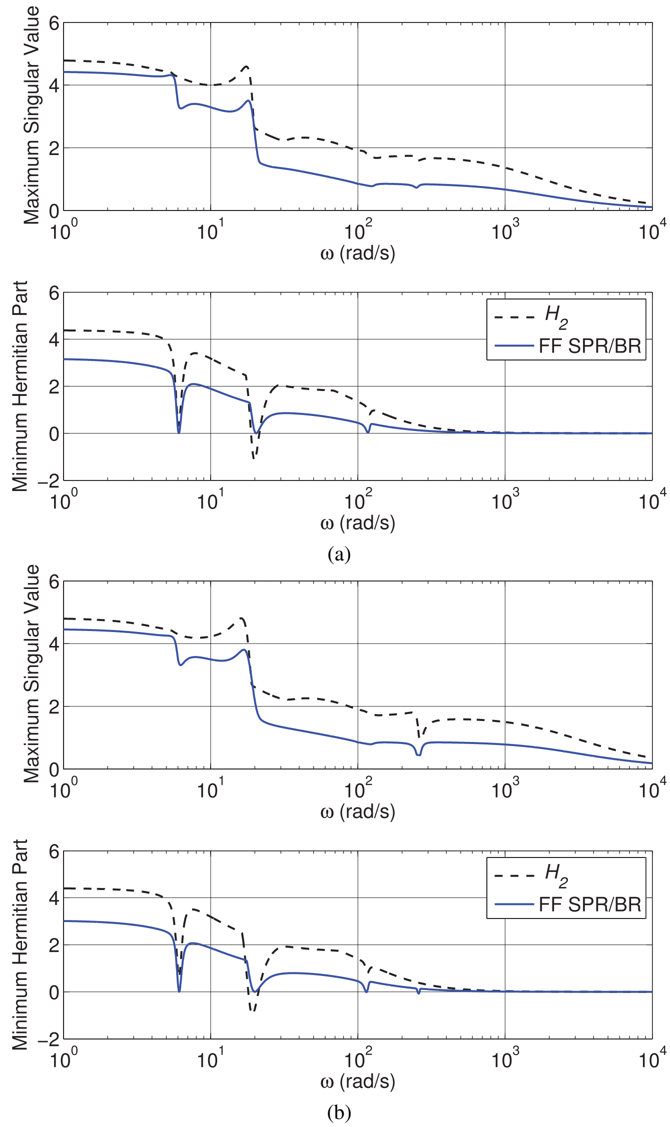

Fig. 4. Frequency response of FF SPR/BR controllers designed about set points 1 and 2. (a) Controller designed about set point 1, $\mathbf{G}_{21}(s)$. (b) Controller designed about set point 2, $\mathbf{G}_{22}(s)$.

respectively. The closed-loop $\mathcal{H}_2$ norm using the $\mathcal{H}_2$ and FF SPR/BR controllers about set point 2 is 221.94 and 247.57, respectively. As expected, when using the FF SPR/BR controllers, the value of the closed-loop $\mathcal{H}_2$ norm is larger because the FF SPR/BR controllers, although designed to mimic the standard $\mathcal{H}_2$ controllers, are constrained and unable to exactly match the value of the closed-loop $\mathcal{H}_2$ norm attained when using the standard $\mathcal{H}_2$ controllers. Shown in Fig. 5(a) and (b) are the maximum singular values of the sensitivity functions associated with the linearized plant controlled by both the $\mathcal{H}_2$ and FF SPR/BR controllers designed about set points 1 and 2. Notice that the difference between the sensitivity functions at low frequency is small owing to the fact that the $\mathcal{H}_2$ and FF SPR/BR controllers do not differ significantly at low frequency.

B. Experimental Results

The FF SPR/BR controllers of Fig. 4(a) and (b) have been used within the scheduling architecture described above to

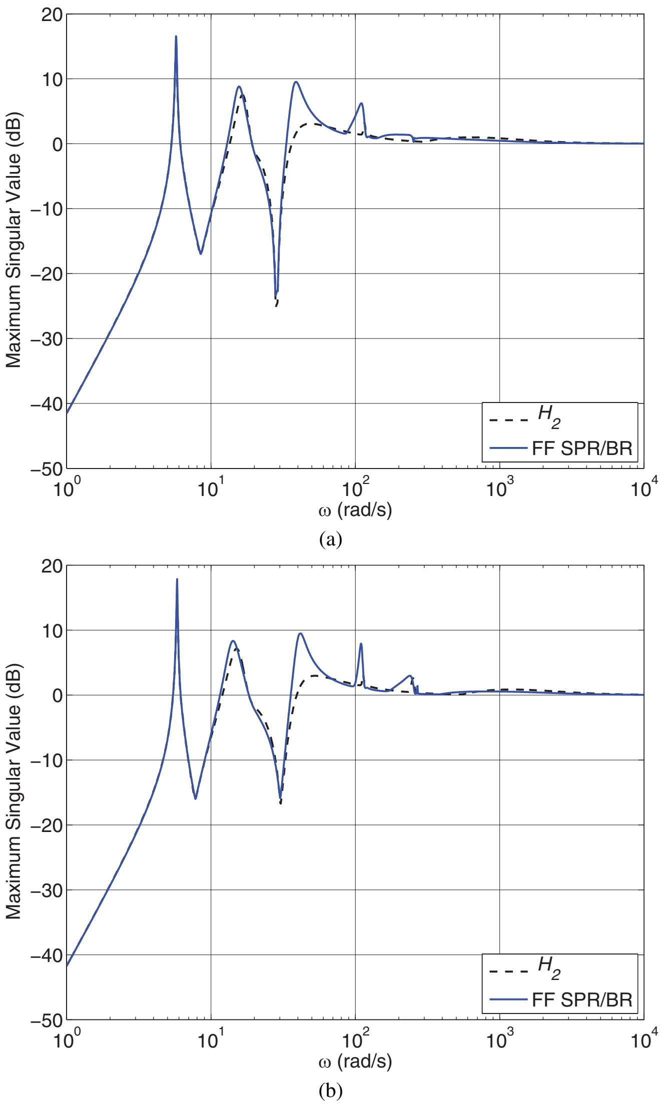

Fig. 5. Maximum singular value of the sensitivity function versus frequency for both $\mathcal{H}_2$ and FF SPR/BR control about set points 1 and 2. (a) Sensitivity function frequency responses about set point 1. (b) Sensitivity function frequency responses about set point 2.

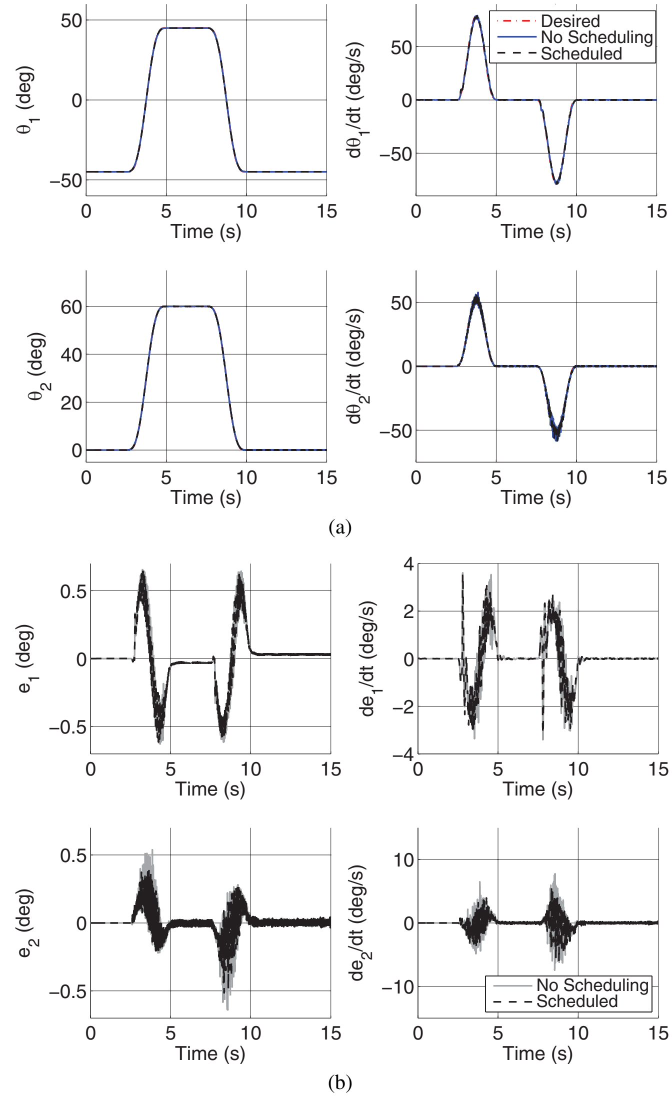

Fig. 6. Two-link experimental results. (a) System response using hybrid control and gain-scheduled hybrid control. (b) Joint position and rate errors.

control the two-link manipulator. The manipulator is to follow a desired trajectory starting at set point 1, moving to set point 2, and then moving back to set point 1. The desired trajectory between set points is

$$\boldsymbol{\theta}_D = \left[10\left(\frac{t}{t_f}\right)^3 - 15\left(\frac{t}{t_f}\right)^4 + 6\left(\frac{t}{t_f}\right)^5\right](\boldsymbol{\theta}_f - \boldsymbol{\theta}_i) + \boldsymbol{\theta}_i$$

where t_f is 2.5 s, $\boldsymbol{\theta}_f$ is the final angular position, and $\boldsymbol{\theta}_i$ is the initial angular position. Between maneuvers, there is a 2.5 s dwell.

Fig. 6(a) shows the position and rate response of the system controlled by set point 1 controller alone (i.e., there is no controller scheduling, and only $\mathbf{G}_{21}(s)$ is used) and the gain-scheduled controller. Fig. 6(b) shows the position and rate error of the system controlled by each scheme where $\mathbf{e} = \boldsymbol{\theta} - \boldsymbol{\theta}_D$. The root mean square (RMS) errors are presented in Table I.

TABLE I
POSITION AND RATE RMS ERRORS

	θ_1 RMS error (rad)	θ_2 RMS error (rad)	$\dot{\theta}_1$ RMS error (rad/s)	$\dot{\theta}_2$ RMS error (rad/s)
Unscheduled	3.586×10^{-3}	1.894×10^{-3}	15.841×10^{-3}	19.095×10^{-3}
Scheduled	3.404×10^{-3}	1.510×10^{-3}	15.670×10^{-3}	15.085×10^{-3}

Although it is perhaps hard to visually discern the quality of the controlled responses, Table I clearly shows that the scheduled FF SPR/BR control scheme realizes lower position and rate errors as compared to the control that uses $\mathbf{G}_{21}(s)$ alone. Although errors associated with the first link do not change significantly (there is only a modest improvement), the errors associated with the second link are improved greatly when the scheduled controller is used.

Controlling the two-link manipulator experimental apparatus using each of the $\mathcal{H}_2$ controllers designed about

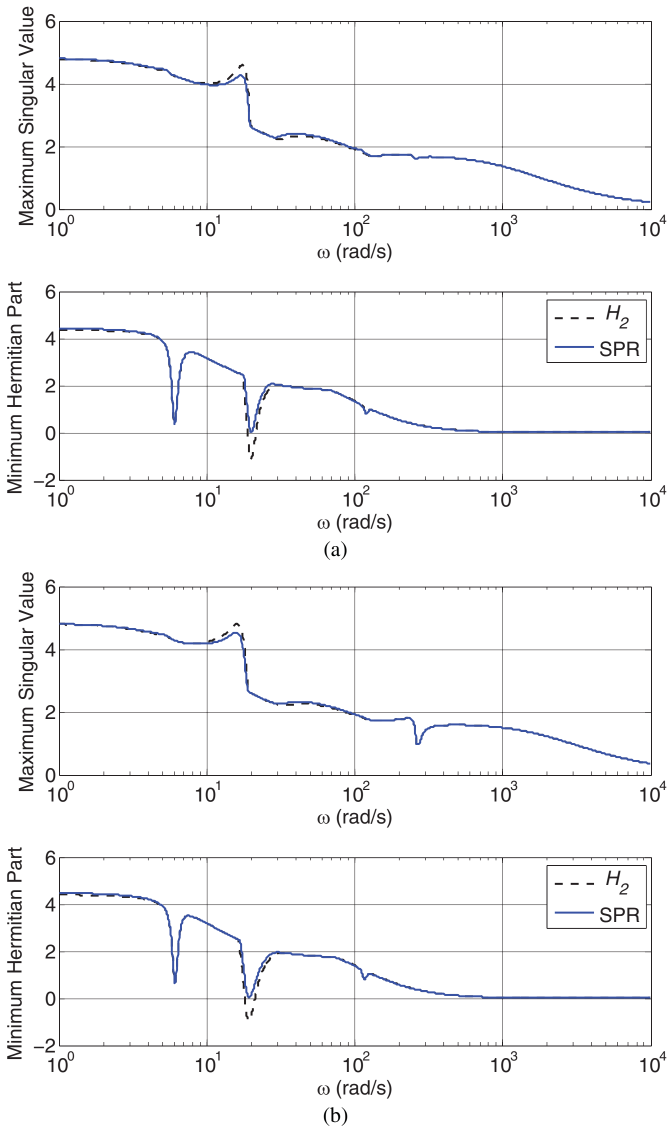

Fig. 7. SPR controllers designed about set points 1 and 2. (a) SPR controller designed about set point 1. (b) SPR controller designed about set point 2.

set points 1 and 2 was attempted, as was gain-scheduling the two $\mathcal{H}_2$ controllers. It was found that the closed loop was unstable when the individual $\mathcal{H}_2$ controllers were used, as well as when each $\mathcal{H}_2$ controller was gain-scheduled. This fact highlights the robust nature of the FF SPR/BR controllers and the gain-scheduling scheme, as well as the inability of the nominal $\mathcal{H}_2$ controllers (designed about set points 1 and 2) to control such a nonideal plant. If more care is taken in modeling the plant, an $\mathcal{H}_2$ controller will be able to control the system. However, as highlighted by these results, the FF SPR/BR controller that mimics the $\mathcal{H}_2$ controller does not need a high-fidelity plant model to successfully control the two-link manipulator system.

In the context of traditional passivity-based control, an SPR controller or two gain-scheduled SPR controllers would be used to control the two-link manipulator system. Not only did we attempt control using FF SPR/BR and standard $\mathcal{H}_2$ controllers, but we also considered closed-loop control using standard SPR controllers (designed about set points 1 and 2) as well as gain-scheduling two SPR controllers. The SPR controllers were synthesized in a very similar way as the FF SPR/BR controllers about set points 1 and 2; see the Appendix for the synthesis procedure. Referring to Fig. 7(a) and (b) in the Appendix, the frequency responses of the two SPR controllers designed about set points 1 and 2 are almost identical to the frequency response of the corresponding traditional $\mathcal{H}_2$ controllers. The SPR controllers have a minimum Hermitian part that is strictly greater than zero, as required. When closed-loop control was attempted, it was found that neither of the SPR controllers designed about set points 1 or 2 could stabilize the closed-loop system, nor could the corresponding gain-scheduled SPR controller. Recall that the two-link system under control has experienced a passivity violation, so neither the individual SPR controllers nor the gain-scheduled SPR controller is assured to stabilize the closed-loop via the traditional passivity theorem.

The most likely reason why $\mathcal{H}_2$, SPR, and corresponding gain-scheduled $\mathcal{H}_2$ and SPR controllers do not stabilize the closed-loop system is related to the high-frequency gain (i.e., the gain above ω_c) of the $\mathcal{H}_2$ and SPR controllers designed about set points 1 and 2. Referring to Figs. 4(a) and (b) and 7(a) and (b), notice that the high-frequency gain of the FF SPR/BR controllers designed about set points 1 and 2 is low compared to both the $\mathcal{H}_2$ and SPR controllers. The unmodeled high-frequency dynamics associated with the passivity violation, as well as other unmodeled dynamics, are most likely destabilized by the high-frequency gain of the $\mathcal{H}_2$ and SPR controllers. Note that, if greater care is taken in modeling the system and better sensors are available (e.g., rate sensors), the $\mathcal{H}_2$ and SPR controllers would be able to stabilize the closed-loop system. What we have shown is that the FF SPR/BR controllers are able to control the system given a low-fidelity plant model and a passivity violation.

VI. Conclusion

In this paper, we investigated the design of optimal FF controllers to control systems, such as flexible robotic manipulators, that have experienced a passivity violation. The contributions of this paper have been: 1) highlighting the connection between the hybrid passivity/finite-gain systems framework and the GKYP lemma; 2) formulating a means to synthesize FF SPR/BR controllers that mimic a nominal $\mathcal{H}_2$ controller to stabilize FF PR/BR plants; and 3) experimentally testing the synthesis procedure by controlling a two-link flexible manipulator.

Future work will focus on other synthesis procedures that yield controllers that are FF SPR/BR. If the LMIs in (11) and (12) are expanded, there are terms such as $\mathbf{A}_c^{\mathsf{T}}\mathbf{Q}_p\mathbf{A}_c$. These product terms make it difficult to parameterize controllers in terms of the dynamic matrix $\mathbf{A}_c$ because the matrix inequality in question is no longer an LMI. In the future, we hope to explore ways to overcome this issue. Another possible avenue we hope to explore is the design of controllers that have three finite-frequency regions with specific SPR or BR properties in each region. For example, motivated by [10] and [11], controllers with low-frequency (i.e., below ω_1) FF

SPR properties, mid-frequency (i.e., between ω_1 and ω_2 where $\omega_1 < \omega_2$) FF SPR and FF BR properties, and high-frequency (i.e., above ω_2) FF BR properties would be of interest. Specifically, the controller synthesis procedure presented in this paper would be applicable provided another mid-frequency LMI constraint was added. It should be noted that FF SPR or FF BR characteristics over mid-frequency ranges (i.e., between ω_1 and ω_2) can be characterized using the GKYP lemma [13], [20].

Appendix

In this appendix, we will consider the optimal design of traditional SPR controllers, i.e., controllers that are SPR over all frequencies. Rather than using one of the synthesis methods presented in [28]–[31], we will modify the FF SPR/BR controller design procedure in Section III in order to present a fair comparison of closed-loop control of the two-link flexible manipulator system using FF SPR/BR controllers and SPR controllers, both used alone and within a gain-scheduling algorithm.

Consider $\mathbf{G}_2(s) = \mathbf{K}_o(s\mathbf{1} - \mathbf{A}_c)^{-1}\mathbf{K}_e$, where the dynamic matrix $\mathbf{A}_c$ and input matrix $\mathbf{K}_e$ are taken from a standard $\mathcal{H}_2$ controller design. The dynamic matrix $\mathbf{A}_c$ is assumed to be Hurwitz, and $(\mathbf{A}_c, \mathbf{K}_e)$ and $(\mathbf{K}_o, \mathbf{A}_c)$ are assumed to be controllable and observable, respectively. During optimization, $\mathbf{A}_c$ and $\mathbf{K}_e$ are held fixed. As in Section III, we wish to design a $\mathbf{K}_o$ that renders the controller $\mathbf{G}_2(s)$ SPR. Additionally, we want $\mathbf{K}_o$ to render $\mathbf{G}_2(s)$ as close to the nominal $\mathcal{H}_2$ controller as possible. Recall that $\mathbf{G}_2(s)$ will be SPR if there exists $\mathbf{P}_c = \mathbf{P}_c^{\mathsf{T}} > 0$ such that [29]

$$\mathbf{P}_c\mathbf{A}_c + \mathbf{A}_c^{\mathsf{T}}\mathbf{P}_c < 0. \tag{25a}$$

$$\mathbf{K}_e^{\mathsf{T}}\mathbf{P}_c = \mathbf{K}_o. \tag{25b}$$

The LMI given in (25a) will be used to constrain $\mathbf{G}_2(s)$ to be SPR. Consider the following objective function:

$$\begin{aligned}\mathcal{J} &= \operatorname{tr}\left[(\mathbf{K}_o - \mathbf{K}_c)(\mathbf{K}_o - \mathbf{K}_c)^{\mathsf{T}}\right] \\ &= \operatorname{tr}\left[(\mathbf{K}_e^{\mathsf{T}}\mathbf{P}_c - \mathbf{K}_c)(\mathbf{K}_e^{\mathsf{T}}\mathbf{P}_c - \mathbf{K}_c)^{\mathsf{T}}\right] \end{aligned} \tag{26}$$

where $\mathbf{K}_c$ is the state-feedback gain from the nominal $\mathcal{H}_2$ controller, and (25b) has been used to replace $\mathbf{K}_o$ with $\mathbf{K}_e^{\mathsf{T}}\mathbf{P}$. Using the slack variable $g \in \mathbb{R}^+$ and the positive-semidefinite slack matrix $\mathbf{Z} \in \mathbb{R}^{n_u \times n_u}$, we can write the objective function given in (26) as

$$\mathcal{J} = g$$

$$\operatorname{tr}[\mathbf{Z}] \le g \tag{27a}$$

$$\begin{bmatrix} \mathbf{Z} & (\mathbf{K}_e^{\mathsf{T}}\mathbf{P}_c - \mathbf{K}_c) \\ (\mathbf{K}_e^{\mathsf{T}}\mathbf{P}_c - \mathbf{K}_c)^{\mathsf{T}} & \mathbf{1} \end{bmatrix} \ge 0. \tag{27b}$$

In summary, the SPR synthesis procedure is as follows:

$$\begin{aligned} \text{minimize} \quad & \mathcal{J}(\mathbf{K}_o, \mathbf{P}_c, g, \mathbf{Z}) = g \\ \text{with respect to} \quad & \mathbf{K}_o,\ \mathbf{P}_c\ , g\ \mathbf{Z} \\ \text{s.t.} \quad & \mathbf{P}_c = \mathbf{P}_c^{\mathsf{T}} > 0, \quad (25a) \\ & \mathbf{Z} = \mathbf{Z}^{\mathsf{T}} \ge 0, \quad (27a), \quad (27b). \end{aligned}$$

As in Section V, the two-link flexible manipulator dynamics will be linearized about set points 1 and 2, and two SPR controllers will be synthesized using the procedure outlined in this section. The weights used to design traditional $\mathcal{H}_2$ controllers about the same set points are the same weights given in (24). The frequency response of the two SPR controllers are shown in Fig. 7(a) and (b) along with the frequency response of the two $\mathcal{H}_2$ used as the basis for design. Notice that: 1) the minimum Hermitian part of each controller is strictly greater than zero and 2) the SPR controllers mimic the $\mathcal{H}_2$ very closely.

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