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On the Design of Strictly Positive Real Transfer Functions

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Abstract—**The synthesis of strictly positive real transfer functions is considered. For a given Hurwitz polynomial of degree n comprising the denominator polynomial, necessary and sufficient conditions on the numerator which render a rational function strictly positive real are given. In the case where the function is strictly proper, a parameterization of the polynomial numerator by n real numbers satisfying a simple constraint is provided. The approach taken employs factorization of a polynomial into its even and odd parts. The relationship of the results to those provided by the Kalman–Yakubovich Lemma is given and the present method shown to have certain advantages.**

I. INTRODUCTION

An important concept encountered in systems and circuit theory is that of *passivity*. Roughly speaking a system (linear or not) is strictly passive if it "consumes" energy and it is passive if it does not "deliver" energy. This concept was first used in circuit theory motivated by the fact that networks containing RLC elements are passive and it has become a fundamental tool in the stability analysis of feedback systems, [1]. Restricting our attention to causal, linear time-invariant systems, these concepts are closely related to the notions of *positive real* and *strictly positive real* [2], [3]. If the transfer function $H(s)$ of a system is positive real (PR), then the system is passive. Moreover, a feedback interconnection containing a passive subsystem (linear or not), and a strictly proper, strictly positive real (SPR) one, is always closed-loop stable [4].

Definition: Let $\mathcal{P}^n$ denote the set of real polynomials of nth degree in the indetermined variable s. Consider a rational function $H(s) = p(s)/q(s)$, where $p(s) \in \mathcal{P}^n$ and $q(s) \in \mathcal{P}^m$. Then, $H(s)$ is said to be in the class $\mathcal{Q}$ if and only if, (i) $q(s)$ is a Hurwitz polynomial (*i.e.*, all of its roots lie in the open left half of the complex plane) and (ii) $Re[H(\jmath\omega)] > 0$, $\forall\omega \in [0, \infty)$. $H(s)$ is said to be (weak) strictly positive real (SPR) if it is in the class $\mathcal{Q}$ and the degrees of the numerator and denominator polynomials differ by -1, 0, or 1. Assume now that $H(s)$ is SPR and strictly proper. Then, $H(s)$ is said to be strong SPR, or simply SPR, [3], if in addition it satisfies

$$\lim_{\omega\to\infty} \omega^2 Re[H(\jmath\omega)] > 0. \tag{1}$$

Remarks: It is important to recognize the difference between an SPR function and one that is merely in the class $\mathcal{Q}$. Clearly, if $H(s)$ is SPR then $H(s) \in \mathcal{Q}$. The converse is however not true. For example the function $(s+1)^{-1} + s^3$ is in $\mathcal{Q}$ but is not SPR. In fact, it is not even PR. In the sequel, we shall denote by $\mathcal{S}$ the set of SPR rational functions.

Motivated by practical applications in control theory and adaptive control schemes, we are interested in the construction of transfer functions which are strictly positive real. More explicitly, we study the following problems:

- Given a Hurwitz polynomial $q \in \mathcal{P}^n$, find necessary and sufficient conditions for a polynomial p to belong to the set $\mathcal{P} = \{p \in \mathcal{P}^n \mid H(s) = p(s)/q(s) \in \mathcal{Q}\}$
- Given a Hurwitz polynomial q find the subset $\mathcal{P}_{\text{sp}} \subset \mathcal{P}$ given by

  $$\mathcal{P}_{\text{sp}} = \{p \in \mathcal{P} \mid \lim_{s\to\infty} H(s) = 0\}$$

  Hence, if $p \in \mathcal{P}_{\text{sp}}$, then p/q is strictly proper and SPR. Moreover, it will be shown that the set $\mathcal{P}_{\text{sp}}$ can be parameterized using n real numbers that satisfy a simple constraint.
- Given a Hurwitz polynomial q find the subset $\mathcal{SP}_{\text{sp}} \subset \mathcal{P}_{\text{sp}}$ given by

  $$\mathcal{SP}_{\text{sp}} = \{p \in \mathcal{P}_{\text{sp}} \mid \lim_{\omega\to\infty} \omega^2 Re[H(\jmath\omega)] > 0\}.$$

Problems (i) and (ii) are important in the design of adaptive systems [5] and in applications where the plant under control is known to be passive, but there exists large uncertainty in the actual model, such as in the control of flexible structures [6]. Problem (ii) is a realizability condition. Namely, only strictly proper rational functions can be implemented using an actual physical device. SPR transfer functions may be characterized using the Kalman–Yakubovich Lemma [2], [3], but the parameterization presented here has certain advantages over that solution in the case of scalar transfer functions.

II. NECESSARY AND SUFFICIENT CONDITIONS FOR SPR

In this section we solve problems (i)-(iii) as stated above. Our approach to this problem is inspired by the algebraic theory of control initiated by Desoer *et al.*, [7]. There are however substantial differences between the two methologies. While the factorization approach of Desoer *et al.* is based on the fact that the set of proper and stable rational functions forms a ring, the set of SPR functions is not closed under multiplication and therefore does not form a ring.

Theorem 1: Let $q \in \mathcal{P}^n$ be Hurwitz. Then $p \in \mathcal{P}^m$ belongs to the set $\mathcal{P} = \{p \in \mathcal{P}^m \mid H = p/q \in \mathcal{Q}\}$, if and only if there exists functions $u(s), v(s), r(s)$, and $k(s)$ such that

$$p(s) = q(s)r(s) + [u(s) - v(s)]k(s) \tag{2}$$

where $u(s)$ and $v(s)$ are, respectively, even and odd polynomials which satisfy the Bezout identity,

$$q_e(s)u(s) + q_o(s)v(s) = 1. \tag{3}$$

Here, q_e and q_o designate the even and odd parts of $q(s)$. The function $r(s)$ is an arbitrary odd polynomial and $k(s)$ is an even polynomial that satisfies the inequality, $k(\jmath\omega) > 0$.

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Proof: Consider a rational function $H(s) = p(s)/q(s)$ where q is Hurwitz. We partition $p(s)$ and $q(s)$ into their even and odd parts[1]

$$H(s) = \frac{p(s)}{q(s)} = \frac{p_e + p_o}{q_e + q_o} = \frac{p(s)q(-s)}{q(s)q(-s)} = \frac{[p_e(s)q_e(s) - p_o(s)q_o(s)] + [q_e(s)p_o(s) - p_e(s)q_o(s)]}{q_e^2(s) - q_o^2(s)} \tag{4}$$

where p_e, q_e are the even parts of p and q, and p_o, q_o are the odd parts of p and q, respectively. It follows that $[q_e(s)p_o(s) - p_e(s)q_o(s)]$ is the odd part of the numerator of (4) and

$$k(s) = p_e(s)q_e(s) - p_o(s)q_o(s) \tag{5}$$

is the even part. Since $[q_e^2(\jmath\omega) - q_o^2(\jmath\omega)]$ is always positive, it follows that $H(s) \in \mathcal{Q}$ if and only if $q(\cdot)$ is Hurwitz and $k(\jmath\omega) > 0$ where $k(s)$ is an even polynomial.

Given $k(s)$, all possible solutions of (5) can be obtained by adding the homogeneous solution and a particular solution. It is immediate that $p_{eh} = q_o r$ and $p_{oh} = q_e r$ satisfy the homogeneous equation $p_{eh}(s)q_e(s) - p_{oh}(s)q_o(s) = 0$ for any odd polynomial $r(s)$. To find a particular solution we use the fact that q_e and q_o are coprime (see Lemma 1 in the Appendix), i.e., there exists an even function u and an odd function v which satisfy (3). Multiplying (3) by $k(s)$ we obtain, $q_e(s)u(s)k(s) + q_o(s)v(s)k(s) = k(s)$ which when compared to (5) yields

$$p_{ep}(s) = u(s)k(s)\ , \quad p_{op}(s) = -v(s)k(s). \tag{6}$$

It follows that,

$$p_e(s) = p_{eh}(s) + p_{ep}(s) = q_o(s)r(s) + u(s)k(s) \tag{7}$$

$$p_o(s) = p_{oh}(s) + p_{op}(s) = q_e(s)r(s) - v(s)k(s) \tag{8}$$

and necessity is obtained by forming $p = p_e + p_o$. To prove sufficiency, note that (2) implies (5) upon using the factorization (7) and (8). □

Remarks: While Theorem 1 gives necessary and sufficient conditions for a polynomial p to belong to the set $\mathcal{P}$, for a given Hurwitz q, it does not provide a method for finding the polynomial $r(s)$. In general, the solution of $r(s)$ depends on the desired relative order of the polynomial p with respect to the given q. For physical realizability, we concentrate on the subset $\mathcal{P}_{\rm sp}$ of $\mathcal{P}$ which makes H strictly proper. It will be demonstrated that $r(s)$ is uniquely determined by the strictly proper constraint.

Theorem 2: Given a Hurwitz polynomial q, of degree $n \geq 2$, the set $\mathcal{P}_{\rm sp} \subset \mathcal{P}$ of all polynomials satisfying,

$$\mathcal{P}_{\rm sp} = \{p \in \mathcal{P}^{n-1} \mid H(s) = p(s)/q(s) \in \mathcal{S}\ , \ \lim_{s\to\infty} H(s) = 0\}$$

can be parameterized using n real numbers $k_1, k_2, \cdots, k_n$ which are chosen such that

$$\hat{k}(x) = k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n > 0\ , \quad \forall x \geq 0 \tag{9}$$

but are otherwise arbitrary.

Proof: According to Theorem 1, if $p \in \mathcal{P}_{\rm sp} \subset \mathcal{P}$ then p satisfies (2) with r even and k odd. The degree of k must be chosen such that (5) is satisfied. Suppose first that n is even. In this case we have, $\partial(q_e) = n$, $\partial(q_o) = n-1$, $\partial(p_e) = n-2$, and $\partial(p_o) = n-1$. (The symbol $\partial(p)$ denotes the degree of p). It follows from (5) that

$$\partial(k) = \partial(p_e) + \partial(q_e) = \partial(p_o) + \partial(q_o) = 2n - 2. \tag{10}$$

[1]This partition is classical in the networks literature. See, for example, [8] or [9].

Similarly, if n is odd, $\partial(q_e) = n-1$, $\partial(q_o) = n$, $\partial(p_e) = n-1$, $\partial(q_o) = n-2$, and $\partial(k) = 2n-2$. We conclude that, if q has degree n, then k must have degree $2n-2$. We also argue that $\partial(u) = n-2$, and $\partial(v) = n-1$, as can be seen by a simple application of the Euclidean algorithm. Thus, $\partial[(u-v)k] = 3n-3$.

It follows from (2) that $\partial(p) = n-1$ if and only if $\partial(r) = 2n-3$ and $r(\cdot)$ is chosen such that the coefficients of the first $2n-2$ powers contained in qr are cancelled with the corresponding coefficients contained in $(u-v)k$. Since r is an odd polynomial, we define it as follows,

$$r(s) = r_1 s^{2n-3} + r_2 s^{2n-5} + \cdots + r_{n-1}s. \tag{11}$$

Similarly, an even polynomial k that satisfies $k(\jmath\omega) > 0$ can be represented by

$$k(s) = [(-1)^{n-1}k_1 s^{2n-2} + (-1)^{n+2}k_2 s^{2n-4} + \cdots - k_{n-1}s^2 + k_n]. \tag{12}$$

The constraints imposed on the k_i, $i = 1 \cdots n$, by $k(\jmath\omega) > 0$ will be addressed later.

Let $q = s^n + a_1 s^{n-1} + \cdots + a_n$, and assume without loss of generality that n is even. Then

$$q_e = s^n + a_2 s^{n-2} + \cdots + a_n\ , \quad q_o = a_1 s^{n-1} + a_3 s^{n-3} + \cdots + a_{n-1}s. \tag{13}$$

To find $u(s)$ and $v(s)$, we use the Euclidean algorithm. We have,

$$q_e = q_o\Delta_0 + \xi_1\ , \quad 0 \leq \partial(\xi_1) < \partial(q_o) \tag{14}$$

$$\begin{aligned} q_o &= \xi_1\Delta_1 + \xi_2\ , \quad 0 \leq \partial(\xi_2) < \partial(\xi_1) \\ \xi_1 &= \xi_2\Delta_2 + \xi_3\ , \quad 0 \leq \partial(\xi_3) < \partial(\xi_2) \\ &\vdots \\ \xi_{n-3} &= \xi_{n-2}\Delta_{n-2} + \xi_{n-1}\ , \quad \xi_{n-1} \in R/\{0\} \\ &\qquad \text{(last nonzero remainder)}. \end{aligned} \tag{15}$$

Here each $\Delta_i(s)$, being the quotient of an even (odd) polynomial of degree $m \leq n$, and an odd (even) polynomial of degree $m-1$, has the form

$$\Delta_i(s) = \delta_i s\ , \quad \delta_i \neq 0\ , \ i = 1, \cdots, n-1 \tag{16}$$

u and v can be obtained by solving (15) for ξ_{n-1} and substituting backward in the remainders of the previous equations, i.e.,

$$\begin{aligned} \xi_{n-1} &= \xi_{n-3} - \xi_{n-2}\Delta_{n-2} \\ \Rightarrow 1 &= [\xi_{n-3} - \xi_{n-2}\Delta_{n-2}]/\xi_{n-1} \\ &= [\xi_{n-3}(1 + \Delta_{n-3}\Delta_{n-2}) - \xi_{n-4}\Delta_{n-2}]/\xi_{n-1} \end{aligned} \tag{17}$$

and so on. The final result has the following form

$$1 = \Big\{[\delta_1\delta_2\cdots\delta_{n-2}s^{n-2} + \cdots]q_e(s) - [\delta_0\delta_1\delta_2\cdots\delta_{n-2}s^{n-1} + \cdots]q_o(s)\Big\}/\xi_{n-1}.$$

Therefore,

$$\begin{aligned} u(s) &= [\delta_1\delta_2\cdots\delta_{n-2}s^{n-2} + \cdots]/\xi_{n-1}, \\ v(s) &= -[\delta_0\delta_1\delta_2\cdots\delta_{n-2}s^{n-1} + \cdots]/\xi_{n-1} \end{aligned} \tag{18}$$

and substituting (11)–(13) and (18) into (2) gives

$$\begin{aligned} p(s) &= qr + (u-v)k \\ &= [s^n + a_1 s^{n-1} + \cdots + a_n] \\ &\quad \times [r_1 s^{2n-3} + r_2 s^{2n-5} + \cdots + r_{n-1}s] \\ &\quad + [\delta_0\delta_1\delta_2\cdots\delta_{n-2}s^{n-1} + \delta_1\delta_2\cdots\delta_{n-2}s^{n-2} + \cdots] \\ &\quad \times [-k_1 s^{2n-2} + k_2 s^{2n-4} + \cdots]/\xi_{n-1} \\ &= s^{3n-3}[r_1 - k_1\delta_0\delta_1\delta_2\cdots\delta_{n-2}/\xi_{n-1}] \\ &\quad + s^{3n-4}[a_1 r_1 - k_1\delta_1\delta_2\cdots\delta_{n-2}/\xi_{n-1}] + \cdots. \end{aligned}$$

To cancel the highest order power of p, we chose

$$r_1 = k_1\delta_0\delta_1\delta_2\cdots\delta_{n-2}/\xi_{n-1}. \tag{19}$$

However, with this election, the coefficient of s^{3n-4} is given by

$$a_1 r_1 - k_1\delta_1\delta_2\cdots\delta_{n-2}/\xi_{n-1} = k_1\delta_1\delta_2\cdots\delta_{n-2}(a_1\delta_0 - 1)/\xi_{n-1}.$$

From (14) we have $q_e = q_o\Delta_0 + \xi_1$ and dividing q_e by q_o we have that $\Delta_0(s) = s/a_1$. Therefore, $\delta_0 = 1/a_1$ so that the coefficient of s^{2n-4} is also zero. The rest of the first $2n-2$ coefficients are cancelled similarly. Therefore $p(s)$ is parameterized in terms of the n parameters k_i. From (9) and (12), $k(j\omega) = \hat{k}(x) > 0$ where $x = \omega^2 \geq 0$.

It remains to show that the if $k(s)$ is selected subject to (9) and (12), that $p(s)$ constructed according to (2) will be a strictly proper SPR function. To see this, realize that (2) necessitates (5). It is then clear from the properties of the k_i, $i = 1\cdots n$, that $Re\{H(j\omega)\} > 0$. Given the parameters k_i, $i = 1\cdots n$, the coefficients of $p(s)$ can be determined directly from (2). Let us take

$$p_e = c_2 s^{n-2} + \cdots + c_n\,, \quad p_o = c_1 s^{n-1} + c_3 s^{n-3} + \cdots + c_{n-1}s.$$

Substituting for p_e, p_o, q_e, and q_o into (5) and matching powers of s leads to the algebraic expression

$$\begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ -a_3 & a_2 & -a_1 & 1 & \cdots & \vdots \\ a_5 & -a_4 & a_3 & -a_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \vdots \\ 0 & 0 & -a_n & a_n - 1 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ \vdots \\ k_{n-1} \\ k_n \end{bmatrix}. \tag{20}$$

The degree of $p(s)$ will be $n-1$ and $Re[H(j\omega)] > 0$ by construction. □

Comments: The parameterization described in Theorems 1 and 2 generates all weak and strong SPR transfer functions that have a given denominator. From (9), it is sufficient that $k_i > 0$, $i = 1\cdots n$. By considering $\omega = 0$, it is necessary that $k_n > 0$. In general, $\hat{k}(x)$ must have no real, positive roots. This is a classical problem which has a solution in the form of Sturm's Theorem [9], [10]. For a given polynomial, the theorem provides a test yielding the number of roots in an open interval. By requiring no real roots in the interval $[0, \infty]$, one can generate the required conditions on the k_i. This problem has been solved in [10] for $n = 2$, 3, 4 but for $n \geq 5$ the Sturm test is very difficult to use symbolically. However, the constraint in (9) is amenable to numerical implementation by enforcing $\hat{k}(x_i) \geq \epsilon > 0$ at N discrete points $x_i > 0$ in addition to $k_n \geq \epsilon$. These relationships define simple convex, in fact linear, constraints on the k_i. Notice that, although the polynomial $r(s)$ played a key role in the proof of Theorem 2, it is not required in practice since the procedure described in Theorem 2 uniquely determines the coefficients of $p(s)$ as a function of the coefficients of $k(s)$ and $q(s)$ using (20).

In stability theory, one is usually interested in strong SPR transfer functions. The following theorem isolates the subset of strong SPR transfer functions from the set $\mathcal{P}_{\text{sp}}$.

Theorem 3: Given a Hurwitz polynomial $q$of degree $n \geq 2$, the polynomial $p \in \mathcal{P}^{n-1}$ belongs to the set $\mathcal{SP}_{\text{sp}} \subset \mathcal{P}_{\text{sp}} \subset \mathcal{P}$ satisfying

$$\mathcal{SP}_{\text{sp}} = \{p \in \mathcal{P}^{n-1} \mid H(s) = p(s)/q(s) \in \mathcal{S}\,,\ \lim_{\omega\to\infty} \omega^2 Re[H(j\omega)] = \rho > 0\}$$

if and only if $k_1 \neq 0$ in the parameterization of Theorem 2.

Proof: From (4), we have $Re[H(j\omega)] = [p_e q_e - p_o q_o]/[q_e^2 - q_o^2]$, and from (5), $p_e q_e - p_o q_o = k$. As $\omega \to \infty$, $k(j\omega) \to k_1\omega^{2n-2}$, and $[q_e^2(j\omega) - q_o^2(j\omega)] \to \omega^{2n}$. Hence,

$$\lim_{\omega\to\infty} \omega^2 Re[H(j\omega)] = k_1$$

and the result follows. □

It is of interest to compare the present characterization of strong SPR functions to that in the following theorem. See reference [11], for example, for the proof.

Theorem 4: (Kalman–Yakubovich). Let

$$H(s) = \frac{p(s)}{q(s)} = \frac{c_1 s^{n-1} + c_2 s^{n-2} + \cdots + c_n}{s_n + a_1 s^{n-1} + \cdots + a_n} = \mathbf{c}^T(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{b} \tag{21}$$

and assume $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{c}^T, \mathbf{A})$ is observable. Then $H(s)$ is strong SPR if and only if there exist positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}\,, \quad \mathbf{Pb} = \mathbf{c}. \tag{22}$$

For scalar H we can without loss in generality take

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}. \tag{23}$$

From (22), all strong SPR functions can be parameterized by the $n(n+1)/2$ free entries in the matrix $\mathbf{Q}$ subject to positivity of the principal minors. However, it is clear from the previous results of the paper, that this parameterization can be replaced by the n parameters k_i subject to the constraints noted. This will be made clear in the context of an example.

To garner further insight into the apparent over parameterization provided by Theorem 4, let $y(t)$ be the output of the system (21) with input $u(t)$. Defining $\mathcal{E}(t) \triangleq \frac{1}{2}\mathbf{x}(t)\mathbf{Px}$, it is readily shown using (21) and (22) that

$$\dot{\mathcal{E}}(t) = y(t)u(t) - \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx}.$$

Willems [12] has termed $\mathcal{E}$ the storage function and the quadratic form involving $\mathbf{Q}$ the dissipation rate. Using the above, we can write

$$\int_0^T yu\,dt = \mathcal{E}(T) + \tfrac{1}{2}\mathbf{x}^T(T)\mathbf{Qx}(T) \tag{24}$$

assuming $\mathbf{x}(0) = \mathbf{0}$. Clearly $\mathbf{Q}$ uniquely determines $\mathbf{P}$ and hence $\mathbf{c}$ using (22). In [12], the problem of determining $\mathbf{P}$ (and thus $\mathbf{Q}$) which satisfies (22) for given $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ was studied and shown to not necessarily have a unique solution. In general, the admissible $\mathbf{P}$ form a subset of a linear manifold of dimension $(n-m)(n-m+1)/2$ ($m = \operatorname{rank}\mathbf{b}$). Furthermore, these values of $\mathbf{P}$ satisfy $\mathbf{O} \leq \mathbf{P}^- \leq \mathbf{P} \leq \mathbf{P}^+$ where the matrices $\mathbf{P}^+$ and $\mathbf{P}^-$ are uniquely determined by the system and satisfy $(\mathbf{P}^+ - \mathbf{P}^-)\mathbf{b} = \mathbf{0}$. Hence, if $n = m$ and $\operatorname{rank}\mathbf{b} = n$, then $\mathbf{P} = \mathbf{P}^+ = \mathbf{P}^-$ is uniquely determined and therefore so is $\mathbf{Q}$.

In the SISO case treated here, $m = 1$ and there will be at most $n(n-1)/2$ extra parameters in $\mathbf{P}$ and $\mathbf{Q}$. These extra degrees of freedom do not affect the left-hand side of (24) (they don't alter the input–output relationship) but they do change the balance between the stored energy and dissipation rate in the internal states of the system. In contrast to this, the parameterization presented in (20) uniquely determines the n parameters k_i given $\mathbf{c}$ and vice versa since we have directly exploited the input–output representation of $H(s)$.

III. Examples

Example 1: Consider the following polynomial $q \in \mathcal{P}^3$,

$$q(s) = s^3 + as^2 + bs + c \tag{25}$$

and assume that $q(s)$ is Hurwitz. We want to find the set $\mathcal{P}_{\rm sp}$ that corresponds to this polynomial. Separating $q$into its even and odd parts and using the Euclidean algorithm we find,

$$u(s) = \frac{1}{c} + \frac{as^2}{abc - c^2}, \quad v(s) = \frac{-a^2 s}{abc - c^2}.$$

Since $\partial(q) = 3$, we must choose $\partial(r) = 3$, and $\partial(k) = 4$. Thus,

$$r(s) = r_1 s^3 + r_2 s, \quad k(s) = k_1 s^4 - k_2 s^2 + k_3$$

and (9) with $x = \omega^2$ implies that

$$k_1 \geq 0, \quad k_3 > 0, \quad k_2 > -2\sqrt{k_1 k_3}. \tag{26}$$

From (2), any $p(s) \in \mathcal{P}_{\rm sp}$ is given by

$$p(s) = q(s)r(s) + [u(s) - v(s)]k(s).$$

Expanding the products we obtain

$$\begin{aligned} p(s) = &\left[r_1 + \frac{ak_1}{abc - c^2}\right]s^6 + \left[ar_1 + \frac{a^2 k_1}{abc - c^2}\right]s^5 \\ &+ \left[br_1 + r_2 + \frac{k_1}{c} - \frac{ak_2}{abc - c^2}\right]s^4 \\ &+ \left[ar_2 + cr_1 - \frac{a^2 k_2}{abc - c^2}\right]s^3 + \left[br_2 + \frac{ak_3}{abc - c^2} - \frac{k_2}{c}\right]s^2 \\ &+ \left[cr_2 + \frac{a^2 k_3}{abc - c^2}\right]s + \frac{k_3}{c}. \end{aligned} \tag{27}$$

Choosing

$$r_1 = -\frac{ak_1}{abc - c^2}, \quad r_2 = \frac{k_1 c + ak_2}{abc - c^2} \tag{28}$$

we can eliminate the coefficients of s^6, s^5, s^4, and s^3. Substituting (28) into (27) we obtain

$$p(s) = \left[\frac{bck_1 + ck_2 + ak_3}{abc - c^2}\right]s^2 + \left[\frac{c^2 k_1 + ack_2 + a^2 k_3}{abc - c^2}\right]s + \frac{k_3}{c} \tag{29}$$

or

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{abc - c^2}\begin{bmatrix} bc & c & a \\ c^2 & ac & a^2 \\ 0 & 0 & ab - c \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}. \tag{30}$$

It is readily verified that the inverse of the coefficient matrix coincides with that given by (20) when $n = 3$.

Let us compare this solution with that furnished by Theorem 4. Form the matrices $\mathbf{A}$ and $\mathbf{b}$ according to (23) and write $\mathbf{Q} = \text{matrix}\{q_{ij}\}$. The solution of the Lyapunov equation (22) for $\mathbf{P}$ yields

$$\begin{aligned} P_{11} &= \frac{q_{11}bc + (q_{22} - 2q_{13})c + q_{33}a}{2(abc - c^2)}, \\ P_{12} &= \frac{q_{11}c^2 + (q_{22} - 2q_{13})ac + q_{33}a^2}{2(abc - c^2)}, \\ P_{13} &= \frac{q_{33}}{2c}. \end{aligned} \tag{31}$$

Therefore, the solution for $p(s)$ is given by

$$\begin{aligned} p(s) = &\left[\frac{bcq_{11} + (q_{22} - 2q_{13})c + aq_{33}}{2(abc - c^2)}\right]s^2 \\ &+ \left[\frac{c^2 q_{11} + ac(q_{22} - 2q_{13}) + a^2 q_{33}}{2(abc - c^2)}\right]s + \frac{q_{33}}{2c}. \end{aligned} \tag{32}$$

Making the identifications

$$k_1 = \frac{q_{11}}{2}, \quad k_2 = \frac{q_{22}}{2} - q_{13}, \quad k_3 = \frac{q_{33}}{2} \tag{33}$$

it is clear that this is equivalent to (29).

The remaining question is whether the positive definiteness of $\mathbf{Q}$ implies the required conditions on k_1, k_2, k_3. Clearly $q_{11} > 0$ and $q_{33} > 0$ imply that $k_1 > 0$ and $k_3 > 0$. Interchanging the second and third rows of $\mathbf{Q}$ and examination of the second principal minor leads to the condition $q_{11}q_{33} - q_{13}^2 > 0$. In terms of the k_i, this becomes $4k_1k_3 - [k_2 - (q_{22}/2)]^2 > 0$ or $|k_2 - (q_{22}/2)| < 2\sqrt{k_1 k_3}$. Since $q_{22} > 0$, this is equivalent to that in (26) for k_2. We conclude that the approach taken here yields a result which is equivalent to the Kalman–Yakubovich Lemma. The former approach has $n = 3$ free parameters with simple constraints whereas the latter involves six parameters in $\mathbf{Q}$, not all of which are required.

Example 2: Our second example is designed to emphasize the distinction between weak and strong SPR as well as illustrate the nonuniqueness of $\mathbf{Q}$ given $H(s)$. Consider the second degree polynomial $q(s) = (s + a)(s + b)$, $a > 0$, $b > 0$. Proceeding as in Example 1 we find that any member of the set $\mathcal{P}_{\rm sp}$ is given by

$$p(s) = c_1 s + c_2 = \frac{(abk_1 + k_2)s + (a + b)k_2}{ab(a + b)}, \quad k_1 \geq 0, k_2 > 0 \tag{34}$$

and $p(s) \in \mathcal{SP}_{\rm sp}$ if and only if $k_1 \neq 0$. In other words, $p(s)/q(s)$ is weak SPR but not strong SPR if and only if $k_1 = 0$. In this simple example the condition for weak SPR is well known, [3]. Namely, $H(s) = k(s+c)/[(s+a)(s+b)]$ is weak SPR if and only if $c = a+b$. This result is in agreement with Theorem 3 and (34) with $k_1 = 0$.

Alternatively, using Theorem 4, we find that

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix} = \frac{1}{2ab(a + b)}\begin{bmatrix} q_{22} + abq_{11} \\ (a + b)q_{22} \end{bmatrix}, \\ P_{22} &= -q_{12} + \frac{(a + b)^2 q_{22} + ab(q_{22} + abq_{11})}{2ab(a + b)} \end{aligned}$$

where $q_{11} > 0$, $q_{11}q_{22} > q_{12}^2$. Idenfifying $q_{ii}/2$ with k_i, $i = 1, 2$, provides equivalence between the two approaches when $q_{12} = 0$. Since c_1 and c_2 do not depend on q_{12}, nontrivial values of q_{12} satisfying the inequality constraint generate the same SPR function as $q_{12} = 0$ for given q_{11} and q_{22}; however, the balance between energy storage and dissipation in (24) will change. Note that $H(s)$ completely determines q_{11} and q_{22} via a, b, c_1, and c_2, but not q_{12}.

IV. Conclusion

A parameterization of all possible polynomials p that make p/q weak or strong SPR for a given q was obtained. Important characteristics of the solution are the small number of parameters and the simplicity of the constraints placed on them. The distinction between weak and strong SPR has been further clarified for transfer functions of arbitrary degree. Although our ultimate interest is in robust control of flexible structures, the results are important in several areas, including adaptive control and circuit theory. Future work will focus on the design of optimal SPR compensation for passive plants. The n free parameters in the parameterization developed here provide an ideal basis for optimization of SPR transfer functions.

Appendix

The following property, known as the Hermite-Biehler property, will be used in the proof of Lemma 1. See [13] for a proof. We assume for simplicity that n is even. A similar result applies to the case where n is odd.

Property 1: Let $a \in \mathcal{P}^n$ and separate a into its even and odd parts, a_e and a_o,

$$a(s) = a_e + a_o = a_e(s^2) + s\bar{a}_o(s^2) \tag{35}$$

where the notation $a_e(s^2)$ and $\bar{a}_o(s^2)$ is used to enhance the fact that a_e and $\bar{a}_o$ contain only even powers. Then, a is Hurwitz if and only if there exists λ_i, ξ_j, and $c \in R$ satisfying

$$a_e(-\omega^2) = (\lambda_1 - \omega^2)(\lambda_2 - \omega^2)\cdots(\lambda_{n/2} - \omega^2) \tag{36}$$

$$\bar{a}_o(-\omega^2) = c(\xi_1 - \omega^2)(\xi_2 - \omega^2)\cdots(\xi_{n/2-1} - \omega^2) \tag{37}$$

where $c > 0$ and $0 < \lambda_1 < \xi_1 < \lambda_2 < \xi_2 < \cdots < \lambda_{n/2}$.

Lemma 1: Consider a Hurwitz polynomial q and let q_e and q_o denote its even and odd parts. Then there exists an even function u and an odd function v satisfying

$$q_e(s)u(s) + q_o(s)v(s) = 1. \tag{38}$$

Proof: Equation (38) is a Bezout identity and its satisfaction is equivalent to the statement that $q_e(s)$ and $q_o(s)$ are coprime. To show this, we reason by contradiction. Suppose q_e and q_o are not coprime. In this case we must have,

$$q_e(s) = a(s)f(s) \tag{39}$$

$$q_o(s) = b(s)f(s) \tag{40}$$

for some nontrivial polynomial $f(s)$. In this case, one of the following must be true,

$$f \text{ even} \Rightarrow a \text{ even and } b \text{ odd} \tag{41}$$

$$f \text{ odd} \Rightarrow a \text{ odd and } b \text{ even.} \tag{42}$$

We assume without loss of generality that (41) holds and q is even. In this case, q_e and q_o can be rewritten as follows,

$$\begin{aligned} q_e(s^2) &= a(s^2)f(s^2) = (a_1 + a_2s^2 + a_4s^4 + \cdots) \\ &\quad \times (f_o + f_2s^2 + f_4s^4 + \cdots) \\ q_o(s^2) &= s\bar{b}(s^2)f(s^2) = s(b_1 + b_3s^2 + b_5s^4 + \cdots) \\ &\quad \times (f_o + f_2s^2 + f_4s^4 + \cdots) = s\bar{q}_o(s^2). \end{aligned}$$

Thus, if $\{\zeta_1, \zeta_2, \cdots, \zeta_m\}$ are the roots of $f(-\omega^2)$, we have

$$\begin{aligned} q_e(-\omega^2) &= a(-\omega^2)(\zeta_1 - \omega^2)(\zeta_2 - \omega^2)\cdots(\zeta_m - \omega^2) \\ \bar{q}_o(-\omega^2) &= b(-\omega^2)(\zeta_1 - \omega^2)(\zeta_2 - \omega^2)\cdots(\zeta_m - \omega^2). \end{aligned}$$

It follows that q is not Hurwitz, since the m roots of $q_e(-\omega^2)$ and $\bar{q}_o(-\omega^2)$ contained in $f(-\omega^2)$ do not satisfy Property 1. This contradicts the assumptions. To complete the proof of Lemma 1 there remains to show that u and v are respectively even and odd. This is a straightforward consequence of the Euclidean algorithm (see, for example, [14]), by which u and v can be determined, and the even property of $mu + nv = 1$. □

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On the Advantages of the LMS Spectrum Analyzer Over Nonadaptive Implementations of the Sliding-DFT

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Abstract— **Based on the least mean squares (LMS) algorithm, the LMS spectrum analyzer can be used to recursively calculate the discrete Fourier transform (DFT) of a sliding window of data. In this paper, we compare the LMS spectrum analyzer with the straightforward nonadaptive implementation of the recursive DFT. In particular, we demonstrate the robustness of the LMS spectrum analyzer to the propagation of roundoff errors, a property that is not shared by other recursive DFT algorithms.**

I. Introduction

In some signal processing applications, a discrete time signal must be continuously analyzed in the frequency domain. At each instant, the N most recent samples of the input sequence are transformed by an N-point DFT. As a new data sample becomes available, the input window is shifted by one position forward in time, and a new DFT is evaluated. This is sometimes refered to as the *sliding-DFT* [1]. To save computations, the new DFT can be calculated recursively from the previous one. However, the propagation and accumulation of noise due for example to roundoff errors in floating point arithmetic makes it necessary to often reset the DFT. This increases the overall number of computations and adds to the complexity of the circuitry.

The *LMS spectrum analyzer* [2] can also perform the recursive computation of a sliding-DFT but because it relies on an adaptive

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