Optimal strictly positive real approximations for stable transfer functions

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Abstract: The problem of finding the optimal strictly positive real (SPR) approximation to a given stable transfer function is considered. The transfer function is further assumed to be strictly proper and the SPR approximation is constrained to have the same pole structure. The optimisation is carried out using the (weighted) $\mathbf{H}_2$-norm and the problem is reduced to a strictly convex quadratic programming problem with linear inequality constraints. At the heart of the method is a parametrisation for all SPR compensators which possess a given denominator polynomial. Motivation for the problem stems from the robust stability provided by SPR compensation for passive plants such as flexible structures with collocated sensing and actuation. Numerical examples are provided, as well as the experimental implementation of an optimal approximation to the control of a single-flexible-link manipulator.

1 Introduction

An important result from input-output stability theory is the passivity theorem [1] which states that the feedback interconnection of a passive system and a strictly passive one is input-output stable. Restricting our attention to causal linear time-invariant (LTI) systems, the concepts of passivity and strict passivity are closely related to the notions of positive real and strictly positive real [2, 3]. Specifically, the transfer function of an LTI system is positive real if and only if it is passive. An SPR system with positive-definite high-frequency gain is strictly passive. Of greater interest is the case where the SPR system is restricted to be strictly proper. In this case, one can show that the feedback interconnection of a passive subsystem (linear or not) and a strictly proper SPR one is always closed-loop stable.

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These observations are of paramount importance in applications where the plant under control is known to be passive, but where there exists large uncertainty in the actual model, such as in the control of flexible manipulators [4] and large space structures [5]. In these cases, the passivity theorem guarantees that any SPR compensator provides closed-loop stability for any possible plant. Motivated by these ideas, a great deal of research has focused on the design of feedback loops for passive plants using SPR compensators [6–8]. A possible design procedure begins by obtaining a finite-dimensional LTI model of the original, possibly high order, nonlinear plant. Then one would design a linear controller $K(s)$ subject to the constraint that $K(s)$ is SPR.

The main difficulty with this approach is that the SPR condition imposed on the compensator is rather difficult to use as a design constraint. An obvious way to ensure that the compensator is SPR is to choose a lead-lag type control law. However, owing to the complexity of the plants under consideration, and increased demands for performance, optimal techniques are usually required. The results which have appeared in the literature consider the linear quadratic Gaussian (LQG) [7] or $\mathbf{H}_\infty$ problems [8] and concentrate on selecting the weighting matrices to ensure that the resulting compensator satisfies the Kalman-Yakubovich lemma (i.e. is SPR). Since the design parameters are *a priori* limited, the final properties of the overall design may not be guaranteed. Other approaches to the problem have considered compensator designs which render the closed-loop system positive real [9–11], which provides robust stability with respect to additive passive uncertainties.

In this paper, a radically new approach is proposed. To begin it is assumed that, following any design procedure, a stable controller $K(s)$ is obtained. No restrictions are placed on $K(s)$ other than stability. In particular, $K(s)$ can be obtained using any optimal control technique, such as LQG or $\mathbf{H}_\infty$. An $\mathbf{H}_2$-optimal SPR approximation $K_{spr}(s)$ to the original compensator $K(s)$ is then found. The approximation is further assumed to possess the same pole structure as $K(s)$; this ensures that the closed-loop properties are close to those of the original design. This approach is analogous to the so-called indirect methods of controller reduction [12]. There, a full-order controller is designed and then the order is reduced so as to approximate the controller and its closed-loop properties. The related work [13] considered optimal $\mathbf{H}_2$-approximation with prescribed poles for general stable transfer functions.

2 Definitions and problem statement

The following notation is used: $\mathbf{P}^n$ is the set of polynomials of exactly nth degree and $\mathbf{H}_\infty$ is the Banach space of complex-valued functions of a complex variable that are analytic and essentially bounded in the open right halfplane (RHP). $\mathbf{H}_2$ the Banach space of complex-valued functions of a complex variable $H(s)$ that are analytic in the open RHP and for which the integrals $\int_{-\infty}^{\infty} |H(\sigma + j\omega)|^2 \, d\omega$ are essentially bounded for all $\sigma > 0$. $\mathbf{H}_2$ may also be identified with the Laplace transforms $H(s)$ of the time signals $h(t)$ contained in $L_2 = \{h : R^+ \rightarrow R | \int_0^\infty h^2(t)\, dt < \infty\}$. The usual norms on $\mathbf{H}_2$ and $\mathbf{H}_\infty$ are denoted by $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively. $\mathbf{RH}_2$ is the set of strictly proper, stable, real rational functions and $\mathbf{RH}_\infty$ is the set of proper, stable, real rational functions. L_{2e} is the set of time functions whose truncations are square-integrable, i.e. $L_{2e} = \{x : R^+ \rightarrow R | \int_0^T x^2(t)\, dt < \infty, \forall T \geq 0\}$

Definition 1: Consider a system with input $u \in L_{2e}$ and output $y = \mathbf{G}u \in L_{2e}$ where $\mathbf{G}$: $L_{2e} \rightarrow L_{2e}$ is a (possibly nonlinear) operator. The system $\mathbf{G}$ is passive if $\int_0^T y(t)u(t)\, dt \geq 0$, $\forall u \in L_{2e}$, $\forall T \geq 0$.

Definition 2: A function of a complex variable $G(s)$ is positive real if:

(i) $G(s)$ is analytic in the open RHP;

(ii) $G(s)$ is real for real s, and

(iii) $\text{Re}[G(s)] \geq 0$ when $\text{Re}[s] > 0$.

We say that $G(s)$ is strictly positive real (SPR) if $G(s-\epsilon)$ is positive real for some $\epsilon > 0$. It is known that an LTI system $\mathbf{G}$ is passive if and only if its transfer function $G(s)$ is positive real. In the LTI case, we do not distinguish $G(s)$ from its corresponding convolution operator $\mathbf{G}$.

Definition 3: Let $H(s) \in \mathbf{RH}_2$. Then $H(s)$ is weak strictly positive real (weak SPR) if

$$\forall \omega \in [0, \infty), \text{Re}[H(\jmath\omega)] > 0$$

and it is strong SPR, or simply SPR, if in addition

$$\lim_{\omega \rightarrow \infty} \omega^2 \text{Re}[H(\jmath\omega)] > 0 \quad (1)$$

$\mathbf{V}^n$ will denote the set of strictly positive real transfer functions contained in $\mathbf{RH}_2$ with denominator in $\mathbf{P}^n$.

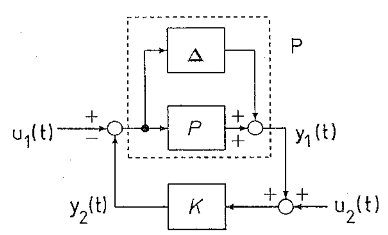

Fig. 1 *Feedback system*

Now consider the system in Fig. 1 and assume that $\mathbf{P}$ corresponds to a passive, possibly nonlinear, system. $P(s)$ is a positive real transfer function corresponding to a finite-dimensional LTI approximation of the plant $\mathbf{P}$ and $\Delta = \mathbf{P}-P$ is the error introduced by the approximation. In certain situations, such as modal truncation of high order models of flexible structures, Δ corresponds to the truncated modes and its transfer function will be positive real. In these cases, selecting $K(s)$ to make the closed-loop system $P(s)\ \{1+P(s)K(s)\}^{-1}$ SPR guarantees closed-loop stability for all positive real Δ [9–11]. However, if the original system $\mathbf{P}$ is passive but nonlinear, then Δ is typically not passive. Hence, making the nominal closed-loop system SPR will not provide robust stability in this case.

In general, we advocate making $K(s)$ SPR according to the following procedure. Let $K(s) = c(s)/a(s)$ be a stabilising compensator for $P(s)$ which has been optimised with respect to the performance of $P(s)$. Although $K(s)$ will stabilise $P(s)$ by design, there is no such guarantee that it will stabilise the nonlinear system $\mathbf{P} = P + \Delta$. However, on the basis of the passivity theorem, for an SPR controller $K_{spr}(s)$, stability of the nonlinear system will also be assured. See, for example, [14].

Define the sensitivity functions $S(s)$ and $S_{spr}(s)$ and the complementary sensitivity functions $T(s)$ and $T_{spr}(s)$ as

$$S(s) = \{1 + P(s)K(s)\}^{-1}$$
$$T(s) = P(s)K(s)\{1 + P(s)K(s)\}^{-1}$$
$$S_{spr}(s) = \{1 + P(s)K_{spr}(s)\}^{-1}$$
$$T_{spr}(s) = P(s)K_{spr}(s)\{1 + P(s)K_{spr}(s)\}^{-1}$$

The aim is to approximate $K(s)$ using an SPR compensator $K_{spr}(s)$ such that S and S_{spr} on the one hand, and T and T_{spr} on the other, are close in the $\mathbf{H}_2$ sense. In other words, it is not enough to ensure that K_{spr} resembles K. It is required to ensure that the closed-loop properties of the overall design are maintained when K is replaced by K_{spr}.

The main problem can now be stated. Given a frequency-dependent weighting function $W(s) \in \mathbf{RH}_\infty$ and given $K(s) = c(s)/a(s) \in \mathbf{RH}_2$, where $c(s)$ and $a(s)$ are coprime polynomials and $a(s) \in \mathbf{P}^n$ is Hurwitz, find $K_{spr}(s) = \hat{c}(s)/a(s) \in \mathbf{V}^n$ so as to minimise

$$J(\hat{c}) = \|W(s)\{K(s) - K_{spr}(s)\}\|_2^2$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |W(\jmath\omega)\{K(\jmath\omega) - K_{spr}(\jmath\omega)\}|^2 d\omega \quad (2)$$

Note that K_{spr} and K have arbitrarily been allowed to share the same denominator polynomial. Simple manipulations then show that

$$(T - T_{spr}) = -(S - S_{spr}) = S_{spr}SP(K - K_{spr})$$

Letting $\hat{W}(s) \in \mathbf{RH}_\infty$ denote a weighting function and defining $W = \hat{W}SPS \in \mathbf{RH}_\infty$ to first order in $(K-K_{spr})$, we can write

$$\|\hat{W}(T - T_{spr})\|_2 = \|\hat{W}(S - S_{spr})\|_2$$
$$\doteq \|W(K - K_{spr})\|_2$$
$$\leq \|W\|_\infty \|K - K_{spr}\|_2$$

Minimising the difference between K and K_{spr}, subject to the same pole structure, places an upper bound on the right-hand side of the inequality. If the indicated choice for W is used in eqn. 2, the weighted approximation K_{spr} provides (weighted) closed-loop properties similar to the original design. Note that this result is true regardless of the norm selected in eqn. 2.

The above problem has a nice circuit theoretic interpretation. Given an active impedance $K(s)$, we find the closest impedance $K_{spr}(s)$ with the same poles is found which can be implemented with passive components. 'Close' is measured using the energy in the difference of their impulse responses [assuming that $W(s) = 1$].

3 Parametrisation of SPR transfer functions

In preparation for solving the optimisation problem of Section 2, all SPR transfer functions with a given denominator polynomial are now parametrised. Given a Hurwitz polynomial $a(s) \in \mathbf{P}^n$, the set

$$\mathbf{Q} = \{\hat{c}(s) \in \mathbf{P}^{n-1} | K(s) = \hat{c}(s)/a(s) \in \mathbf{V}^n\} \quad (3)$$

is sought. In other words, the set of polynomials $\hat{c}(s)$ of degree n–1 is sought which makes the strictly proper function $\hat{c}(s)/a(s)$ SPR. This problem was previously studied by the authors in [15].

Begin by writing $a(s) = s^n + a_1 s^{n-1} + \ldots + \mathbf{a}_n$, $\hat{c}(s) = \hat{c}_1 s^{n-1} + \ldots + \hat{c}_n$, and defining $\hat{\mathbf{c}} = [\hat{c}_1\ \hat{c}_2\ \ldots\ \hat{c}_n]^T$ and $\mathbf{k} = [k_1\ k_2\ \ldots\ k_n]^T$. Further, use the coefficients of $a(s)$ to form

$$\mathbf{X} = \begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ -a_3 & a_2 & -a_1 & 1 & \cdots & \vdots \\ a_5 & -a_4 & a_3 & -a_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \vdots \\ 0 & 0 & -a_n & a_{n-1} & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \in \mathbf{R}^{n \times n} \quad (4)$$

It is now demonstrated that the set $\mathbf{Q}$ has a simple parametrisation in terms of n real numbers.

Theorem 1: Suppose that $a(s)$ is Hurwitz of degree $n \geq 2$. Suppose that $K(s)$ is defined by a given polynomial $\hat{c}(s) \in \mathbf{P}^{n-1}$. If K is SPR, then there exist n real values $k_1\ k_2\ \ldots\ k_n$ such that,

$$\forall x \geq 0 \quad k(x) = k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n > 0 \quad (5)$$

and such that

$$\mathbf{X}\hat{\mathbf{c}} = \mathbf{k} \quad (6)$$

Furthermore, $\mathbf{X}$ is nonsingular, and, given $k_1\ \ldots\ k_n$ satisfying eqn. 5, if $\hat{c}(s)$ is chosen according to eqn. 6, then $K(s) = \hat{c}(s)/a(s)$ is SPR.

Proof: Suppose that $K(s)$ has the form $c(s)/a(s)$. Following [15], decompose $a(s)$ and $c(s)$ into their even and odd parts, denoted by a_e, a_o, c_e, and c_o, respectively:

$$K(s) = \frac{c_e + c_o}{a_e + a_o} = \frac{(c_e + c_o)(a_e - a_o)}{(a_e + a_o)(a_e - a_o)}$$
$$= \frac{\{c_e(s)a_e(s) - c_o(s)a_o(s)\} + \{a_e(s)c_o(s) - c_e(s)a_o(s)\}}{a_e^2(s) - a_o^2(s)}$$

It is immediate that

(i) $a_e^2(j\omega) - a_o^2(j\omega)$ is real and nonnegative, and, since $a(s)$ is Hurwitz, $a_e^2(j\omega) - a_o^2(j\omega) > 0$;

(ii) $\kappa(s) = c_e(s)a_e(s) - c_o(s)a_o(s)$ is an even polynomial and hence $\kappa(j\omega)$ is real;

(iii) $\gamma(s) = a_e(s)c_o(s) - c_e(s)a_o(s)$ is an odd polynomial and hence $\gamma(j\omega)$ is imaginary. It follows that $\mathrm{Re}[K(j\omega)] = \kappa(j\omega)/[a_e^2(j\omega) - a_o^2(j\omega)]$ and, since the denominator is greater than zero, $\mathrm{Re}[K(j\omega)] > 0$ if and only if $\kappa(j\omega) > 0$.

(*Necessity*): It is assumed, without loss of generality, that n, the degree of $a(s)$, is even. Writing

$$\kappa(s) = (-1)^{n-1} k_1 s^{2n-2} + (-1)^{n-2} k_2 s^{2n-4} + \cdots + k_n \quad (7)$$

$\kappa(j\omega) = k(\omega^2)$ where $k(x)$ is defined by eqn. 5. Furthermore, $K(s)$ is SPR, so $\kappa(j\omega) > 0$, which implies that $\forall x \geq 0$, $k(x) > 0$.

In addition, it is possible to write

$$a_e(s) = s^n + a_2 s^{n-2} + \cdots + a_n$$
$$a_o(s) = a_1 s^{n-1} + a_3 s^{n-3} + \cdots + a_{n-1} s$$
$$c_e(s) = c_2 s^{n-2} + c_4 s^{n-4} + \cdots + c_n$$
$$c_o(s) = c_1 s^{n-1} + c_3 s^{n-3} + \cdots + c_{n-1} s \quad (8)$$

Substituting eqn. 8 into $\kappa = c_e a_e - c_o a_o$ and matching coefficients with eqn. 7 $\mathbf{k} = \mathbf{Xc}$ is obtained, where $\mathbf{X}$ is given by eqn. 4.

(*Sufficiency*): Assume that there exists real $k_1, \ldots, k_n$ satisfying eqn. 5. Again, assume that n is even, and form the polynomial $\kappa(s)$ defined by eqn. 7. Clearly, eqn. 5 implies that $\kappa(j\omega) > 0$ since $\kappa(j\omega) = k(\omega^2) > 0$.

The numerator $c(s)$ is now determined by solving the equation $c_e a_e - c_o a_o = \kappa$ for the even and odd parts of $c(s)$. Forcing $c(s) \in \mathbf{P}^{n-1}$ and matching coefficients, again, leads to $\mathbf{k} = \mathbf{Xc}$ where $\mathbf{X}$ is given by eqn. 4. It can be shown that $\mathbf{X}$ is nonsingular if $a(s)$ is Hurwitz (see [16], pp.284–286); hence $\mathbf{c}$ is uniquely determined by $\mathbf{k}$. But $\mathrm{Re}[K(j\omega)] > 0$ since $\kappa(j\omega) > 0$. Recalling definition 3, this implies that $K(s)$ is weak SPR. It is readily shown that $\lim_{\omega \to \infty} \omega^2 \mathrm{Re}[K(j\omega)] = k_1$ which shows that $K(s)$ is strong SPR if $k_1 > 0$.

Remarks: The above proof is much shorter than that originally presented in [15] for a more general problem. It is also demonstrated in [15] that $k_1 > 0$ is equivalent to the condition of eqn. 1 for strong SPR. Hence, if $k_1 = 0$, the parametrisation yields all weak SPR compensators which are not strong SPR. For realisability, the class of compensators has been restricted to be strictly proper. Some authors may prefer to relax this condition by allowing K (and hence K_{spr}) to be biproper. This is possible with only minor changes. An obvious analogous problem to that solved in this paper is to study optimal approximation using the $\mathbf{H}_\infty$-norm rather than the $\mathbf{H}_2$-norm. The major motivation for choosing the latter is the numerical simplicity of the solution to be presented.

4 Optimal SPR approximation

Again using the coefficients of $a(s)$, define

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and set $\mathbf{b} = \mathbf{e}_1 = [1\ 0\ 0 \ldots 0]^T$. Denote a minimal realisation of the stable weighting function $W(s)$ by

$$W(s) = \mathbf{c}_w^T (s\mathbf{I} - \mathbf{A}_w)^{-1} \mathbf{b}_w + d_w$$

and let the given controller be represented as

$$K(s) = \frac{c(s)}{a(s)} = \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}$$

Then, a realisation of the transfer function $W(s)K(s)$ is $\{\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}^T, 0\}$, where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{b}\mathbf{c}_w^T \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \quad \bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b}d_w \\ \mathbf{b}_w \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{c}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix} \quad (9)$$

Writing $K_{spr}(s) = \hat{c}(s)/a(s)$, the function $W(s)K_{spr}(s)$ can be realised in the same way, with $\hat{\mathbf{c}}$ replacing $\mathbf{c}$. Using a standard result for the calculation of the $\mathbf{H}_2$-norm, the cost functional in eqn. 2 can be expressed as

$$J(\hat{\mathbf{c}}) = (\mathbf{c} - \hat{\mathbf{c}})^T \mathbf{L} (\mathbf{c} - \hat{\mathbf{c}}) \quad (10)$$

where $\mathbf{L}$ represents the upper $n \times n$ partition of the matrix $\bar{\mathbf{L}}$ which satisfies the Lyapunov equation

$$\bar{\mathbf{A}}\bar{\mathbf{L}} + \bar{\mathbf{L}}\bar{\mathbf{A}}^T = -\bar{\mathbf{b}}\bar{\mathbf{b}}^T \tag{11}$$

Under the given assumptions, there is a unique positive-definite solution $\bar{\mathbf{L}}$ to eqn. 11 and hence $\mathbf{L}$ is positive-definite. It is important to realise that $\mathbf{L}$ can be calculated *a priori* from the given functions $a(s)$ and $W(s)$.

Using theorem 1, the constraints on the optimisation are easily realised. $\hat{\mathbf{c}} = \mathbf{X}^{-1}\mathbf{k}$, where the k_i satisfy eqn. 5. The constraints placed on the k_i by $k(x) > 0$ can be made explicit using the classical Sturm test [17]. This is straightforward for given numerical values of $\mathbf{k}$ but symbolic calculations are very tedious for $n > 3$ (see [17] for $n = 4$). Numerical enforcement of the constraints is proposed. At $x = 0$, eqn. 5 implies that $k_n > 0$ and as $x \to \infty$, it implies that $k_1 > 0$. In the intermediate regime, eqn. 5 can be imposed at N discrete values $x_i > 0$, $i = 1,..., N$. Defining

$$\mathbf{x}_i = [\,x_i^{n-1} \quad x_i^{n-2} \quad \cdots \quad x_i \quad 1\,]^T$$

the constraints are invoked as

$$k_1 \geq \epsilon \quad k_n \geq \epsilon \quad \mathbf{k}^T\mathbf{x}_i \geq \epsilon \quad i = 1, \ldots, N \tag{12}$$

where $\epsilon > 0$ is a small prescribed number. In the sequel, the N values of x_i are prescribed using M decades and m logarithmically spaced values per decade.

Using the parametrisation in eqn. 6, the cost function eqn. 10 becomes

$$J(\mathbf{k}) = (\mathbf{c} - \mathbf{X}^{-1}\mathbf{k})^T\mathbf{L}(\mathbf{c} - \mathbf{X}^{-1}\mathbf{k}) \tag{13}$$

which is a strictly convex (quadratic) function. It is subject to the convex (linear) constraints in eqn. 12. The resulting quadratic-programming problem has a unique global minimum which can be obtained using specialised numerical approaches. Here the method of Goldfarb and Idnani [18] is used as implemented in [19]. This algorithm is well known for its reliability and efficiency [20]. To ensure the validity of the solutions, an independent approach was taken using the optimisation software package Minos [21].

By way of example, consider the stable transfer function

$$K(s) = \frac{(s+25)(s+35)(s+38)(s+180)(s+185)}{(s+1)(s+3)(s+90)^2(s+95)(s+100)} \tag{14}$$

where $K(j\omega)$ exhibits multiple crossovers of the imaginary axis and hence is not SPR. First consider the unweighted optimisation problem ($W = 1$). Using the numerical procedure in Section 3, the constraint was enforced using m logarithmically distributed values of x_i using M decades beginning with 0.01 (rad/s)^2; $\epsilon = 10^{-6}$. The resulting optimal value of J for several cases is given in Table 1.

Table 1 Optimal value of *J* for varying constraints

Case	Decades (M)	Points/decade (m)	$\|\|K - K_{spr}\|\|_2^2$ (J)
1	3	10	0.497884
2	4	10	0.498253
3	5	2	0.500811
4	5	10	0.504091
5	5	20	0.504122
6	5	40	0.504163
7	6	10	0.504091
8	6	40	0.504163

The Bode plots showing both the given function and the optimal approximation ($M = 6$, $m = 40$) are given in Figs. 2 and 3. The approximations obtained for cases 4–8 in Table 1 were graphically indistinguishable from one another. Also shown is the case where $W(s) = H_4(s)$, a fourth order lowpass Butterworth transfer function with unity DC gain and corner frequency 10 rad/s. The weighted case exhibits better magnitude agreement in the passband. Enforcement of the SPR constraint is clear for both weightings. For cases 1–3, the SPR constraint was not enforced at some frequencies.

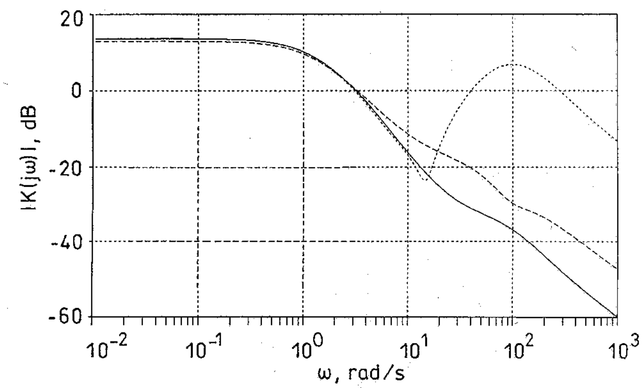

Fig. 2 *Numerical example: magnitude against frequency*
—— $K(s)$
- - - - K_{spr}, $W = 1$
.... K_{spr}, $W = H_4$

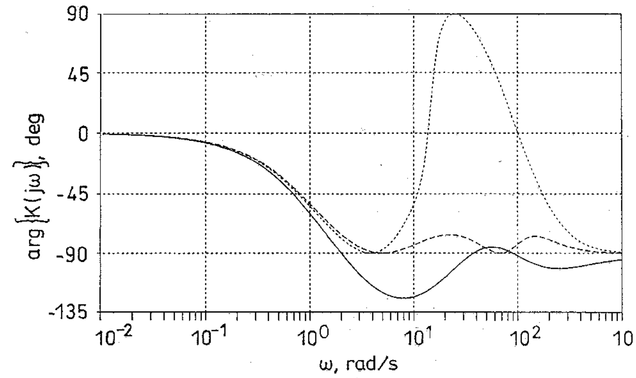

Fig. 3 *Numerical example: phase against frequency*
—— $K(s)$
- - - - K_{spr}, $W = 1$
.... K_{spr}, $W = H_4$

5 Controller-design example

Now consider a controller design for an experimental apparatus consisting of a single flexible beam (aluminum, 1000 × 50.8 × 3.1 mm) mounted to the output shaft of a direct-drive DC motor. Let $\theta(t)$ denote the motor angle and $\tau(t)$ be the applied torque. Sensors consist of a 51 500 pulse/rev encoder and an analogue tachometer measurement created using frequency-to-voltage conversion of the encoder pulses. The motor torque is commanded with a 12-bit digital-analogue converter. The controller to be developed below is implemented in fixed-point arithmetic using a 16-bit 80C196KB microcontroller.

The transfer function for this single-link robot is given by

$$P(s) = \frac{s\theta(s)}{\tau(s)} = \frac{1}{I_0 s} + \sum_{\alpha=1}^{m} \frac{s\theta_\alpha^2}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2} \tag{15}$$

The first term is the rigid mode and $I_0 = 0.1449\,\text{kg}\cdot\text{m}^2$ is the moment of inertia of the link plus motor rotor.

Subsequent terms represent the unconstrained (pinned-free) modes of the link. Good agreement was obtained between the measured natural frequencies and those obtained from the analytical model (ω_α) which uses clamped-free mode shapes for discretisation of the link deflection. The damping ratios ζ_α were experimentally measured by forcing the link at the natural frequencies and then observing the free decay. The values of the mode slopes θ_α represent the joint participation in the α^{th} mode shape. The values of the various parameters are given in Table 2. Note that $P(s)$ is passive (positive real) regardless of the number of modes that are kept in eqn. 15.

Table 2 Properties of the single flexible link

Mode α	ω_α (rad/s)	ζ_α	θ_α	$\bar{\omega}_\alpha$ (rad/s)
0	0	0.0	—	4.034
1	65.35	0.0230	8.465	66.68
2	162.6	0.0110	13.06	163.95
3	316.3	0.0103	7.688	316.57
4	582.2	0.0022	3.841	582.21

Of ultimate interest is the joint-angle tracking; it is proposed that the controller be given by

$$K(s) = \frac{K_p}{s} + K_d(s)$$

where the proportional loop K_p is closed first and then $K_d(s)$ is designed for this closed-loop plant. Since the damping ratios are typically poorly known, set them to zero for design of the controllers. Hence, the closure of the proportional loop merely shifts the vibration frequencies (including the rigid zero-frequency mode) along the imaginary axis. The new vibration frequencies $\bar{\omega}_\alpha$ are also given in Table 2 for K_p = 2.5Nm/rad.

The closure of the proportional loop does not alter the passivity of the plant since this represents the feedback interconnection of positive real transfer functions. Let $\{\mathbf{A}_p, \mathbf{b}_p, \mathbf{c}_p^T, 0\}$ denote a realisation of this modified plant. The nominal controller $K_d(s)$ is selected to be

$$K_d(s) = \mathbf{k}_c^T[s\mathbf{I} - (\mathbf{A}_p - \mathbf{b}_p\mathbf{k}_c^T - \mathbf{k}_e\mathbf{c}_p^T)]^{-1}\mathbf{k}_e \qquad (16)$$

where the controller/estimator feedback gains $\mathbf{k}_c$ and $\mathbf{k}_e$ are designed using pole placement. The plant poles are placed so as to preserve the undamped natural frequencies given in Table 3 ($\bar{\omega}_\alpha$) but with a damping ratio of 1.0. The estimator eigenvalues have the same damping ratio but are 2.2 times faster. The closed-loop eigenvalues for the nominal plant (rigid plus first two vibration modes) and the resulting sixth-order controller are given in Table 3. Also shown are the eigenvalues for this controller in feedback with the full-order plant (rigid mode plus four vibration modes). For this calculation, the open-loop damping ratios have been included in the plant model. As can be gleaned from the Table 3, a spillover instability has occured.

Now obtain the $\mathbf{H}_2$-optimal SPR approximation, $K_{spr}(s)$, to the controller, $K_d(s)$. The Bode diagrams for the pole placement controller and its SPR approximation [M = 6 starting with 0.01 (rad/s)^2, m = 30, $\epsilon = 10^{-6}$] are given in Figs. 3, 4 and 5. The SPR controller guarantees stability for any number modes, regardless of the values of ω_α, θ_α, ζ_α and the proportional feedback gain $K_p > 0$. The closed-loop eigenvalues using it also appear in Table 3 and no instabilities are present.

Table 3 Closed-loop eigenvalues

Nominal plant plus $K(s)$	Full plant plus $K(s)$	Full plant plus $K_{spr}(s)$
−4.03, −4.03	−3.11, −22.1	−1.44±j1.81
−66.7, −66.7	−4.64±j2.63	−6.78±j9.70
−163.9, −163.9	−29.7±j14.0	−14.4±j29.5
−9.28, −9.28	−49.6±j81.3	−26.5±j104.1
−153.4, −153.4	−87.4±j200.5	−13.1±j311.5
−377.1, −377.1	0.91±j315.7	−136.9±j302.0
	−5.61±j573.6	−6.87±j573.5
	−594±j547.1	−575.8±j510.7

The sensitivity and complementary sensitivity functions are given in Figs. 6 and 7 for the nominal controller and its approximation. They are calculated assuming that the loop transfer function is given by $P(s)K(s) = s^{-1}P(s)\{K_p + sK_d(s)\}$. As expected, the two sets of curves exhibit close agreement.

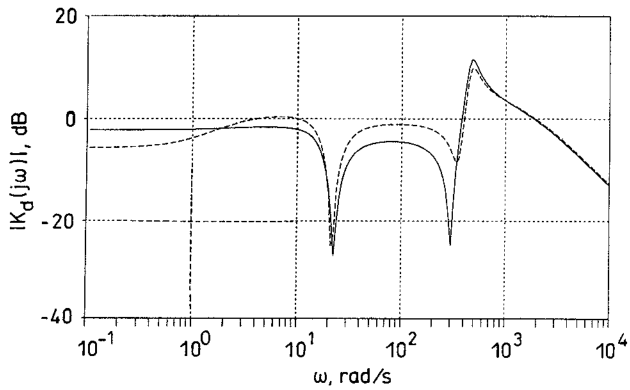

Fig.4 *Controller designs: magnitude against frequency*
——— nominal
- - - - optimal approximation

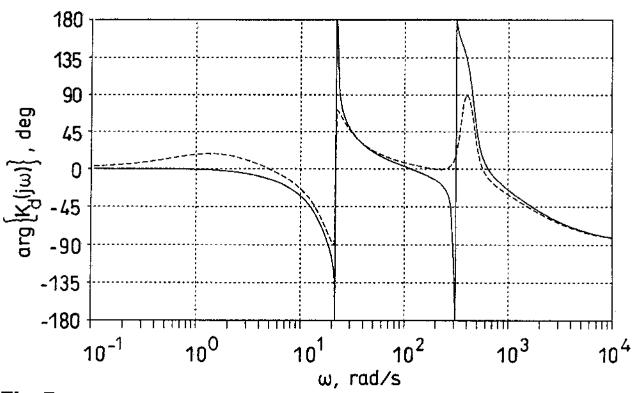

Fig.5 *Controller designs: phase against frequency*
——— nominal
- - - - optimal approximation

For experimental implementation of the controller, bilinear transformation of K_{spr} using a sample period of 0.0025s was used. To avoid numerical problems, the resulting discrete compensator $K_{spr,D}(z)$ was factored into a cascade of three biquadratic transfer functions. The proportional controller was not changed by the discretisation. It is important to realise that the zero-order hold employed in applying the control torque to the plant effectively destroys the passivity of the plant model. However, the bilinear transformation preserves the passivity of the controller. Both the simulated and experimental responses to a step command for the joint angle ($\theta_d = \pi/6$) are shown in Fig. 8. No attempt was made to compensate for the static friction and hence the experimental result exhibits significant steady-state error for this value of K_p. However, the stability properties predicted using continuous-time arguments are readily apparent as is the instability incurred when using the nominal controller.

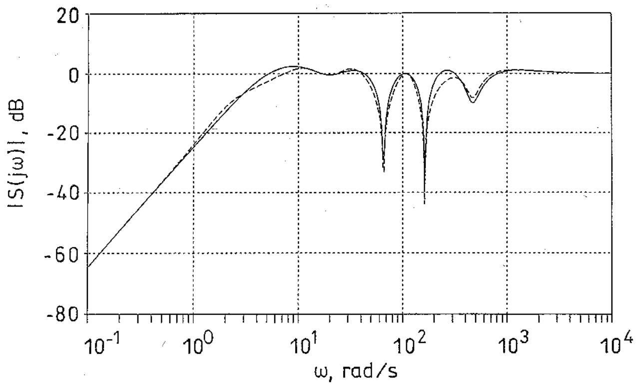

Fig. 6 *Controller properties: sensitivity against frequency*
——— nominal
- - - - optimal approximation

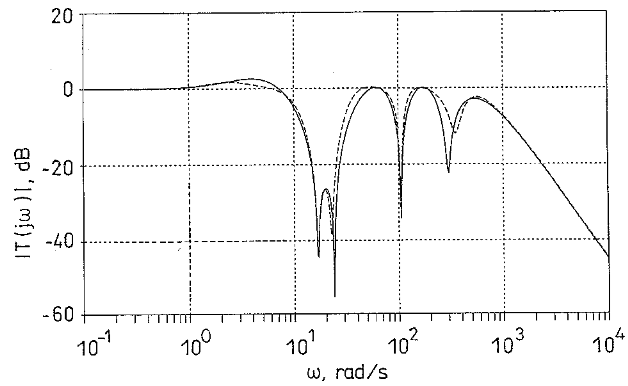

Fig. 7 *Controller properties: complementary sensitivity against frequency*
——— nominal
- - - - optimal approximation

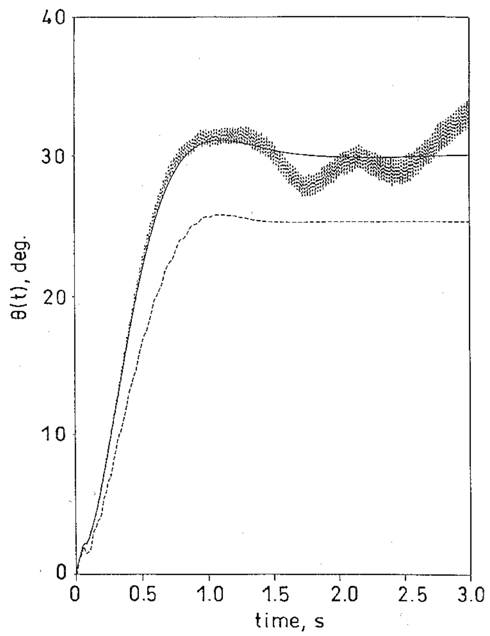

Fig. 8 *Step response*
——— simulation (SPR)
········ simulation (non-SPR)
- - - - experimental (SPR)

6 Concluding remarks

In this paper, a systematic methodology has been developed for finding the closest SPR transfer function to a given stable transfer function. The approximation is assumed to possess the same poles as the given function; this is in order to maintain the closed-loop properties of a controller design. More importantly, the representation of all SPR functions in this case is as a linear parametrisation in terms of values k_i, $i = 1, ..., n$. By discretising the constraints placed on the k_i, it is possible to reduce the constrained $\mathbf{H}_2$-optimisation to a standard quadratic-programming problem.

The primary motivation in seeking the SPR approximations is the robust stability which they afford for passive systems. The example involving an experimental flexible structure demonstrated clearly that the SPR approximation can be used to remove the spillover phenomenon from an otherwise good controller design. Although the results of the paper are single-input/single-output in character, they can be directly applied to decentralised feedback loops.

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