AER506
Spacecraft Dynamics and Control I

A course presented at the
UNIVERSITY OF TORONTO

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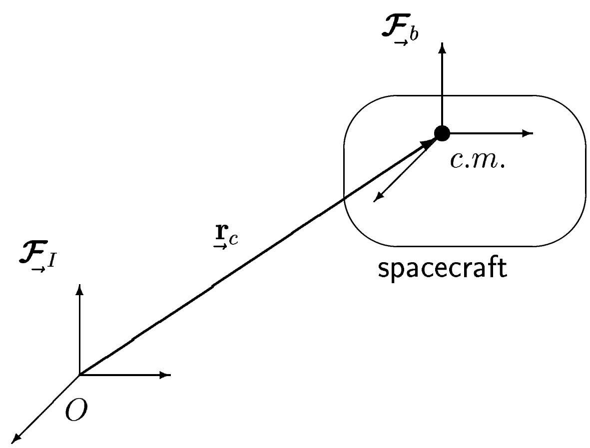
Course notes are available at:
http://arrow.utias.utoronto.ca/~damaren/aer506h.html

1 Kinematics of Spacecraft Motion

Spacecraft are free bodies; that is they are free to translate and rotate. Mathematically, they have six degrees of freedom: three in translation and three in rotation. The study of spacecraft translational motion is described within the framework of *orbital dynamics*. The study of its rotational motion is the subject of *attitude dynamics*. We shall see later on in the course that, for all intents and purposes, the two classes of motion are uncoupled and can be treated separately.

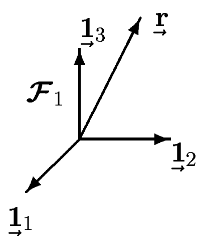
Before writing down the equations that describe the dynamics of attitude motion, we must examine ways of describing it. In other words, we shall study the *kinematics* of rotational motion. For example, the position of a point mass can be described with a vector consisting of three components. Rotational motion is described by expressing the orientation of one reference frame with respect to another.



In the context of spacecraft dynamics, one reference frame is fixed to the spacecraft ($\underrightarrow{\mathcal{F}}_b$) and we keep track of the frame's motion with respect to a nonspinning, inertial reference frame ($\underrightarrow{\mathcal{F}}_I$). Orbital dynamics is the study of how the position of the spacecraft centre of mass ($\underrightarrow{\mathbf{r}}_c$) evolves in time. Attitude dynamics is concerned with the orientation of $\underrightarrow{\mathcal{F}}_b$ with respect to $\underrightarrow{\mathcal{F}}_I$. This course will deal both subjects. We endeavour to provide a complete description of a rigid body moving in the gravitational field of the earth.

1.1 Reference Frames and Rotation Matrices

We shall take a *vector* to be a quantity $\underrightarrow{\mathbf{r}}$. This vector can be expressed in a reference frame as



$$\begin{aligned}
\underrightarrow{\mathbf{r}} &= r_1\underrightarrow{\mathbf{1}}_1 + r_2\underrightarrow{\mathbf{1}}_2 + r_3\underrightarrow{\mathbf{1}}_3 \\
&= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \\ \underrightarrow{\mathbf{1}}_2 \\ \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \\
&= \mathbf{r}^T \underrightarrow{\boldsymbol{\mathcal{F}}}_1
\end{aligned} \tag{1}$$

The quantity

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

is a column matrix containing the *components* of $\underrightarrow{\mathbf{r}}$. The quantity

$$\underrightarrow{\boldsymbol{\mathcal{F}}}_1 = \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \\ \underrightarrow{\mathbf{1}}_2 \\ \underrightarrow{\mathbf{1}}_3 \end{bmatrix}$$

is a column containing the basis (or unit) vectors forming the reference frame $\boldsymbol{\mathcal{F}}_1$. We shall refer to $\underrightarrow{\boldsymbol{\mathcal{F}}}_1$ as a vectrix.

Note that (1) can also be written as

$$\begin{aligned}
\underrightarrow{\mathbf{r}} &= \left[\underrightarrow{\mathbf{1}}_1 \;\; \underrightarrow{\mathbf{1}}_2 \;\; \underrightarrow{\mathbf{1}}_3\right] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \\
&= \underrightarrow{\boldsymbol{\mathcal{F}}}_1^T \mathbf{r}
\end{aligned}$$

<u>Dot Product</u>

Consider two vectors $\underrightarrow{\mathbf{r}}$ and $\underrightarrow{\mathbf{s}}$ expressed in the same reference frame $\underrightarrow{\boldsymbol{\mathcal{F}}}_1$:

$$\underrightarrow{\mathbf{r}} = [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \\ \underrightarrow{\mathbf{1}}_2 \\ \underrightarrow{\mathbf{1}}_3 \end{bmatrix}, \quad \underrightarrow{\mathbf{s}} = \left[\underrightarrow{\mathbf{1}}_1 \;\; \underrightarrow{\mathbf{1}}_2 \;\; \underrightarrow{\mathbf{1}}_3\right] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

The dot product is given by

$$\begin{aligned}
\underrightarrow{\mathbf{r}} \cdot \underrightarrow{\mathbf{s}} &= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \\ \underrightarrow{\mathbf{1}}_2 \\ \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \cdot \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\
&= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \cdot \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_1 \cdot \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_1 \cdot \underrightarrow{\mathbf{1}}_3 \\ \underrightarrow{\mathbf{1}}_2 \cdot \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_2 \cdot \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_2 \cdot \underrightarrow{\mathbf{1}}_3 \\ \underrightarrow{\mathbf{1}}_3 \cdot \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_3 \cdot \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_3 \cdot \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}
\end{aligned}$$

But

$$\underrightarrow{\mathbf{1}}_1 \cdot \underrightarrow{\mathbf{1}}_1 = \underrightarrow{\mathbf{1}}_2 \cdot \underrightarrow{\mathbf{1}}_2 = \underrightarrow{\mathbf{1}}_3 \cdot \underrightarrow{\mathbf{1}}_3 = 1$$

and

$$\underrightarrow{\mathbf{1}}_1 \cdot \underrightarrow{\mathbf{1}}_2 = \underrightarrow{\mathbf{1}}_2 \cdot \underrightarrow{\mathbf{1}}_3 = \underrightarrow{\mathbf{1}}_3 \cdot \underrightarrow{\mathbf{1}}_1 = 0$$

Therefore,

$$\underrightarrow{\mathbf{r}} \cdot \underrightarrow{\mathbf{s}} = \mathbf{r}^T \mathbf{1} \mathbf{s} = \mathbf{r}^T \mathbf{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$$

The notation $\mathbf{1}$ will be used to designate the identity matrix. Its dimension can be inferred from context.

Cross Product

The cross product of two vectors expressed in the same reference frame is given by:

$$\begin{aligned}
\underrightarrow{\mathbf{r}} \times \underrightarrow{\mathbf{s}} &= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 \times \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_1 \times \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_1 \times \underrightarrow{\mathbf{1}}_3 \\ \underrightarrow{\mathbf{1}}_2 \times \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_2 \times \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_2 \times \underrightarrow{\mathbf{1}}_3 \\ \underrightarrow{\mathbf{1}}_3 \times \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_3 \times \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_3 \times \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\
&= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \underrightarrow{\mathbf{0}} & \underrightarrow{\mathbf{1}}_3 & -\underrightarrow{\mathbf{1}}_2 \\ -\underrightarrow{\mathbf{1}}_3 & \underrightarrow{\mathbf{0}} & \underrightarrow{\mathbf{1}}_1 \\ \underrightarrow{\mathbf{1}}_2 & -\underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{0}} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\
&= \begin{bmatrix} \underrightarrow{\mathbf{1}}_1 & \underrightarrow{\mathbf{1}}_2 & \underrightarrow{\mathbf{1}}_3 \end{bmatrix} \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_1^T \mathbf{r}^{\times} \mathbf{s}
\end{aligned}$$

Hence, if $\underrightarrow{\mathbf{r}}$ and $\underrightarrow{\mathbf{s}}$ are expressed in the same reference frame, the 3×3 matrix

$$\mathbf{r}^{\times} \triangleq \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

can be used to construct the components of the cross product. This matrix is skew-symmetric, that is,

$$(\mathbf{r}^\times)^T = -\mathbf{r}^\times$$

You should convince yourself that

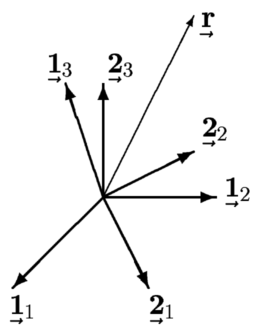
$$\mathbf{r}^\times \mathbf{r} = \mathbf{0}$$

where $\mathbf{0}$ is a column matrix of zeros and

$$\mathbf{r}^\times \mathbf{s} = -\mathbf{s}^\times \mathbf{r}$$

Rotation Matrices

Let us consider two reference frames $\underrightarrow{\mathcal{F}}_1$ and $\underrightarrow{\mathcal{F}}_2$ and express $\underrightarrow{\mathbf{r}}$ in each frame:



$$\underrightarrow{\mathbf{r}} = \underrightarrow{\mathcal{F}}_1^T \mathbf{r}_1 = \underrightarrow{\mathcal{F}}_2^T \mathbf{r}_2$$

We seek to discover a relationship between the components in $\mathcal{F}_1$, $\mathbf{r}_1$ and those in $\mathcal{F}_2$, $\mathbf{r}_2$. Proceed as follows:

$$\begin{aligned} \underrightarrow{\mathcal{F}}_2^T \mathbf{r}_2 &= \underrightarrow{\mathcal{F}}_1^T \mathbf{r}_1 && (2) \\ \underrightarrow{\mathcal{F}}_2 \cdot \underrightarrow{\mathcal{F}}_2^T \mathbf{r}_2 &= \underrightarrow{\mathcal{F}}_2 \cdot \underrightarrow{\mathcal{F}}_1^T \mathbf{r}_1 \\ \mathbf{r}_2 &= \mathbf{C}_{21} \mathbf{r}_1 && (3) \end{aligned}$$

We have defined

$$\begin{aligned} \mathbf{C}_{21} &= \underrightarrow{\mathcal{F}}_2 \cdot \underrightarrow{\mathcal{F}}_1^T && (4) \\ &= \begin{bmatrix} \underrightarrow{2}_1 \\ \underrightarrow{2}_2 \\ \underrightarrow{2}_3 \end{bmatrix} \cdot \begin{bmatrix} \underrightarrow{1}_1 & \underrightarrow{1}_2 & \underrightarrow{1}_3 \end{bmatrix} \\ &= \begin{bmatrix} \underrightarrow{2}_1 \cdot \underrightarrow{1}_1 & \underrightarrow{2}_1 \cdot \underrightarrow{1}_2 & \underrightarrow{2}_1 \cdot \underrightarrow{1}_3 \\ \underrightarrow{2}_2 \cdot \underrightarrow{1}_1 & \underrightarrow{2}_2 \cdot \underrightarrow{1}_2 & \underrightarrow{2}_2 \cdot \underrightarrow{1}_3 \\ \underrightarrow{2}_3 \cdot \underrightarrow{1}_1 & \underrightarrow{2}_3 \cdot \underrightarrow{1}_2 & \underrightarrow{2}_3 \cdot \underrightarrow{1}_3 \end{bmatrix} && (5) \end{aligned}$$

The matrix $\mathbf{C}_{21}$ is called a *rotation matrix*. It is sometimes referred to as a "direction cosine matrix" since the dot product of two unit vectors is just the angle between them.

The unit vectors in $\boldsymbol{\mathcal{F}}_2$ can be related to those in $\boldsymbol{\mathcal{F}}_1$ by inserting (3) into (2):

$$\underrightarrow{\mathcal{F}}_1^T = \underrightarrow{\mathcal{F}}_2^T \mathbf{C}_{21} \tag{6}$$

Rotation matrices possess some special properties. From Eq. (3),

$$\mathbf{r}_1 = \mathbf{C}_{21}^{-1}\mathbf{r}_2 = \mathbf{C}_{12}\mathbf{r}_2$$

But, from (5), $\mathbf{C}_{21}^T = \mathbf{C}_{12}$. Hence,

$$\mathbf{C}_{12} = \mathbf{C}_{21}^{-1} = \mathbf{C}_{21}^T \tag{7}$$

We say that $\mathbf{C}_{21}$ is an *orthonormal* matrix because its inverse is equal to its transpose.

Consider three reference frames $\underrightarrow{\mathcal{F}}_1$, $\underrightarrow{\mathcal{F}}_2$, and $\underrightarrow{\mathcal{F}}_3$. The components of a vector $\underrightarrow{r}$ in these three frames are $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Now,

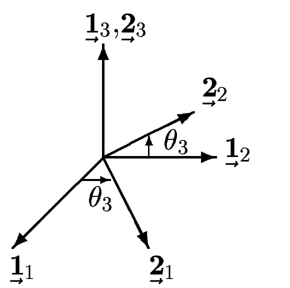
$$\mathbf{r}_3 = \mathbf{C}_{32}\mathbf{r}_2 = \mathbf{C}_{32}\mathbf{C}_{21}\mathbf{r}_1$$

But, $\mathbf{r}_3 = \mathbf{C}_{31}\mathbf{r}_1$, and therefore

$$\mathbf{C}_{31} = \mathbf{C}_{32}\mathbf{C}_{21}$$

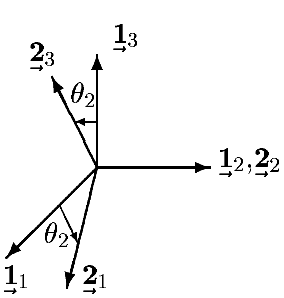
Principal Rotations

The most important rotations of one frame with respect to another are those about one of the coordinate axes. The situation where $\boldsymbol{\mathcal{F}}_2$ has been rotated from $\boldsymbol{\mathcal{F}}_1$ through a rotation about the 3-axis is shown below. The rotation matrix in this case is



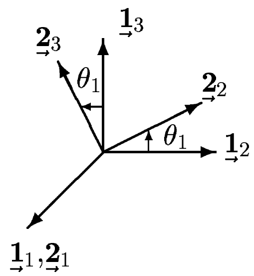
$$\mathbf{C}_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

For a rotation about the 2-axis:



$$\mathbf{C}_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{9}$$

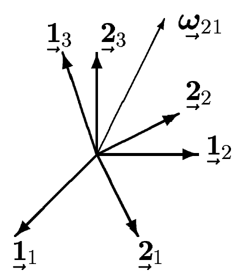
The rotation matrix for a rotation about the 1-axis is given by



$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{10}$$

1.2 Angular Velocity

Let frame $\boldsymbol{\mathcal{F}}_2$ rotate with respect to frame $\boldsymbol{\mathcal{F}}_1$. The angular velocity of frame 2 with respect to frame 1 is denoted by $\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$. The angular velocity of frame 1 with respect to 2 is $\underset{\rightarrow}{\boldsymbol{\omega}}_{12} = -\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$.



The magnitude of $\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$, $|\underset{\rightarrow}{\boldsymbol{\omega}}_{21}| = \sqrt{(\underset{\rightarrow}{\boldsymbol{\omega}}_{21} \cdot \underset{\rightarrow}{\boldsymbol{\omega}}_{21})}$, is the rate of rotation. The direction of $\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$, *i.e.*, the unit vector in the direction of $\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$ ($|\underset{\rightarrow}{\boldsymbol{\omega}}_{21}|^{-1}\,\underset{\rightarrow}{\boldsymbol{\omega}}_{21}$) is the *instantaneous* axis of rotation.

Observers in the frames $\boldsymbol{\mathcal{F}}_2$ and $\boldsymbol{\mathcal{F}}_1$ do not see the same motion because of their own relative motions. Let us denote the vector time derivative as seen in $\boldsymbol{\mathcal{F}}_1$ by $(\dot{\ })$ and that seen in $\boldsymbol{\mathcal{F}}_2$ by $(\circ)$. Therefore,

$$\dot{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}_1 = \underset{\rightarrow}{\mathbf{0}}\,, \quad \overset{\circ}{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}_2 = \underset{\rightarrow}{\mathbf{0}}$$

It can be shown that

$$\dot{\underset{\rightarrow}{2}}_1 = \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{2}_1\;,\;\; \dot{\underset{\rightarrow}{2}}_2 = \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{2}_2\;,\;\; \dot{\underset{\rightarrow}{2}}_3 = \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{2}_3$$

or equivalently

$$\left[\dot{\underset{\rightarrow}{2}}_1 \;\; \dot{\underset{\rightarrow}{2}}_2 \;\; \dot{\underset{\rightarrow}{2}}_3\right] = \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \left[\underset{\rightarrow}{2}_1 \;\; \underset{\rightarrow}{2}_2 \;\; \underset{\rightarrow}{2}_3\right]$$

or

$$\dot{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}{}_2^T = \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}{}_2^T \tag{11}$$

We want to determine the time derivative of an arbitrary vector expressed in both frames:

$$\underset{\rightarrow}{\mathbf{r}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}{}_1^T \mathbf{r}_1 = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}{}_2^T \mathbf{r}_2$$

Therefore, the time derivative as seen in $\boldsymbol{\mathcal{F}}_1$ is

$$\dot{\underset{\rightarrow}{\mathbf{r}}} = \dot{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}_1^T \mathbf{r}_1 + \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1^T \dot{\mathbf{r}}_1 = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1^T \dot{\mathbf{r}}_1 \tag{12}$$

In a similar way,

$$\overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}{}_2^T \mathbf{r}_2 + \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \overset{\circ}{\mathbf{r}}_2 = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \overset{\circ}{\mathbf{r}}_2 = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{r}}_2 \tag{13}$$

(Note that for nonvectors, $(\dot{\ }) = (\circ)$, *i.e.*, $\overset{\circ}{\mathbf{r}}_2 = \dot{\mathbf{r}}_2$).

Alternatively, the time derivative as seen in $\boldsymbol{\mathcal{F}}_1$, but expressed in $\boldsymbol{\mathcal{F}}_2$, is

$$\begin{aligned}
\dot{\underset{\rightarrow}{\mathbf{r}}} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{r}}_2 + \dot{\underset{\rightarrow}{\boldsymbol{\mathcal{F}}}}_2^T \mathbf{r}_2 \\
&= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{r}}_2 + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \mathbf{r}_2 \\
&= \overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{r}}
\end{aligned} \tag{14}$$

The above is true for any vector $\underset{\rightarrow}{\mathbf{r}}$. The most important application occurs when $\underset{\rightarrow}{\mathbf{r}}$ denotes position, $\boldsymbol{\mathcal{F}}_1$ is a nonrotating inertial reference frame, and $\boldsymbol{\mathcal{F}}_2$ is a frame that rotates with a body, particle, etc. In this case, Eq. (14) expresses the velocity in the inertial frame in terms of the motion in the second frame.

Now, express the angular velocity in $\boldsymbol{\mathcal{F}}_2$:

$$\underset{\rightarrow}{\boldsymbol{\omega}}_{21} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \boldsymbol{\omega}_{21} \tag{15}$$

Therefore, using (12) and (15), (14) becomes

$$\begin{aligned}
\dot{\underset{\rightarrow}{\mathbf{r}}} = \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_1^T \dot{\mathbf{r}}_1 &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{r}}_2 + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{r}} \\
&= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{r}}_2 + \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T \boldsymbol{\omega}_{21}^{\times} \mathbf{r}_2 \\
&= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_2^T (\dot{\mathbf{r}}_2 + \boldsymbol{\omega}_{21}^{\times} \mathbf{r}_2)
\end{aligned} \tag{16}$$

If we want to express the 'inertial time derivative' (that seen in $\boldsymbol{\mathcal{F}}_1$) in $\boldsymbol{\mathcal{F}}_1$, then we can use the rotation matrix $\mathbf{C}_{12}$:

$$\dot{\mathbf{r}}_1 = \mathbf{C}_{12}(\dot{\mathbf{r}}_2 + \boldsymbol{\omega}_{21}^{\times} \mathbf{r}_2) \tag{17}$$

Acceleration

Let us denote the *velocity* by

$$\underset{\rightarrow}{\mathbf{v}} = \dot{\underset{\rightarrow}{\mathbf{r}}} = \overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{r}}$$

The acceleration can be calculated by applying (14) to $\underset{\rightarrow}{\mathbf{v}}$:

$$\begin{aligned}
\ddot{\underset{\rightarrow}{\mathbf{r}}} = \dot{\underset{\rightarrow}{\mathbf{v}}} &= \overset{\circ}{\underset{\rightarrow}{\mathbf{v}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{v}} \\
&= (\overset{\circ\circ}{\underset{\rightarrow}{\mathbf{r}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} + \overset{\circ}{\underset{\rightarrow}{\boldsymbol{\omega}}}_{21} \times \underset{\rightarrow}{\mathbf{r}}) + (\underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{r}}) \\
&= \overset{\circ\circ}{\underset{\rightarrow}{\mathbf{r}}} \ + \ 2\underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \overset{\circ}{\underset{\rightarrow}{\mathbf{r}}} \ + \ \overset{\circ}{\underset{\rightarrow}{\boldsymbol{\omega}}}_{21} \times \underset{\rightarrow}{\mathbf{r}} \ + \ \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\boldsymbol{\omega}}_{21} \times \underset{\rightarrow}{\mathbf{r}}
\end{aligned} \tag{18}$$

The matrix equivalent in terms of components can be had by using (13), (15) and by making the following substitutions:

$$\underset{\to}{\ddot{\mathbf{r}}} = \underset{\to}{\boldsymbol{\mathcal{F}}}_1^T \ddot{\mathbf{r}}_1 \ , \quad \underset{\to}{\overset{\circ\circ}{\mathbf{r}}} = \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \ddot{\mathbf{r}}_2 \ , \quad \underset{\to}{\overset{\circ}{\boldsymbol{\omega}}}_{21} = \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \dot{\boldsymbol{\omega}}_{21}$$

The result for the components is

$$\ddot{\mathbf{r}}_1 = \mathbf{C}_{12} \left[\ddot{\mathbf{r}}_2 + 2\boldsymbol{\omega}_{21}^\times \dot{\mathbf{r}}_2 + \dot{\boldsymbol{\omega}}_{21}^\times \mathbf{r}_2 + \boldsymbol{\omega}_{21}^\times \boldsymbol{\omega}_{21}^\times \mathbf{r}_2 \right] \tag{19}$$

The various terms in the expression for the acceleration have been given special names:

$\underset{\to}{\overset{\circ\circ}{\mathbf{r}}}$	acceleration w.r.t. $\boldsymbol{\mathcal{F}}_2$
$2\underset{\to}{\boldsymbol{\omega}}_{21} \times \underset{\to}{\overset{\circ}{\mathbf{r}}}$	coriolis accleration
$\underset{\to}{\overset{\circ}{\boldsymbol{\omega}}}_{21} \times \underset{\to}{\mathbf{r}}$	angular acceleration
$\underset{\to}{\boldsymbol{\omega}}_{21} \times \underset{\to}{\boldsymbol{\omega}}_{21} \times \underset{\to}{\mathbf{r}}$	centripetal acceleration

Angular Velocity Given Rotation Matrix

Begin with Eq. (6) which relates relates two reference frames via the rotation matrix:

$$\underset{\to}{\boldsymbol{\mathcal{F}}}_1^T = \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \mathbf{C}_{21}$$

Now take the time derivative of both sides as seen in $\boldsymbol{\mathcal{F}}_1$:

$$\underset{\to}{\mathbf{0}} = \underset{\to}{\dot{\boldsymbol{\mathcal{F}}}}_2^T \mathbf{C}_{21} + \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{C}}_{21}$$

Substitute Eq. (11) for $\dot{\boldsymbol{\mathcal{F}}}_2$:

$$\underset{\to}{\mathbf{0}} = \underset{\to}{\boldsymbol{\omega}}_{21} \times \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \mathbf{C}_{21} + \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{C}}_{21}$$

Now use (15) to get

$$\begin{aligned} \underset{\to}{\mathbf{0}} &= \boldsymbol{\omega}_{21}^T \underset{\to}{\boldsymbol{\mathcal{F}}}_2 \times \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \mathbf{C}_{21} + \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \dot{\mathbf{C}}_{21} \\ &= \underset{\to}{\boldsymbol{\mathcal{F}}}_2^T \left(\boldsymbol{\omega}_{21}^\times \mathbf{C}_{21} + \dot{\mathbf{C}}_{21} \right) \end{aligned}$$

Therefore, we conclude that

$$\dot{\mathbf{C}}_{21} = -\boldsymbol{\omega}_{21}^\times \mathbf{C}_{21} \tag{20}$$

Given the angular velocity as measured in the frame $\boldsymbol{\mathcal{F}}_2$, the rotation matrix relating $\boldsymbol{\mathcal{F}}_1$ to $\boldsymbol{\mathcal{F}}_2$ can be determined by integrating the above expression. This is termed 'strapdown navigation' because the sensors that measure $\boldsymbol{\omega}_{21}$ are strapped down in the rotating frame $\boldsymbol{\mathcal{F}}_2$.

Eqn. (20) can be written as an explicit function of $\boldsymbol{\omega}_{21}$:

$$\begin{aligned} \boldsymbol{\omega}_{21}^\times &= -\dot{\mathbf{C}}_{21} \mathbf{C}_{21}^{-1} \\ &= -\dot{\mathbf{C}}_{21} \mathbf{C}_{21}^T \end{aligned} \tag{21}$$

which gives the angular velocity when the rotation matrix is known as a function of time.

1.3 Attitude Parameterization

To this point we have seen one way of discussing the orientation of one reference frame with respect to another: the rotation matrix. This requires 9 parameters (they are not independent). There are a number of other alternatives.

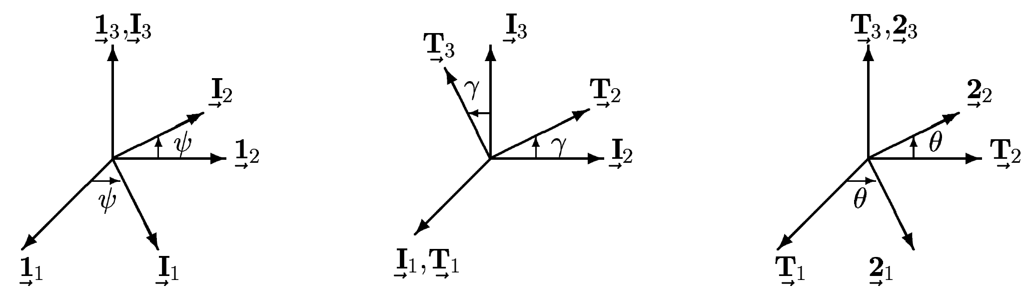
Euler Angles

The orientation of one reference frame with respect to another can also be specified by a sequence of three principal rotations. One possible sequence is as follows:

1. A rotation ψ about the original 3-axis.

2. A rotation γ about the intermediate 1-axis.

3. A rotation θ about the transformed 3-axis.



This tranformation is called a 3-1-3 transformation and is the sequence originally used by Euler. In the classical mechanics literature, the angles are referred to by the following names:

$$
\begin{aligned}
\theta &= \text{spin angle} \\
\gamma &= \text{nutation angle} \\
\psi &= \text{precession angle}
\end{aligned}
$$

The rotation matrix from frame 1 to frame 2 is given by:

$$
\begin{aligned}
\mathbf{C}_{21}(\theta,\gamma,\psi) &= \mathbf{C}_{2T}\mathbf{C}_{TI}\mathbf{C}_{I1} \\
&= \mathbf{C}_3(\theta)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi) \\
&= \begin{bmatrix} c_\theta c_\psi - s_\theta c_\gamma s_\psi & s_\psi c_\theta + c_\gamma s_\theta c_\psi & s_\gamma s_\theta \\ -c_\psi s_\theta - c_\theta c_\gamma s_\psi & -s_\psi s_\theta + c_\theta c_\gamma c_\psi & s_\gamma c_\theta \\ s_\psi s_\gamma & -s_\gamma c_\psi & c_\gamma \end{bmatrix}
\end{aligned} \tag{22}
$$

We have made the abbreviations $s = \sin$, $c = \cos$.

Another possible sequence that can be used is as follows:

1. A rotation θ_3 about the original 3-axis ('yaw' rotation).

2. A rotation θ_2 about the intermediate 2-axis ('pitch' rotation).

3. A rotation θ_1 about the transformed 1-axis ('roll' rotation).

This sequence which is very common in aerospace applications is called the 3-2-1 attitude sequence or the 'yaw-pitch-roll' convention.

In this case, the rotation matrix from frame 1 to frame 2 is given by:

$$\begin{aligned}\mathbf{C}_{21}(\theta_1, \theta_2, \theta_3) &= \mathbf{C}_1(\theta_1)\mathbf{C}_2(\theta_2)\mathbf{C}_3(\theta_3) \\ &= \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 \end{bmatrix}\end{aligned} \tag{23}$$

where $s_i = \sin\theta_i$, $c_i = \cos\theta_i$.

The above transformations have singularities. If $\gamma = 0$ for the 3-1-3, then the angles θ and ψ become associated with the same degree of freedom and cannot be uniquely determined.

For the 3-2-1, a singularity occurs when $\theta_2 = \pi/2$. In this case:

$$\mathbf{C}_{21}(\theta_1, \frac{\pi}{2}, \theta_3) = \begin{bmatrix} 0 & 0 & -1 \\ \sin(\theta_1 - \theta_3) & \cos(\theta_1 - \theta_3) & 0 \\ \cos(\theta_1 - \theta_3) & -\sin(\theta_1 - \theta_3) & 0 \end{bmatrix}$$

Therefore, θ_1 and θ_3 are associated with the same rotation.

Infinitesimal Rotations

Consider the 3-2-1 transformation, (23), when the angles satisfy $|\theta_1|$, $|\theta_2|$, $|\theta_3| \ll 1$, *i.e.*, small angles. In this case, we make the approximations $c_i \doteq 1$, $s_i \doteq \theta_i$ and neglect products of small angles, $\theta_i\theta_j \doteq 0$:

$$\begin{aligned}\mathbf{C}_{21} &= \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \\ &= \mathbf{1} - \boldsymbol{\theta}^\times\end{aligned} \tag{24}$$

where

$$\boldsymbol{\theta}^T = [\theta_1 \;\; \theta_2 \;\; \theta_3]$$

It is easy to show that the form of the rotation matrix for infinitesmal rotations ('small angle approximation') does not depend on the order in which the rotations are performed. For example, show that the same result is obtained for a 2-1-3 Euler sequence.

Euler Parameters

Euler's Theorem. *The most general motion of a rigid body with one point fixed is a rotation about an axis through that point.*

Let us denote, the axis by $\mathbf{a} = [a_1 \;\; a_2 \;\; a_3]^T$ and assume that it is a unit vector:

$$\mathbf{a}^T\mathbf{a} = a_1^2 + a_2^2 + a_3^2 \equiv 1 \tag{25}$$

The angle of rotation is φ. We state, without proof, that the rotation matrix in this case is given by

$$\mathbf{C}_{21} = \cos\varphi \mathbf{1} + (1 - \cos\varphi)\mathbf{a}\mathbf{a}^T - \sin\varphi \mathbf{a}^\times \tag{26}$$

It does not matter in which frame $\mathbf{a}$ is expressed because

$$\mathbf{C}_{21}\mathbf{a} = \mathbf{a} \tag{27}$$

The combination of variables

$$\eta = \cos\frac{\varphi}{2}, \quad \boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{\varphi}{2} = \begin{bmatrix} a_1 \sin(\varphi/2) \\ a_2 \sin(\varphi/2) \\ a_3 \sin(\varphi/2) \end{bmatrix} \tag{28}$$

is particularly useful. The four parameters $\{\eta, \boldsymbol{\varepsilon}\}$ are called the *Euler parameters* associated with a rotation. They are not independent because they satisfy the constraint

$$\eta^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = 1$$

The rotation matrix (26) can be expressed in terms of the Euler parameters as

$$\begin{aligned} \mathbf{C}_{21} &= (\eta^2 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - 2\eta\boldsymbol{\varepsilon}^\times \\ &= \begin{bmatrix} 1 - 2(\varepsilon_2^2 + \varepsilon_3^2) & 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\eta) & 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\eta) \\ 2(\varepsilon_2\varepsilon_1 - \varepsilon_3\eta) & 1 - 2(\varepsilon_3^2 + \varepsilon_1^2) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\eta) \\ 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\eta) & 2(\varepsilon_3\varepsilon_2 - \varepsilon_1\eta) & 1 - 2(\varepsilon_1^2 + \varepsilon_2^2) \end{bmatrix} \end{aligned} \tag{29}$$

Euler parameters are used most often in actual space applications. There are no singularities associated with them and the calculation of the rotation matrix does not involve trigonometric functions (compare (29) with (22) and (23)) which is a significant numerical advantage. The only drawback is the use of four parameters instead of three as is the case with Euler angles.

1.4 Attitude Solution Given Angular Velocity

In the last section, we showed that the orientation of one frame $\boldsymbol{\mathcal{F}}_2$ with respect to another $\boldsymbol{\mathcal{F}}_1$ could be parameterized in different ways. In other words, the rotation matrix could be written as a function of Euler angles or Euler parameters. However, in most applications the attitude changes with time and we must solve a different problem. Usually, the angular velocity of one frame with respect to another, $\boldsymbol{\omega}_{21}$, is known and one must calculate the attitude from this information.

An example of this type of problem is Eq. (20),

$$\dot{\mathbf{C}}_{21} = -\boldsymbol{\omega}_{21}^\times \mathbf{C}_{21}$$

If the initial condition of $\mathbf{C}_{21}$ is known and $\boldsymbol{\omega}_{21}(t)$ is given, then the equation can be integrated to yield the attitude history in the form $\mathbf{C}_{21}(t)$. If $\mathbf{C}_{21}(t)$ is known than one can calculate the corresponding Euler angles or Euler parameters.

Euler Angles

Consider the 3-2-1 Euler angle sequence and its associated rotation matrix (23). If this is substituted into (21), one can show (eventually) that

$$\begin{aligned} \boldsymbol{\omega}_{21} &= \underbrace{\begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix}}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ &= \mathbf{S}(\theta_1,\theta_2)\dot{\boldsymbol{\theta}} \end{aligned} \tag{30}$$

which gives the angular velocity in terms of the Euler angles and the *Euler rates*, $\{\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\}$. By inverting the matrix $\mathbf{S}$, we arrive at a system of differential equations which can be integrated to yield the Euler angles assuming $\boldsymbol{\omega}_{21}$ is known:

$$\begin{aligned} \dot{\boldsymbol{\theta}} &= \mathbf{S}^{-1}(\theta_1,\theta_2)\boldsymbol{\omega}_{21} \\ &= \begin{bmatrix} 1 & \sin\theta_1\tan\theta_2 & \cos\theta_1\tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1\sec\theta_2 & \cos\theta_1\sec\theta_2 \end{bmatrix}\boldsymbol{\omega}_{21} \end{aligned} \tag{31}$$

Infinitesimal Rotations

The angular velocity for a sequence of small rotations can be calculated using (21) and (24):

$$\begin{aligned} \boldsymbol{\omega}_{21}^{\times} &= -\dot{\mathbf{C}}_{21}\mathbf{C}_{21}^T \\ &= \dot{\boldsymbol{\theta}}^{\times}(\mathbf{1}+\boldsymbol{\theta}^{\times}) \\ &= \dot{\boldsymbol{\theta}}^{\times} + \dot{\boldsymbol{\theta}}^{\times}\boldsymbol{\theta}^{\times} \end{aligned}$$

If we assume that the angular rates $\dot{\theta}_i$ are small and neglect products of small angles and rates, then the above reduces to $\boldsymbol{\omega}_{21}^{\times} = \dot{\boldsymbol{\theta}}^{\times}$ or

$$\boldsymbol{\omega}_{21} = \dot{\boldsymbol{\theta}} \tag{32}$$

Therefore,

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t \boldsymbol{\omega}_{21}(\tau)\, d\tau$$

For small angles *and* rates, the components of angular velocity can be integrated directly to yield the (small) attitude angles. In general, however, one must integrate (20) for $\mathbf{C}_{21}$ or an equation of the form of (31) for the Euler angles.